Translation lengths of right-angled Artin groups on extension graphs

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Translation length

(Definition) Suppose a group G acts on a metric space (X, d) isometrically. For $g \in G$, the asymptotic translation length of g is

$$\tau(g) = \lim_{n \to \infty} \frac{d(x, g^n x)}{n},$$

where $x \in X$. The element g is called loxodromic if $\tau(g) > 0$.

(Remark)
$\tau(g)$ is independent of choice of $x \in X$;
$\tau(g^n) = n\,\tau(g)$ for $n \geqslant 1$;
$\tau(g^h) = \tau(g)$ for $h \in G$, where $g^h = h^{-1}gh$.

Length spectrum

(Definition) Length spectrum of G is

$$\mathrm{Spec}(G) = \{\tau(g) \mid g \in G\}.$$

Gromov (1987), Delzant (1996)
If G is a hyperbolic group, $\mathrm{Spec}(G) \subset \frac{1}{m}\mathbb{Z}$ for some $m \geqslant 1$.

(Translation lengths are rational with uniformly bounded denominators.)

Bowditch (2008)
If $G = \mathrm{Mod}(S)$ for a hyperbolic surface S and $X = \mathcal{C}(S)$, the curve graph of S, then $\mathrm{Spec}(G) \subset \frac{1}{m}\mathbb{Z}$ for some $m \geqslant 1$.

Conner (1997)
There exists a finitely presented group G whose action on its Cayley graph has an irrational length element, i.e. $\mathrm{Spec}(G) \not\subset \mathbb{Q}$.

Minimal asymptotic translation length

(Definition) Minimal asymptotic translation length of G is

$$\mathrm{MinTL}(G) = \inf\{\mathrm{Spec}(G) \setminus \{0\}\}.$$

Gadre and Tsi (2011) $\mathrm{MinTL}(\mathrm{Mod}(S_g)) \asymp \frac{1}{g^2}$.

Kin and Shin (2019) $\mathrm{MinTL}(B_n) \asymp \frac{1}{n^2}$.

Baik and Shin (2020) $\mathrm{MinTL}(PB_n) \asymp \frac{1}{n}$.

S_g denote a orientable surface of genus g.
$\mathrm{Mod}(S_g)$ acts on the curve graph $\mathcal{C}(S_g)$.
B_n and PB_n act on the curve graph $\mathcal{C}(D_n)$ of n-punctured disk D_n.

Right-angled Artin group

We are interested in the asymptotic translation lengths when

G = the right-Angled Artin group $A(\Gamma)$,

X = the extension graph Γ^e,

where Γ is a finite simple graph.

Right-angled Artin group

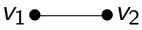
Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite simplicial graph.
The right-angled Artin group $A(\Gamma)$ is

$$A(\Gamma) = \langle\, v \in V(\Gamma) : [v_i, v_j] = 1 \;\text{ for each } \{v_i, v_j\} \in E(\Gamma)\,\rangle.$$

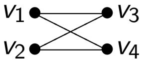
(Here, $[a, b]$ denotes the commutator $a^{-1}b^{-1}ab$.)

$\Gamma \;=$ 

$\Rightarrow \; A(\Gamma) = \langle v_1, v_2 \mid \; \rangle \simeq F_2$

$\Gamma \;=$ 

$\Rightarrow \; A(\Gamma) = \langle v_1, v_2 \mid v_1 v_2 = v_2 v_1 \rangle \simeq \mathbb{Z}^2$

$\Gamma \;=$ 

$\Rightarrow \; A(\Gamma) = \langle v_1, v_2 \mid \; \rangle \times \langle v_3, v_4 \mid \; \rangle \simeq F_2 \times F_2$

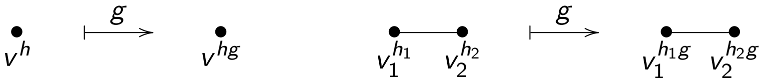
Extension graph Γ^e

The extension graph $\Gamma^e = (V(\Gamma^e), E(\Gamma^e))$ is the graph such that

$$V(\Gamma^e) = \{v^g : v \in V(\Gamma),\ g \in A(\Gamma)\} \subset A(\Gamma),$$
$$E(\Gamma^e) = \{\, \{v_1^{g_1}, v_2^{g_2}\} \ :\ [v_1^{g_1}, v_2^{g_2}] = 1 \text{ in } A(\Gamma)\,\}.$$

Here, v^g denotes the conjugate $g^{-1}vg$.
$V(\Gamma^e)$ is the set of all elements of $A(\Gamma)$ that are conjugate to a vertex, and two vertices $v_1^{g_1}$ and $v_2^{g_2}$ are adjacent in Γ^e if and only if they commute when considered as elements of $A(\Gamma)$.

- Γ^e is usually infinite and locally infinite.
- (Kim-Koberda) Γ^e is a quasi-tree, hence δ-hyperbolic.
- $A(\Gamma)$ acts on Γ^e from right by conjugation.



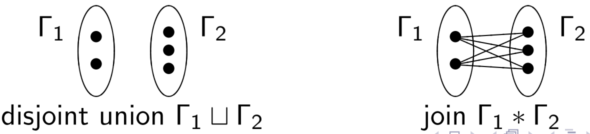
Let $\bar{\Gamma}$ denote the complement graph of Γ, i.e. $V(\bar{\Gamma}) = V(\Gamma)$ and $\{v_1, v_2\} \in E(\bar{\Gamma})$ if and only if $\{v_1, v_2\} \notin E(\Gamma)$ for $v_1 \neq v_2$.

If Γ is disconnected, then Γ^e is also disconnected.

(If $\Gamma = \Gamma_1 \sqcup \Gamma_2$, then Γ^e is a countable union of copies of Γ_1^e and Γ_2^e.)

If $\bar{\Gamma}$ is disconnected, then diameter$(\Gamma^e) \leqslant 2$.

(If $\Gamma = \Gamma_1 * \Gamma_2$, then $\Gamma^e = \Gamma_1^e * \Gamma_2^e$, hence diameter$(\Gamma^e) \leqslant 2$.)



Extension graph Γ^e

Theorem (Kim-Koberda, 2013)

Γ^e is connected and has infinite diameter if and only if $|V(\Gamma)| \geqslant 2$ and both Γ and $\bar{\Gamma}$ are connected.

From now on, we assume that
(i) both Γ and $\bar{\Gamma}$ are connected,
(ii) $|V(\Gamma)| \geqslant 2$, hence $|V(\Gamma)| \geqslant 4$.

Length spectrum of the action of $A(\Gamma)$ on Γ^e

Let C_n denote the cycle with n vertices.
Let girth(Γ) denote the length of a shortest cycle contained in Γ.

Theorem (Baik-Seo-Shin (2022))

For any Γ, $\mathrm{Spec}(A(\Gamma)) \subset \mathbb{Q}$.
If $\mathrm{girth}(\Gamma) \geqslant 6$, *then* $\mathrm{Spec}(G) \subset \frac{1}{m}\mathbb{Z}$ *for some* $m \geqslant 1$.

Theorem (Baik-Seo-Shin (2022))

If Γ *is a tree, then* $\mathrm{Spec}(A(\Gamma)) \subset 2\mathbb{Z}$.
If $\Gamma = C_{2m}$ *with* $m \geqslant 3$, *then* $\mathrm{Spec}(A(\Gamma)) \subset \mathbb{Z}$.
If $\Gamma = C_{2m+1}$ *with* $m \geqslant 3$, $\mathrm{Spec}(A(\Gamma)) \subset \frac{1}{2}\mathbb{Z}$ *and* $\mathrm{Spec}(A(\Gamma)) \not\subset \mathbb{Z}$.

Minimal translation length of the action of $A(\Gamma)$ on Γ^e

Theorem (Baik-Seo-Shin (2022))

If diameter$(\Gamma) \geqslant 3$, *then* $\mathrm{MinTL}(A(\Gamma)) \leqslant 2$.

Theorem (Lee-L (2023))

For any Γ, $\mathrm{MinTL}(A(\Gamma)) \leqslant 2$.

Theorem (Lee-L (2023))

If diameter$(\bar{\Gamma}) = d \geqslant 3$, $\mathrm{MinTL}(A(\Gamma)) \leqslant \frac{2}{d-2}$.

Theorem (Lee-L (2022))

For any Γ, $\mathrm{MinTL}(A(\Gamma)) \geqslant \frac{1}{|V(\Gamma)|-2}$.

Idea Proof of "MinTL$(A(\Gamma)) \leqslant 2$"

From now on, we will explain the idea of the proof of

$$\mathrm{MinTL}(A(\Gamma)) \leqslant 2,$$

i.e. there exists an element $g \in A(\Gamma)$ such that

$$0 < \tau(g) \leqslant 2.$$

Idea Proof of "MinTL$(A(\Gamma)) \leqslant 2$"

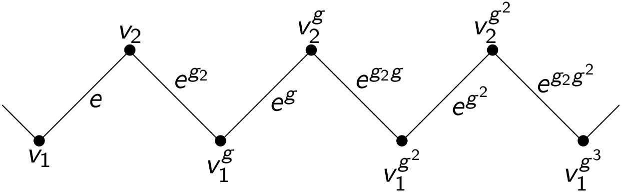
Lemma

Suppose that $e = \{v_1, v_2\} \in E(\Gamma)$, $g_1 \in Z(v_1)$ and $g_2 \in Z(v_2)$. Let $g = g_1 g_2$. Then $\tau(g) \leqslant 2$.

(Proof) Since $v_1^g = (v_1^{g_1})^{g_2} = v_1^{g_2}$ and $v_2 = v_2^{g_2}$,

$$\{v_1^g, v_2\} = \{v_1^{g_2}, v_2^{g_2}\} = e^{g_2} \in E(\Gamma^e).$$

Therefore, $d(v_1, v_1^{g^n}) \leqslant 2n$ and $\tau(g) = \lim_{n\to\infty} \frac{d(v_1, v_1^{g^n})}{2n} \leqslant 2.$



Idea Proof of "MinTL$(A(\Gamma)) \leqslant 2$"

For $g \in A(\Gamma)$, let supp(g) denote the set of generators that appear in a shortest word representing g.
For example, $\mathrm{supp}(v_1^{-1}v_2^3v_3^{-2}) = \{v_1, v_2, v_3\}$.

(Kim-Koberda)
The following are equivalent for cyclically reduced $g \in A(\Gamma)$.

1. g is loxodromic, i.e. $\tau(g) > 0$;
2. supp(g) is not contained in a subjoin of Γ;
3. $\bar{\Gamma}[\mathrm{supp}(g)]$ is connected and supp(g) dominates $\bar{\Gamma}$.

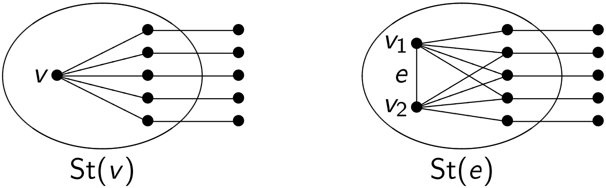
Here, $\bar{\Gamma}[\mathrm{supp}(g)]$ denotes the subgraph of $\bar{\Gamma}$ induced by supp(g).
We say that "supp(g) dominates $\bar{\Gamma}$" if every vertex v is either contained in supp(g) or adjacent in $\bar{\Gamma}$ to a vertex of supp(g).

Idea Proof of "MinTL($A(\Gamma)$) $\leqslant 2$"

For $v \in V(\Gamma)$ and $e = \{v_1, v_2\} \in E(\Gamma)$, define their stars as follows.

$$\mathrm{St}(v) = \{u \in V(\Gamma) : d(u,v) \leqslant 1\},$$
$$\mathrm{St}(e) = \mathrm{St}(v_1) \cup \mathrm{St}(v_2).$$

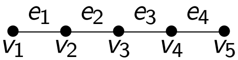
By definition, $\mathrm{St}(v)$ and $\mathrm{St}(e)$ dominate $\bar{\Gamma}$.



Idea Proof of "MinTL($A(\Gamma)$) $\leqslant 2$"

Theorem

Suppose that $|V(\Gamma)| \geqslant 4$ and both Γ and $\bar{\Gamma}$ are connected. Then, there exists an edge e such that $\bar{\Gamma}[\mathrm{St}(e)]$ is connected.

(example) Let $\Gamma =$ 

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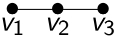
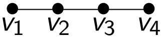
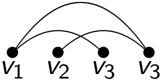
Then $\bar{\Gamma}[\mathrm{St}(e_i)]$ is connected if and only if $i \in \{2, 3\}$.

$\Gamma[\mathrm{St}_\Gamma(e_1)] =$ 

$\Gamma[\mathrm{St}_\Gamma(e_2)] =$ 

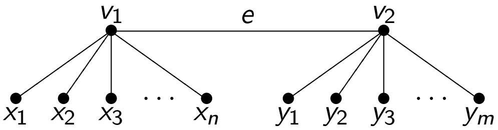
$\bar{\Gamma}[\mathrm{St}_\Gamma(e_1)] =$ 

$\bar{\Gamma}[\mathrm{St}_\Gamma(e_2)] =$ 

Idea Proof of "MinTL($A(\Gamma)$) $\leqslant 2$"

(Proof of "MinTL($A(\Gamma)$) $\leqslant 2$") By previous theorem, there exists an edge $e = \{v_1, v_2\}$ such that $\bar{\Gamma}[\mathrm{St}(e)]$ is connected. Let

$$\mathrm{St}(v_1) = \{v_1, x_1, \ldots, x_n\}, \qquad \mathrm{St}(v_2) = \{v_2, y_1, \ldots, y_m\}.$$



Define $g_1 = v_1 x_1 \cdots x_n$, $g_2 = v_2 y_1 \cdots y_m$ and $g = g_1 g_2$.
Then $g_1 \in Z(v_1)$, $g_2 \in Z(v_2)$ and $\mathrm{supp}(g) = \mathrm{St}(e)$.
Since $g_1 \in Z(v_1)$ and $g_2 \in Z(v_2)$, $\tau(g) \leqslant 2$.

Since $\mathrm{St}(e)$ dominates $\bar{\Gamma}$ and since $\bar{\Gamma}[\mathrm{St}(e)]$ is connected, g is loxodromic, i.e. $\tau(g) > 0$, hence we are done.

The End

Thank You.