Converses to generalized Conway–Gordon type congruences

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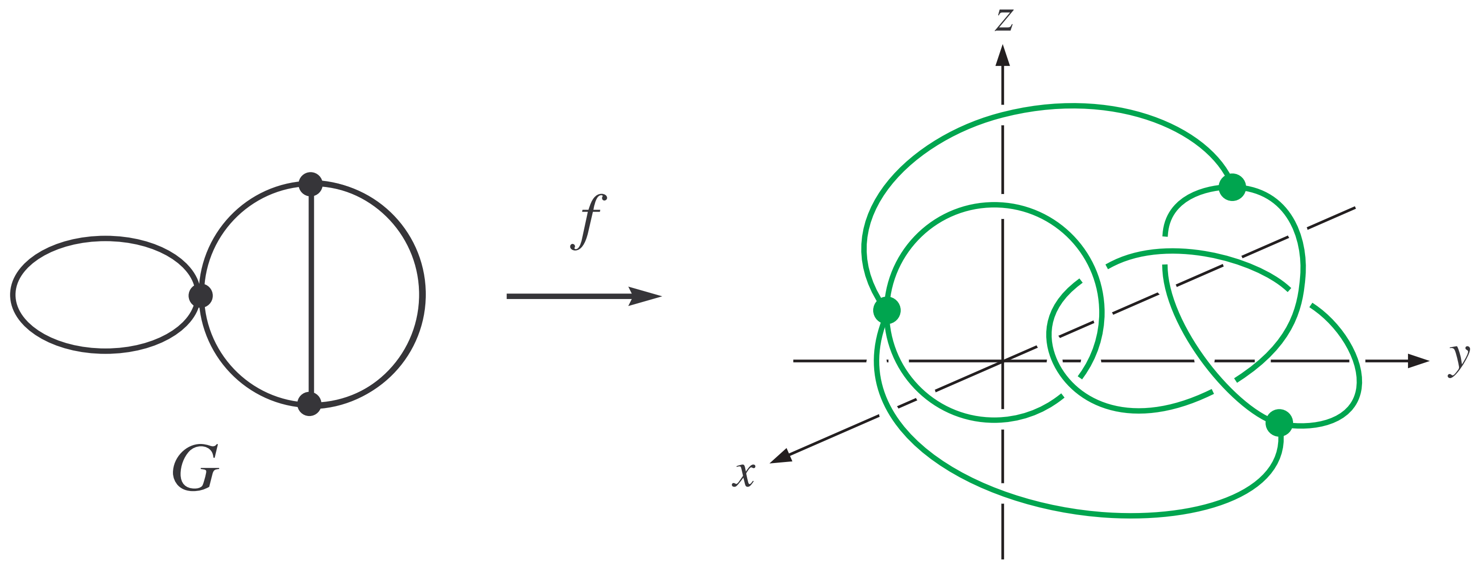
§1. Conway–Gordon type congruences

Spatial graph = The image of a spatial embedding f of a finite graph G into $\mathbb{R}^3$



For a (disjoint union of) *cycle(s)* λ of G, $f(\lambda)$ is called a *constituent knot (link)* of the spatial graph.

$$\mathrm{SE}(G) \overset{\text{def.}}{=} \{\text{embedding } f : G \to \mathbb{R}^3\}$$

$$\Gamma_k(G) \overset{\text{def.}}{=} \{k\text{-cycles of } G\}$$

$$\Gamma_{k,l}(G) \overset{\text{def.}}{=} \{\text{a disjoint pair of } k\text{-cycle and } l\text{-cycle of } G\}$$

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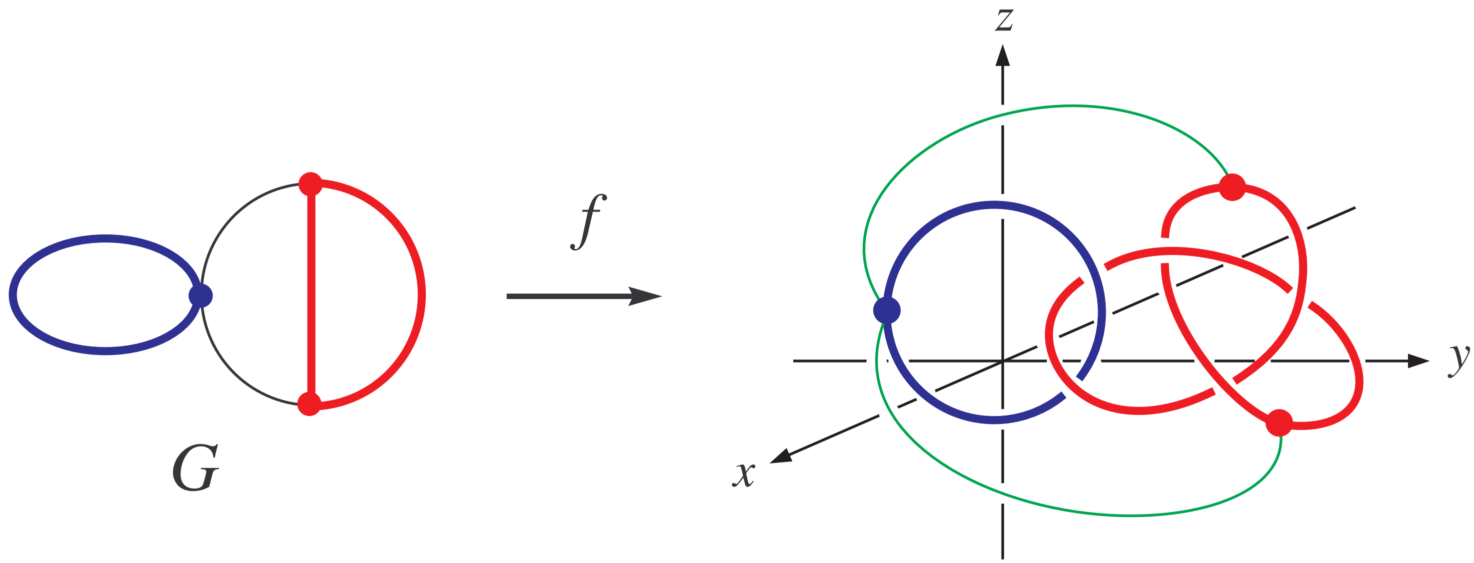

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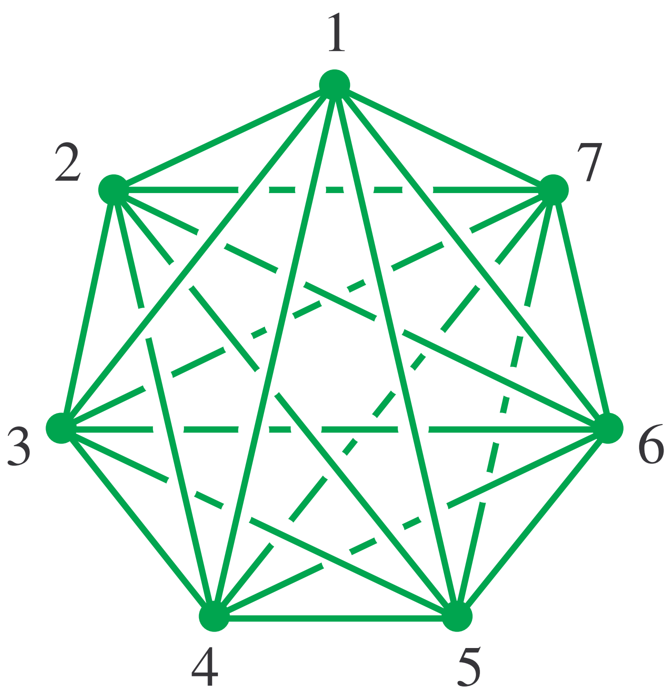
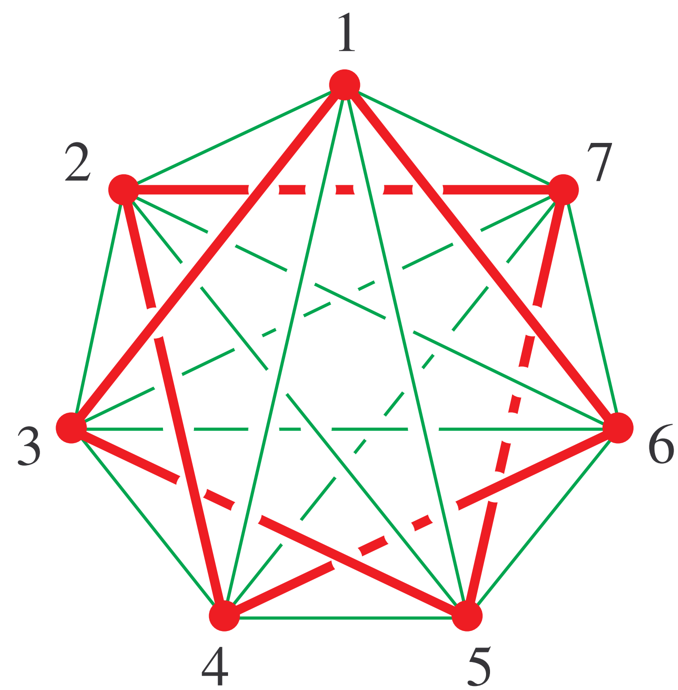
K_n : *complete graph* on n vertices

Theorem 1.1. [Conway–Gordon '83]

$$\forall f \in \mathrm{SE}\,(K_7),\quad \sum_{\gamma \in \Gamma_7(K_7)} a_2\,(f\,(\gamma)) \equiv 1 \pmod 2.$$

Here, a_2: 2nd coefficient of the *Conway polynomial* $\nabla(z)$.





$\forall f\,(K_7) \supset$ nontrivial knot.

Theorem 1.2. [Morishita–Nikkuni '19]

For $n \geq 6$, $\forall f \in \mathrm{SE}(K_n)$,

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) - (n-5)! \sum_{\gamma \in \Gamma_5(K_n)} a_2(f(\gamma))$$
$$= \frac{(n-5)!}{2}\left(\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 - \binom{n-1}{5} \right).$$

Here, lk: *linking number*.

$$n = 6:\ \sum_6 a_2 - \sum_5 a_2 = \frac{1}{2}\sum_{3,3} \mathrm{lk}^2 - \frac{1}{2} \implies \sum_{3,3} \mathrm{lk} \equiv 1 \pmod 2$$

(**Conway–Gordon K_6 theorem**)

$$n = 7:\ \sum_7 a_2 - 2\sum_5 a_2 = \sum_{3,3} \mathrm{lk}^2 - 6 \implies \sum_7 a_2 \equiv 1 \pmod 2$$

(**Conway–Gordon K_7 theorem, Thm. 1.1**)

$$n = 8:\ \sum_8 a_2 - 6\sum_5 a_2 = 3\sum_{3,3} \mathrm{lk}^2 - 63 \implies \sum_8 a_2 \equiv 3 \pmod 6.$$

Corollary 1.3. [Morishita–Nikkuni '19]

For $n \geq 7$, $\forall f \in \mathrm{SE}(K_n)$,

we have the following congruence modulo $(n-5)!$:

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) \equiv \begin{cases} \dfrac{(n-5)!}{2} & (n \equiv 0, 7 \pmod 8) \\ 0 & (n \not\equiv 0, 7 \pmod 8). \end{cases}$$

$n = 7$: $\sum_7 a_2 \equiv \dfrac{(7-5)!}{2} = 1 \pmod 2$ (**Thm. 1.1**)

$n = 8$: $\sum_8 a_2 \equiv \dfrac{(8-5)!}{2} = 3 \pmod 6$ [Foisy '08][Hirano '10]

$n = 9$: $\sum_9 a_2 \equiv 0 \pmod{24}$

Question. Are these congruences the best?

For $n \geq 6$, we define:

$$r_n = \begin{cases} \dfrac{(n-5)!}{2} & (n \equiv 0, 7 \pmod 8) \\ 0 & (n \not\equiv 0, 7 \pmod 8). \end{cases}$$

Theorem 1.4. [N] For integers $n \geq 7$ and m,

$$\exists f \in \mathrm{SE}\,(K_n) \ \text{ s.t. } \sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) = m$$

$$\iff m \equiv r_n \pmod{(n-5)!}.$$

The **ONLY IF** part has already been proven in **Cor. 1.3**.

Our purpose is to show the **IF** part ($\Longleftarrow$), namely the "converses" to Conway–Gordon type congruences.

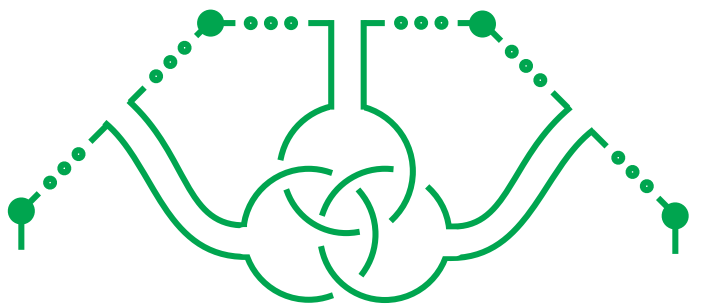
Remark. Theorem 1.4 is also true for $n = 6$.

(i.e. $\forall$ integer can be realized by $\sum_6 a_2$.)

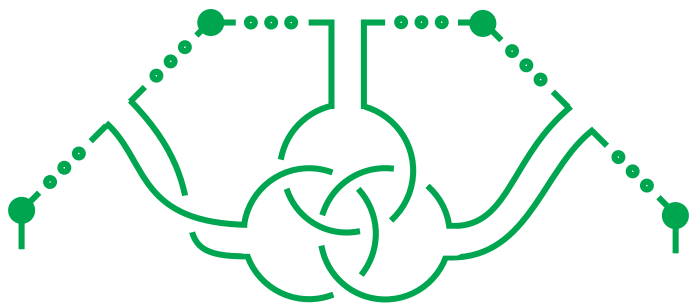
§2. Outline of the proof of Theorem 1.4.

Lemma 2.1. For $n \geq 4$, $\forall f \in \mathrm{SE}(K_n)$, $\exists g \in \mathrm{SE}(K_n)$,

s.t. $$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_n(K_n)} a_2(g(\gamma)) = \pm(n-4)!.$$



(1)



(2)

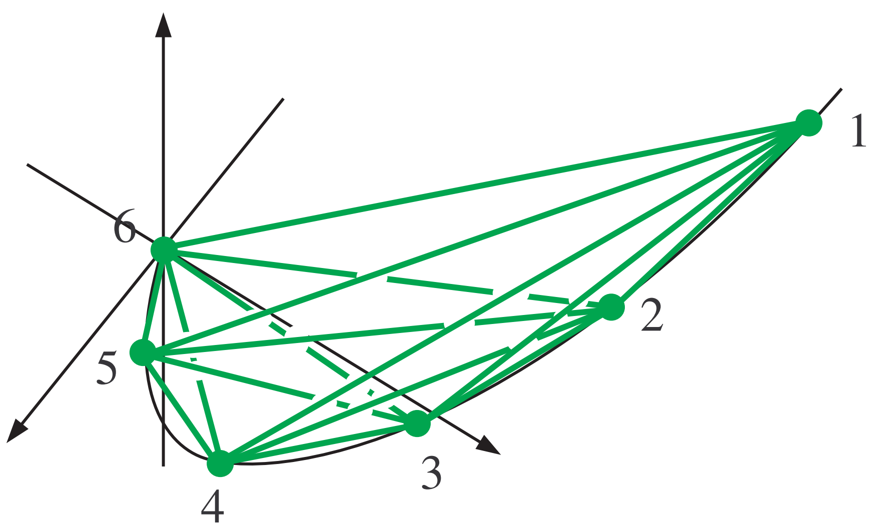
• A 'band sum' of Borromean rings (1) (resp. (2)) causes a_2 to increase (resp. decrease) by exactly one. [Okada '90]

• $\exists (n-4)!$ Hamiltonian cycles of K_n containing a fixed path of length 3.

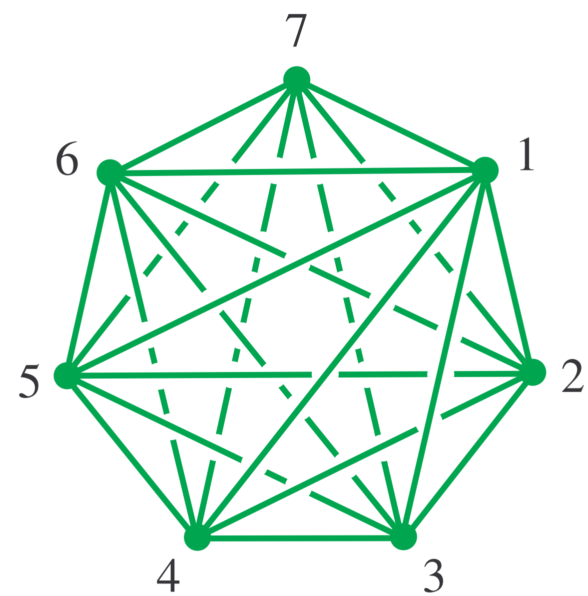
A spatial embedding f_{r} of G is *rectilinear* (or linear) $\overset{\text{def.}}{\Leftrightarrow}$ $\forall$ edge e of G, $f_{\mathrm{r}}(e)$ is a straight line segment in $\mathbb{R}^3$

$\mathrm{RSE}(G) \overset{\text{def.}}{=} \{$rectilinear spatial embeddings of $G\}$

Example. *Standard* rectilinear embedding $h \in \mathrm{RSE}(K_n)$:



$h(K_6)$



$h(K_7)$

Take n vertices on the curve (t, t^2, t^3) and connect every pair of distinct vertices by a straight line segment.

We define:

$$c_n \overset{\text{def.}}{=} \sum_{\gamma \in \Gamma_n(K_n)} a_2(h(\gamma)) \overset{\text{[M–N '19]}}{=} \frac{(n-5)!}{2}\left(\binom{n}{6} - \binom{n-1}{5}\right).$$

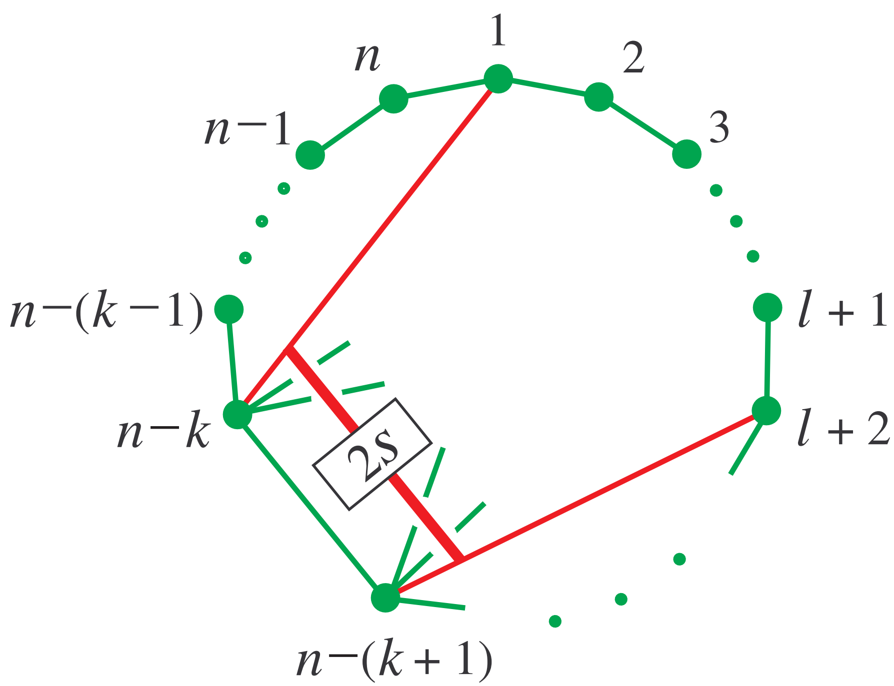
Remark. $c_n \equiv r_n \pmod{(n-5)!}$.

Then we have:

$$\begin{aligned}
&\{(n-5)!q + r_n \mid q \in \mathbb{Z}\} \\
&= \left\{(n-5)!q' + c_n \mid q' \in \mathbb{Z}\right\} \\
&= \left\{(n-5)!((n-4)q'' + s) + c_n \mid q'' \in \mathbb{Z},\ s = 0, 1, \ldots, n-5\right\} \\
&= \left\{(n-4)!q'' + (n-5)!s + c_n \mid q'' \in \mathbb{Z},\ s = 0, 1, \ldots, n-5\right\}.
\end{aligned}$$

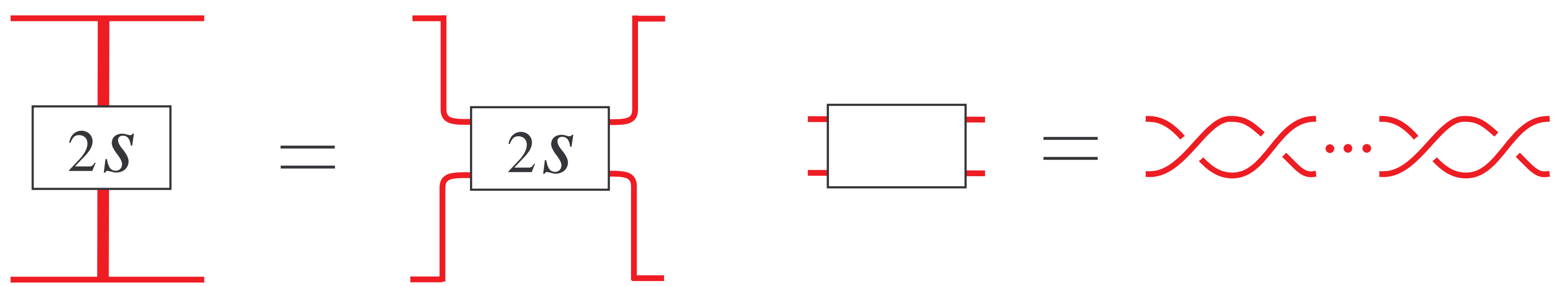
Claim. $\forall s = 0, 1, \ldots, n-5,\ \exists g_s \in \mathrm{SE}(K_n)$

s.t. $\displaystyle\sum_{\gamma \in \Gamma_n(K_n)} a_2(g_s(\gamma)) - c_n \equiv (n-5)!s \pmod{(n-4)!}$.

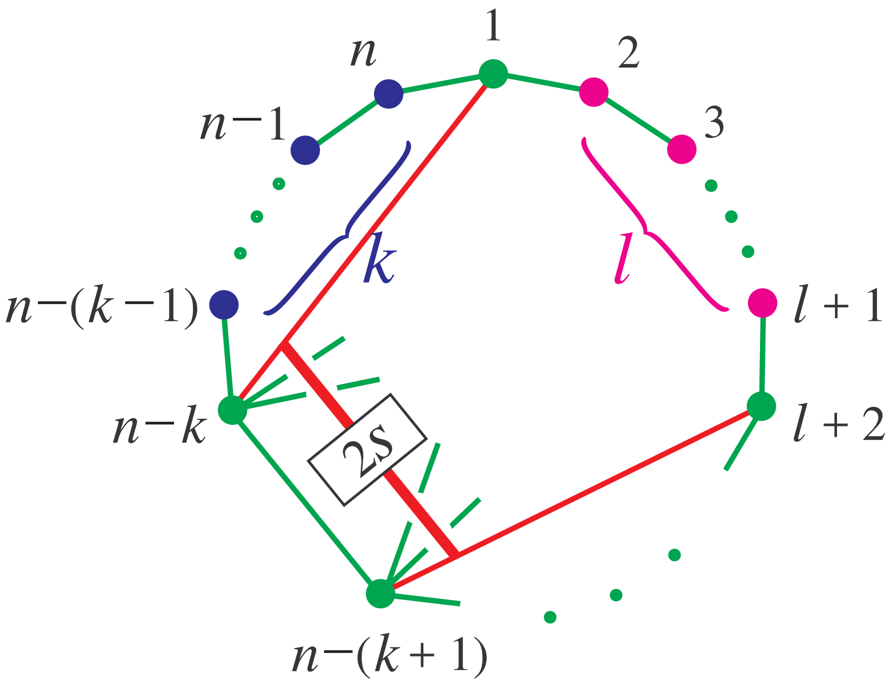


Consider $f_{k,l}^{(s)} \in \mathrm{SE}(K_n)$ $(s, k, l \geq 0,\ k + l \leq n - 4)$:

obtained from h by twisting red edges $2s$ times



Note. This twist is done close enough to the vertices $n-k$ and $n-(k+1)$, and this twisted part has no under crossings with any other edge.



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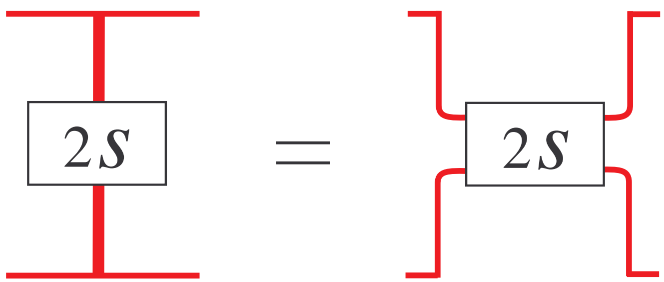
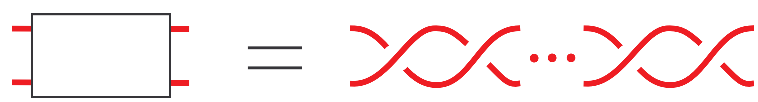




Note. This twist is done close enough to the vertices $n-k$ and $n-(k+1)$, and this twisted part has no under crossings with any other edge.

Lemma 2.2.

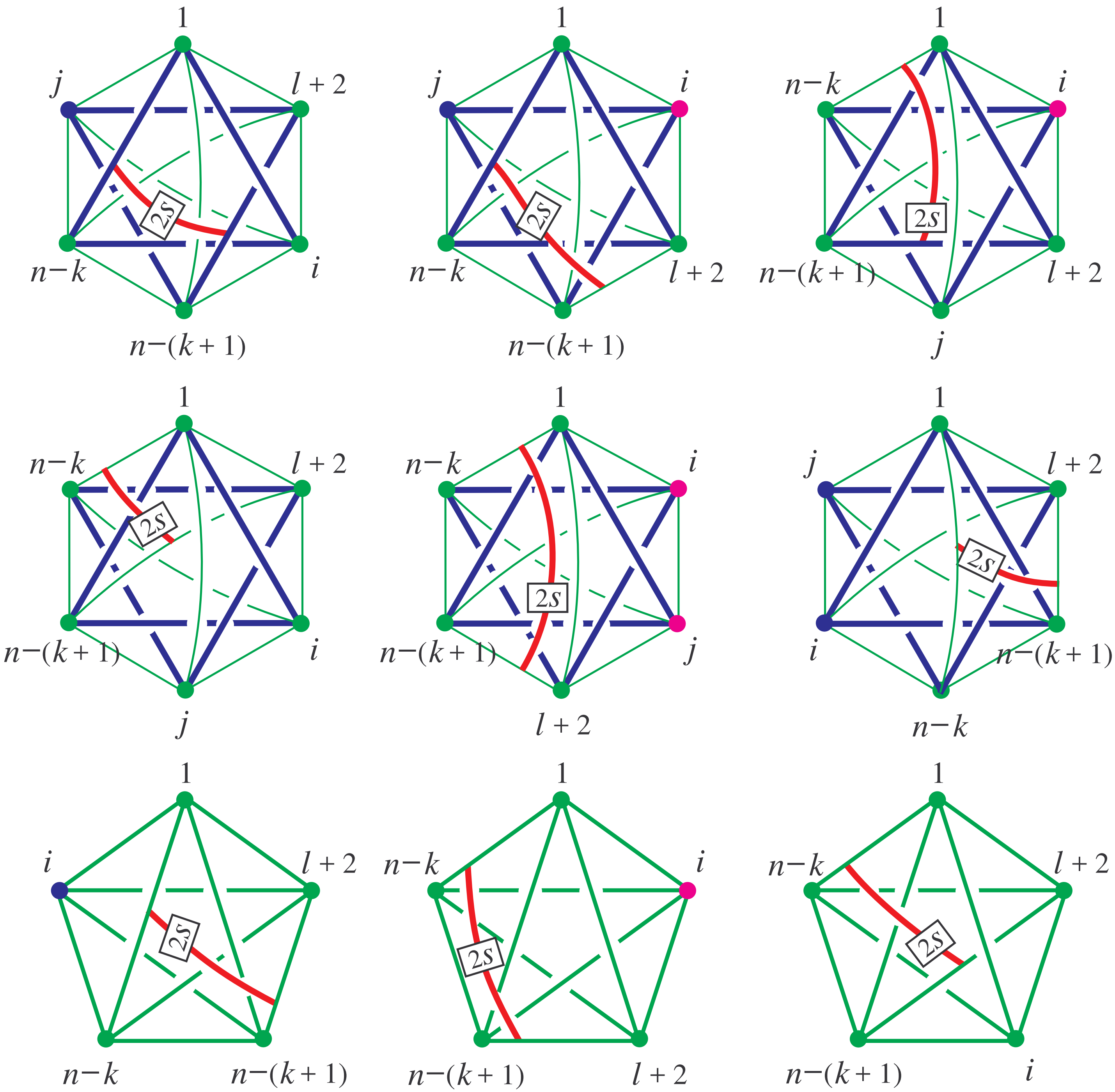
$$\sum_{\gamma\in\Gamma_n(K_n)} a_2\bigl(f_{k,l}^{(s)}(\gamma)\bigr) - c_n = (n-4)!\,\sigma(k,l;s) + (n-5)!\,\tau(k,l;s),$$

where
$$\sigma(k,l;s) = s(s(k+l+1)-k),$$
$$\tau(k,l;s) = s\biggl((1-s)(k^2+kl+l^2) + s\binom{k}{2} + s\binom{n-(k+l+4)}{2} + (s-2)\binom{l}{2}\biggr).$$

Remark. By **Thm. 1.2**, we have

$$\sum_n a_2 = (n-5)!\sum_5 a_2 + \frac{(n-5)!}{2}\Bigl(\sum_{3,3} \mathrm{lk}^2 - \binom{n-1}{5}\Bigr),$$

where $\sum_{3,3} \mathrm{lk}^2 = \sum_{K_n\supset K_6}\sum_{3,3} \mathrm{lk}^2$, $\sum_5 a_2 = \sum_{K_n\supset K_5}\sum_5 a_2$.


1
j
l + 2
2s
n−k
i
n−(k+1)
1
j
i
2s
n−k
l + 2
n−(k+1)
1
n−k
i
2s
n−(k+1)
l + 2
j
1
n−k
l + 2
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n−k
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l + 2
1
n−k
l + 2
2s
n−(k+1)
i

Lemma 2.3. (1) If $n \equiv 1 \pmod 2$, then

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2\big(f_{1,0}^{(s)}(\gamma)\big) - c_n \equiv (n-5)!s \pmod{(n-4)!}.$$

(2) If $n \equiv 0 \pmod 2$, then

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2\Big(f_{\frac{n-6}{2},\frac{n-6}{2}}^{(s)}(\gamma)\Big) - c_n \equiv (n-5)!s \pmod{(n-4)!}.$$

$$\begin{aligned}
(1) \sum_{\gamma \in \Gamma_n(K_n)} & a_2\big(f_{1,0}^{(s)}(\gamma)\big) - c_n \\
&\equiv (n-5)!\tau(1,0;s) \\
&= (n-5)!s\left(-(s-1) + s\binom{n-5}{2}\right) \\
&= (n-5)!s\left(\frac{(n-4)(n-7)s}{2} + 1\right) \\
&\equiv (n-5)!s \pmod{(n-4)!}. \quad \square
\end{aligned}$$

$$
\begin{aligned}
(2) \sum_{\gamma\in\Gamma_n(K_n)} & a_2\Big(f^{(s)}_{\frac{n-6}{2},\frac{n-6}{2}}(\gamma)\Big) - c_n \\
&\equiv (n-5)!\tau\Big(\frac{n-6}{2},\frac{n-6}{2};s\Big) \\
&= (n-5)!s\Big(-(s-1)\cdot\frac{3(n-6)^2}{4} + (2s-2)\binom{\frac{n-6}{2}}{2} + s\Big) \\
&= (n-5)!s\Big(-(s-1)\cdot\frac{(n-4)(n-7)}{2} + 1\Big) \\
&\equiv (n-5)!s \pmod{(n-4)!}. \quad \square
\end{aligned}
$$

Lemma 2.1 + **Lemma 2.3** implies **Theorem 1.4.**

$$n=6:\ \sum_{6} a_2\big(f^{(s)}_{0,0}(\gamma)\big) \equiv s \pmod 2 \ (s=0,1)$$

$$n=7:\ \sum_{7} a_2\big(f^{(s)}_{1,0}(\gamma)\big) \equiv 2s+1 \pmod 6 \ (s=0,1,2)$$

$$n=8:\ \sum_{8} a_2\big(f^{(s)}_{1,1}(\gamma)\big) \equiv 6s+21 \pmod{24} \ (s=0,1,2,3)$$

§3. Related problem

Theorem 3.1. [M–N '19] For $n \geq 6$, $\forall f_{\mathrm{r}} \in \mathrm{RSE}(K_n)$,

$$\frac{(n-5)(n-6)(n-1)!}{2\cdot 6!} \leq \sum_n a_2 \leq \frac{3(n-2)(n-5)(n-1)!}{2\cdot 6!}.$$

$n=6$: $0 \leq \sum_6 a_2 \leq 1$. ($\Longrightarrow$ $\exists$ at most one trefoil knot)

$n=7$: $1 \leq \sum_7 a_2 \leq 15$, $\sum_7 a_2 \equiv 1 \pmod 2$. ($\Longrightarrow$ $\exists$ trefoil)

$n=8$: $21 \leq \sum_8 a_2 \leq 189$, $\sum_8 a_2 \equiv 3 \pmod 6$.

Remark. The lower bound in **Thm. 3.1** is sharp, but the upper bound is not expected to be sharp if $n \geq 7$.

Problem. Determine the integers realized by $\sum_n a_2$ for some $f_{\mathrm{r}} \in \mathrm{RSE}(K_n)$ $(n \geq 7)$.

Example. [Jeon et al. IWSG2010] According to a computer search with the help of oriented matroid theory, it has been announced the following (unpublished):

The number of knots and links in rectilinear K_7						
6-trefoil	7-trefoil	4_1	$(3,3)$-Hopf	$(3,4)$-Hopf	4_1^2	$\sum_7 a_2$
0	1	0	7	14	0	1
1	3	0	9	18	0	3
2	5	0	11	22	0	5
3	7	0	13	22	1	7
3	7	0	13	26	0	7
3	8	1	13	26	0	7
4	9	0	15	26	1	9
4	11	2	15	30	0	9
4	12	3	15	30	0	9
5	11	0	17	30	1	11