# Strong quasipositivity of links


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# Contents

**Definition 1** *Notation:* $\sigma_i$, *for* $i = 1, \dots, n-1$, *is* Artin generator *of the* braid group $B_n$.

*quasipositive link* (Rudolph, Boileau-Orevkov): If zero set of a complex polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ intersects the unit sphere

$$S^3 = \{\, (u,v) \in \mathbb{C}^2 \,:\, |u|^2 + |v|^2 = 1 \,\}$$

transversely, the intersection forms a quasipositive link $L$ in $S^3$.

$L$ is *closure* $\hat{\beta}$ of a braid

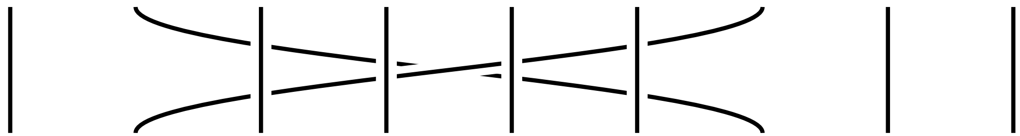
$$\beta = \prod_{k=1}^{l} w_k \sigma_{i_k} w_k^{-1}$$

which is product of conjugates of $\sigma_i$ (but *not* their inverses).

*strongly quasipositive link*: If the words $w_k \sigma_{i_k} w_k^{-1}$ are of the form

$$\sigma_{i,j} = \sigma_i^{-1} \dots \sigma_{j-2}^{-1} \sigma_{j-1} \sigma_{j-2} \dots \sigma_i \,, \tag{1}$$

for $1 \leq i < j \leq n$, "embedded bands" ($\sigma_i = \sigma_{i,i+1}$)

$B_9 \ni \sigma_{2,7}$ =  ,

then

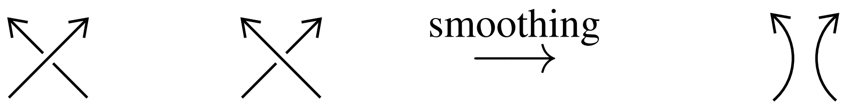
$$\beta = \prod_{k=1}^{l} \sigma_{i_k,j_k} \tag{2}$$

is a *strongly quasipositive braid (word)* and a *positive band representation* of $\beta$.
It gives a *strongly quasipositive surface* of $L = \hat{\beta}$.

More generally, a word $\beta = \prod \sigma_{i_k,j_k}^{\pm 1}$ gives a *band representation* and a *braided surface* (Rudolph) of $L = \hat{\beta}$.

Minimal genus braided surface is called *Bennequin surface* (Birman-Menasco).

All link diagrams and links assumed oriented. Crossings in oriented diagrams are called




positive negative smoothed out (3)

$c_{\pm}(D) :=$ number of positive, respectively negative crossings of a diagram $D$,
$c(D) = c_+(D) + c_-(D)$ *crossing number* and
$w(D) = c_+(D) - c_-(D)$ *writhe*.

After smoothing all crossings, $s(D) :=$ number of *Seifert circles*.

*Crossing number* of a knot $K$:

$$c(K) := \{\, c(D) \,:\, D \text{ is a diagram of } K \,\}\,.$$

**Definition 2** Positive link *is a link with a positive diagram ($c_-(D) = 0$).*

$$\{\text{positive}\} \subset \{\text{strongly quasipositive}\} \subset \{\text{quasipositive link}\}$$

(first inclusion proved by Rudolph, but see below)

**Definition 3** *(maximal)* Thurston-Bennequin number $TB(L)$

$$TB(L) = \max\{\, w(D) - s(D) \; : \; D \textit{ a diagram of } L \,\}\,.$$

## Bennequin's inequality and Bennequin surfaces

**Theorem 4** Bennequin's inequality *states that*

$$TB(L) \leq -\chi(L)\,. \tag{4}$$

*($\chi$ = Euler char.)*

*Improvement: the* slice Bennequin inequality *(Kronheimer-Mrowka, Rudolph, etc.) states that*

$$TB(L) \leq -\chi_4(L) \leq -\chi(L) \tag{5}$$

*($\chi_4$ =* smooth *4-ball Euler char.)*

$\tau$-invariant (for *knots L*, Ozsvath-Szabo), “slice-torus” invariants (Livingston);

$$TB(L) \leq 2\tau(L) - 1 \leq -\chi_4(L)$$

**Definition 5** Bennequin defect $\delta(L) = (-\chi(L) - TB(L))/2$ *(non-negative integer).*

**Definition 6** *We call L* Bennequin-sharp *if* (4) *is equality (i.e.,* $\delta(L) = 0$*).*
*We call L* slice-Bennequin-sharp *if* (5) *is equality.*

$$\begin{array}{ccc} \text{quasipositive} & \Longrightarrow & \text{slice-Bennequin-sharp} \\ \Uparrow & & \Uparrow \\ \text{strongly quasipositive} & \Longrightarrow & \text{Bennequin-sharp} \end{array}$$

**Conjecture 7** *Bennequin-sharp* $\Longleftrightarrow$ *strongly quasipositive*

Bennequin’s inequality $\Rightarrow$ strongly quasipositive surface is minimal genus

**Question 8** (Baker-茂手木) If $L$ is strongly quasipositive, then is *every* minimal genus surface of $L$ strongly quasipositive?

**Remark 9** obviously true for fibered $L$ (or more generally if unique minimal genus surface)

**Definition 10** $b(L)$ braid index:

$$b(L) = \min\{ n : \exists \beta \in B_n , L = \hat{\beta} \} .$$

**Question 11** (Rudolph) If $L$ is strongly quasipositive, does $L$ have a strongly quasipositive surface on minimal ($= b(L)$) strands/strings?

**Remark 12** For Bennequin surfaces, proved by Bennequin (-Birman-Menasco) if $b(L) = 3$, but false for $b(L) = 4$ (平澤-S.)

Answer to Conjecture 7 and Question 11 is affirmative for prime knots up to 16 crossings (see Computations 36 and 37).

**Example 13** $13_{6169} \to 14_{46970}$. *The number of strings on which a strongly quasipositive surface can be realized, can go* down *under positive Hopf plumbing (even for fiber surfaces).*

**Question 14** Does every *alternating* link have a minimal string Bennequin surface?

Generalization of Question 11:

**Conjecture 15** *(伊藤-川室) Every link L has a Bennequin surface on (at most)* $b(L) + \delta(L)$ *strands.*

## 村杉-Przytycki method and graph index

used (in conjunction with 山田-Vogel) to estimate braid index (from above).

**Definition 16** $\Gamma(D)$ Seifert graph *of D: a vertex for every Seifert circle of D, and an edge for every crossing connecting two (distinct) Seifert circles. (Edge is labeled* $\pm$ *by sign of crossing.)*

$\Gamma(D)$ is always bipartite and planar, but not mandatorily simple.

Say two parallel edges in $\Gamma(D)$ come from *Seifert equivalent* crossings in $D$.

Cycles $C$ in $\Gamma(D)$ have even length, and pair of parallel edges form $|C| = 2$.

**Definition 17** *(using Traczyk's* cycle condition*): Edge set $E$ in $\Gamma(D)$ is* independent *if for each cycle $C$ of $\Gamma(D)$ of length $2n$, we have $|E \cap C| < n$.*

**Definition 18**

$$\begin{aligned} \operatorname{ind}(D) &= \textit{maximal size of independent set} \\ \operatorname{ind}_{\mp}(D) &= \textit{maximal size of independent set of negative/positive edges} \end{aligned}$$

**Theorem 19** *("村杉-Przytycki", Traczyk, S.,...)*

$$b(L) \le s(D) - \operatorname{ind}(D)\,.$$

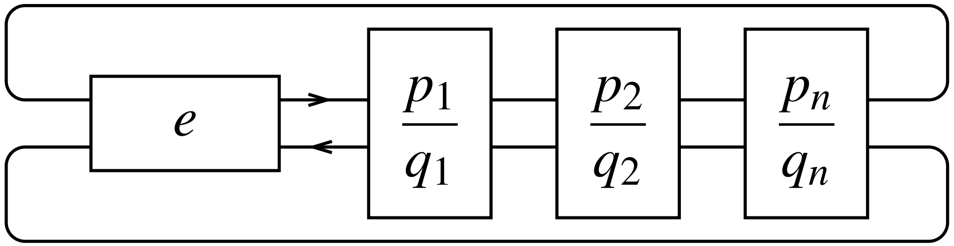
**Conjecture 20** *(村杉-Przytycki) If $D$ is an alternating diagram, then $b(L) = s(D) - \operatorname{ind}(D)$.*

**Example 21** *"Rational (2-bridge) links L", (tested; S.) links with $b(L) \leq 4$, pure or parallel alternating-pretzel links L,* knots *L with $\chi(L) \geq -7$, knots L with at most 18 crossings, ...*

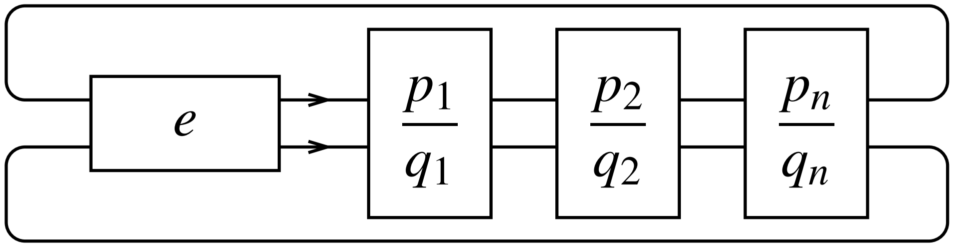
For *Montesinos links*

$$M(e; {}^{p_1}/_{q_1}, \ldots, {}^{p_n}/_{q_n}) \tag{6}$$

($|p_i| < q_i$, $(|p_i|, q_i) = 1$, etc.) orientation issues near the integer part become essential:




*reverse* link




*parallel* link

**Definition 22** *If all* $|p_i| = 1$*, the Montesinos link is a* pretzel link. *A pretzel link is* alternating-pretzel *if* $e \geq 0, p_i/q_i > 0$*, and we say a pretzel link is* pure *if* $e = 0$ *and* $|q_i| > 1$.

**Remark 23** There are alternating (and non-pure) pretzel links which are not alternating-pretzel links.

## 村杉-Przytycki-山田 method for braided surfaces

**Theorem 24** *(S.) If* $E$ *is an independent set of* $\Gamma(D)$ *of a diagram* $D$ *of* $L$ *of* $e_+$ *positive and* $e_-$ *negative edges, then the canonical surface of* $D$ *is a braided surface on* $s(D) - |E|$ *strings with* $c_+(D) - e_+$ *positive bands and* $c_-(D) - e_-$ *negative bands.*

(Proof is constructive.)

For Bennequin surfaces, we are in particular interested when

**Definition 25** *If $\chi(D) = \chi(L)$, then $D$ is* minimal genus diagram *(e.g., alternating, positive, homogeneous, ...). If $L$ has minimal genus diagram $D$, call $L$* canonically minimal.

**Corollary 26** *If conjecture 20 is true, then every alternating link has a minimal string Bennequin surface (Question 14).*

In particular true for special cases (listed in example 21; 2-bridge independently by 平澤).

Without Conjecture 20, can address Conjecture 15 (伊藤-川室) for further alternating links, for example:

**Proposition 27** *Conjecture 15 holds for alternating* knots *$K$ with $\tau(K) \leq 1$ (among others achiral, slice knots, knots of unknotting number one, etc.), and with $\tau(K) = 2$ when $c(K)$ is odd.*

**Definition 28** *Call $D$* pseudo-positive *if its negative crossings are independent ($c_-(D) = \text{ind}_-(D)$). Pseudo-positive link = link with pseudo-positive diagram.*

**Corollary 29** *If D is a pseudo-positive diagram of L, then L is strongly quasipositive (on* $s(D) - c_-(D)$ *strings).*

Converse is also true for minimal genus diagram (but Bennequin-sharp is enough):

**Proposition 30** *D is* minimal genus *diagram of Bennequin-sharp link L* $\Rightarrow$ *D is pseudo-positive.*

Thus can decide from minimal genus diagram if a link is strongly quasipositive.

**Corollary 31** *L is pseudo-positive* $\Longleftrightarrow$ *L is canonically minimal and strongly quasi-positive*

Conjecture 7 holds for canonically minimal *L*.

**Corollary 32** *L is canonically minimal* $\Rightarrow$ *(Bennequin-sharp* $\Longleftrightarrow$ *strongly quasi-positive)*

Question 8 (Baker-茂手木) is answered positively for canonical surfaces:

**Corollary 33** *If a link L is strongly quasipositive, then every minimal genus* canonical *surface of L is strongly quasipositive.*

**Remark 34** Feller-Lewark-Lobb knew corollaries 31, 32 and 33 for *knots*, but $\tau$-invariant technology is not needed here.

## Strong quasipositivity of some families of links

Many knots can be proved strongly quasipositive, by exhibiting a pseudo-positive diagram.

**Example 35** *(FLL; using a check of mine) All strongly quasipositive prime knots up to 13 crossings are pseudo-positive.*

For alternating diagrams, situation is simple:
strongly quasipositive $\Longleftrightarrow$ positive alternating (村杉's *special alternating*).

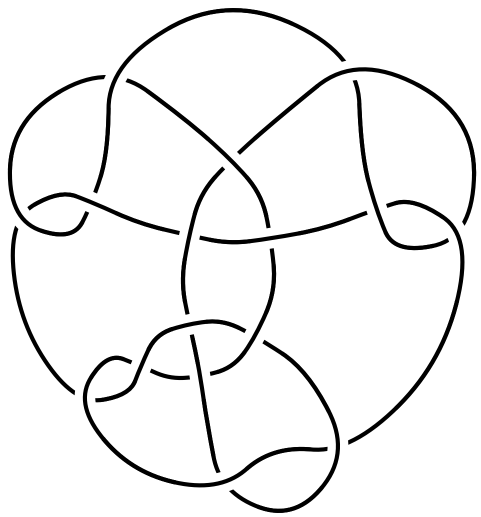
**Computation 36** *There are 22,009 prime non-alternating knots up to 16 crossings which are strongly quasipositive.*

(Previously determined up to 12; for 13 crossings *partial outline* in example 35.)
Data on my website
`http://www.stoimenov.net/stoimeno/homepage/ptab/index.html`.

For *special* alternating diagrams listed in example 21, Rudolph's question 11 is affirmed. Includes alternating knots up to 18 crossings.

**Computation 37** *Rudolph's question 11 is affirmed for all prime* non-alternating $\leq 16$ *crossing knots.*

*Hardest example,* $16_{1057125}$*:*



Major *computational project: ruled out* $b = 5$ *for this knot; a parallelized upgrade of the (m-truncated) HOMFLY polynomial calculation on a 4-cable.*

**Claim 38**
- Can classify strongly quasipositive pretzel *knots* (I need $\tau$ here!; we allow $e \neq 0$ or $q_i = \pm 1$ in (6), non-pure pretzels), and
- have simple (2-page) decision algorithm for (strong quasipositivity of) *parallel* Montesinos *links*.

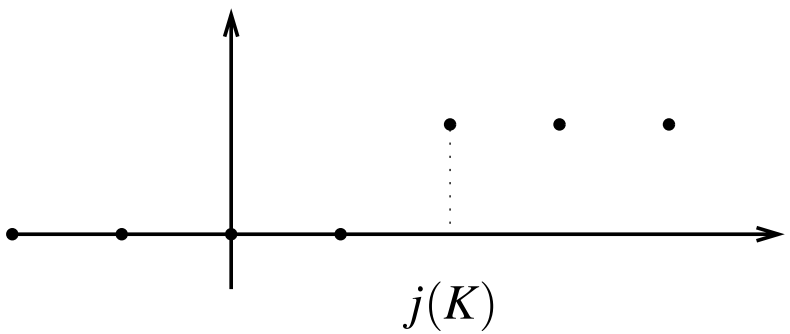
(slightly different method)

**Definition 39** *Whitehead double* $W_{\pm}(K,p)$ *of a knot* $K$ *with framing* $p$ *and positive/negative clasp.*

**Question 40** When is $W_{\pm}(K,p)$ strongly quasipositive?

**Theorem 41** $W_{-}(K,p)$ *is never strongly quasipositive.*
$W_{+}(K,p)$ *is strongly quasipositive* $\iff p \geq -TB(K)$.

$p \mapsto \tau(W_{+}(K,p)) \in \{0,1\}$ is almost constant with one *jump* $j(K) = j_{\tau}(K)$
(studied by 金世久, etc.)



Livingston-Naik do this for "slice torus" invariants $v$.

**Corollary 42** *(Livingston-Naik)* $j_v(K) \leq -TB(K)$.

For the "real" $\tau$, we have $j_\tau(K) = 1 - 2\tau(K)$ (Hedden).

**Corollary 43** *If* $-TB(K) = 1 - 2\tau(K)$ *(incl. if $K$ slice-Bennequin-sharp, in particular quasipositive), then conjecture 7 true for the Whitehead doubles of $K$.*

This is also related to arc index $a(L)$ (gives very simple proof that $a(L) \geq 2b(L)$, etc.); work project with 秦教澤 and 李和靜.

General satellites

**Definition 44** $P \subset$ *solid torus* pattern. *Type* $(a,g)$ *algebraic/geometric* $\cap$ # *(need* $|a| \leq g$ *and* $g - a$ *even; for simplicity assume* $g > 1$, *no connected sums).*
*Say $P$ is* cable type *if* $|a| = g$. *This includes a* braid pattern $P = \beta \in B_n$ *(up to conjugacy), where* $a = g = n$.

**Claim 45** For every knot $K$ and non-cable type $(a,g)$, there is a pattern $P_K$ of type $(a,g)$ with $K * P_K$ strongly quasipositive.

Can also construct many strongly quasipositive cable type satellites:

**Proposition 46** *If $K$ is strongly quasipositive, and $P = \beta$ is strongly quasipositive braid* (2)*, then $K * \beta$ is strongly quasipositive.*

(Works for quasipositive also.) But:

**Example 47** *No (ordinary!, i.e., $P = \beta \in B_2$) 2-cable of figure-8-knot is strongly quasipositive.*

**Conjecture 48** *If $P$ is cable type and $K * P$ is strongly quasipositive, then $K$ is strongly quasipositive.*

**Question 49** When does Seifert's algorithm give a minimal genus surface?

**Definition 50** *Say L is* canonically minimal *if has a minimal genus canonical surface.*

**Example 51** *alternating (村杉-Gabai), alternative (Kauffman), positive (村杉-"Bennequin-山田-Vogel"), homogeneous (Cromwell)*

**Theorem 52** *(FLL for knots, S.) If D is a diagram of a strongly quasipositive link, then D is pseudo-positive if and only if D is minimal genus.*

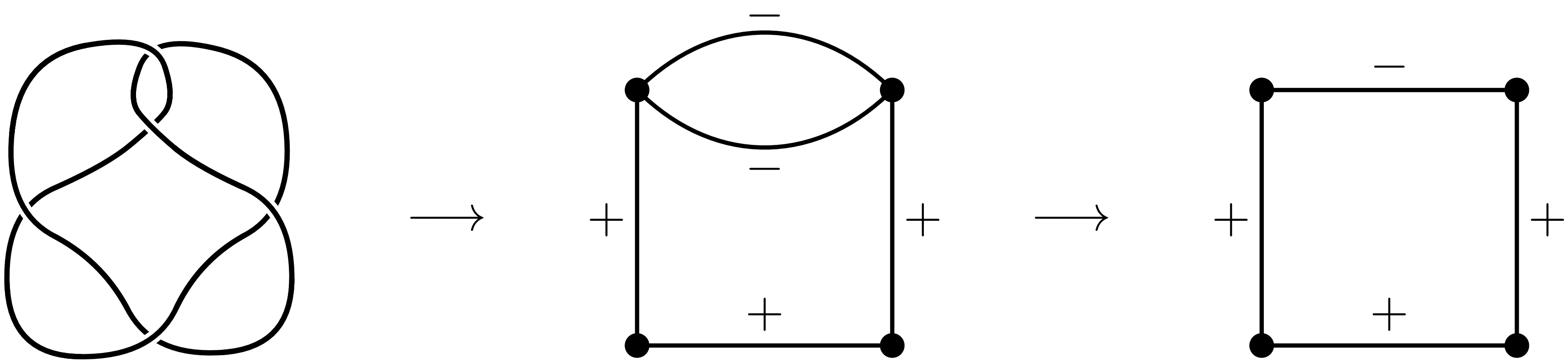
This + Gabai gives a generalization of homogeneous diagrams. (But Alexander polynomial *does not always* work!)

**Definition 53** *A diagram D is* pseudo-homogeneous, *if all blocks are pseudo-positive or -negative.*

*D is* weakly pseudo-homogeneous, *if parallel edges in* $\Gamma(D)$ *have same sign, and when replacing them by a single edge the diagram gets pseudo-homogeneous.*

As usual, X link = link with X diagram (X=...).

**Example 54** *The 5-crossing fig-8-knot diagram is weakly pseudo-homogeneous but not pseudo-homogeneous.*



**Theorem 55**

*canonically minimal* $\supset$ *weakly pseudo-homog.* $\supset$ *pseudo-homogeneous* $\supset$ *homogeneous* $\supset$ *alternative* $\supset$ *alternating* $\supset$ *special alternating*

$\cup$ $\cup$ $\cap$

*quasi-positive* $\supset$ *strongly q.p.* $\supset$ *pseudo-positive* $\supset$ *positive* $\supset$ *positive braid*

**Remark 56** All inclusions *of links* except the red one are known to be proper. (The red one is tricky := not very important but very difficult.)

A (likely) more important top. version:

**Question 57** 1. Is *one* minimal genus surface of every link (for $\chi < 0$) 村杉 sum of strongly quasipositive and strongly quasinegative surfaces?

2. *Every* minimal surface when $\chi$ is small? (I know is *not* when $\chi \geq -1$.)

3. Stable 村杉 sum?

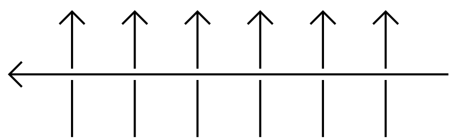
**Example 58** $10_{145}\#3_1$*: pseudo-homogeneous (in fact, -positive) diagrams are not visually prime (but homogeneous may still be: Menasco-小澤 +).*

## (Weakly) successively almost positive links (joint w/ 哲.伊藤)

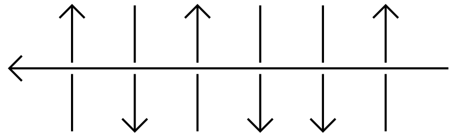
**Problem 59** Can one pose a natural condition on (arbitrary many) negative crossings in a diagram, so that positivity properties can be generalized?

One answer (above): pseudo-positive.
I discuss here another answer (joint w/ 哲.伊藤).

**Definition 60** *("伊藤-茂手木-寺垣内") A diagram is* successively almost positive *(s.a.p.) if all negative crossings appear* consecutively *with their overpass along a single overarc.*



**Definition 61** *(伊藤-S.; "Cromwell") A diagram is* weakly successively almost positive *(w.s.a.p.) if all negative crossings appear with their overpass along a single overarc.*



This extends my old definition:

**Definition 62** *(S.) A diagram D is* almost positive *if* $c_-(D) = 1$.

Obviously:

$$\{\text{w.s.a.p.}\} \supset \{\text{s.a.p.}\} \supset \{\text{almost positive}\} \supset \{\text{positive link}\} \tag{7}$$

(Here I do *not* exclude positive links from almost positive ones.)

**Question 63** $\{\text{w.s.a.p. link}\} \supsetneq \{\text{s.a.p. link}\}$?

We know other inclusions in (7) are proper.

Many properties (like of link polynomials) of positive links extendable to w.s.a.p. links, for example:

**Theorem 64** *The Alexander polynomial detects genus and fibering of a w.s.a.p. link.*

**Remark 65** Does not work for pseudo-positive: $(-3,5,5),(-3,5,7)$-pretzel knots.

**Question 66** (伊藤-S.) Are all (w.) s.a.p. links strongly quasipositive?

**Proposition 67** *(伊藤-S.) s.a.p. $\Rightarrow$ Bennequin sharp*

**Theorem 68** *(Feller-Lewark-Lobb's answer to old question of mine)*
*link $L$ is almost positive $\Rightarrow$ strongly quasipositive.*

**Definition 69** *(FLL, "S.") An almost positive diagram $D$ is*

| | | |
|---|---|---|
| *type 1* | $\Longleftrightarrow$ | *$D$ is minimal genus ( $\Longleftrightarrow$ negative crossing has no Seifert equivalent positive crossing)* |
| *type 2* | $\Longleftrightarrow$ | *$D$ not minimal genus ( $\Longleftrightarrow$ negative crossing has Seifert equivalent positive crossing)* |

Theorem 24 $\Rightarrow$ type 1 is strongly quasipositive on $s(D)-1$ strings

**Theorem 70** *(伊藤-S.) L has s.a.p. diagram "with full marking" $\Rightarrow$ L is strongly quasipositive.*

$\Rightarrow$ type 2 is strongly quasipositive on $s(D)$ strings.

In combination, we have a refinement of Theorem 68:

**Theorem 71** *Let a link L have an almost positive diagram D. Then L has a positive band representation on (at most)*

$$\left\{ \begin{array}{cc} s(D)-1 & \textit{if } D \textit{ is of type 1} \\ s(D) & \textit{if } D \textit{ is of type 2} \end{array} \right\}$$

*strands.*

# Thank you!

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