One-two-way pass-move
for knots and links

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Outline

- **Pass-move and $\#$-move**
- **One-two-way pass-move(= 1-2-move)**
- **Basic properties**
- **Arf invariant**
- **A knot K with $p(K) = nt(K) = 1$**
- **A link L with $p(L) = 1$ and $nt(L) = 2$**

• Local move

A local move is a local change of a knot diagram, which either

preserves,
or changes

the knot type.

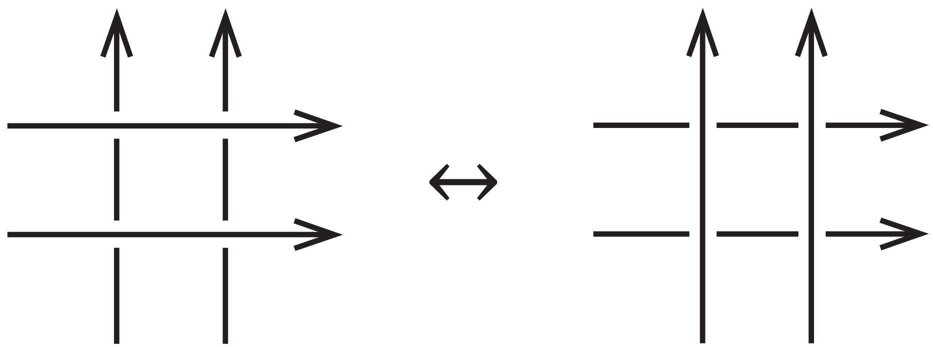
- **Unknotting operation**

A local move is said to be an *unknotting operation* if any knot can be changed to an unknot via a finite sequence of the move.

- **Crossing change**

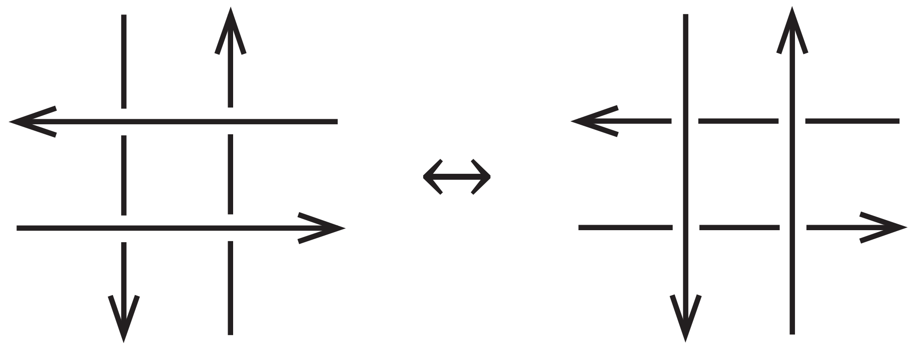
The (classical) crossing change is an unknotting operation.

- **#-move**



A #-move is an unknotting operation [Murakami].

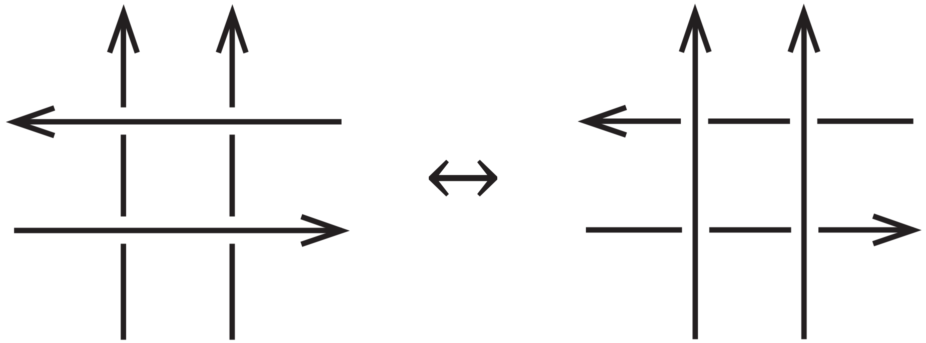
- **Pass-move**



A pass-move is not an unknotting operation.

A knot K is pass-move equivalent to an unknot (a trefoil resp.) if and only if the Arf invariant of K is 0 (1 resp.) [Kauffman].

- **One-two-way pass-move**



Briefly,

One-two-way pass-move = 1-2-*move*

It is a hybrid of the pass-move and the #-move.

• Proper link

K_i: a component of a link L

A link L is a *proper link* if
the linking number $\mathrm{lk}(K_i, L - K_i) = 0 \pmod 2$ for every i.

In particular, we regard a knot as a proper link.

Proposition 1. Suppose that L_1 and L_2 are proper links. Then L_1 and L_2 are pass-move equivalent if and only if $\mathrm{Arf}(L_1) = \mathrm{Arf}(L_2)$.

Theorem 1. Suppose that L_1 and L_2 are proper links. Then L_1 and L_2 are 1-2-move equivalent if and only if $\mathrm{Arf}(L_1) = \mathrm{Arf}(L_2)$.

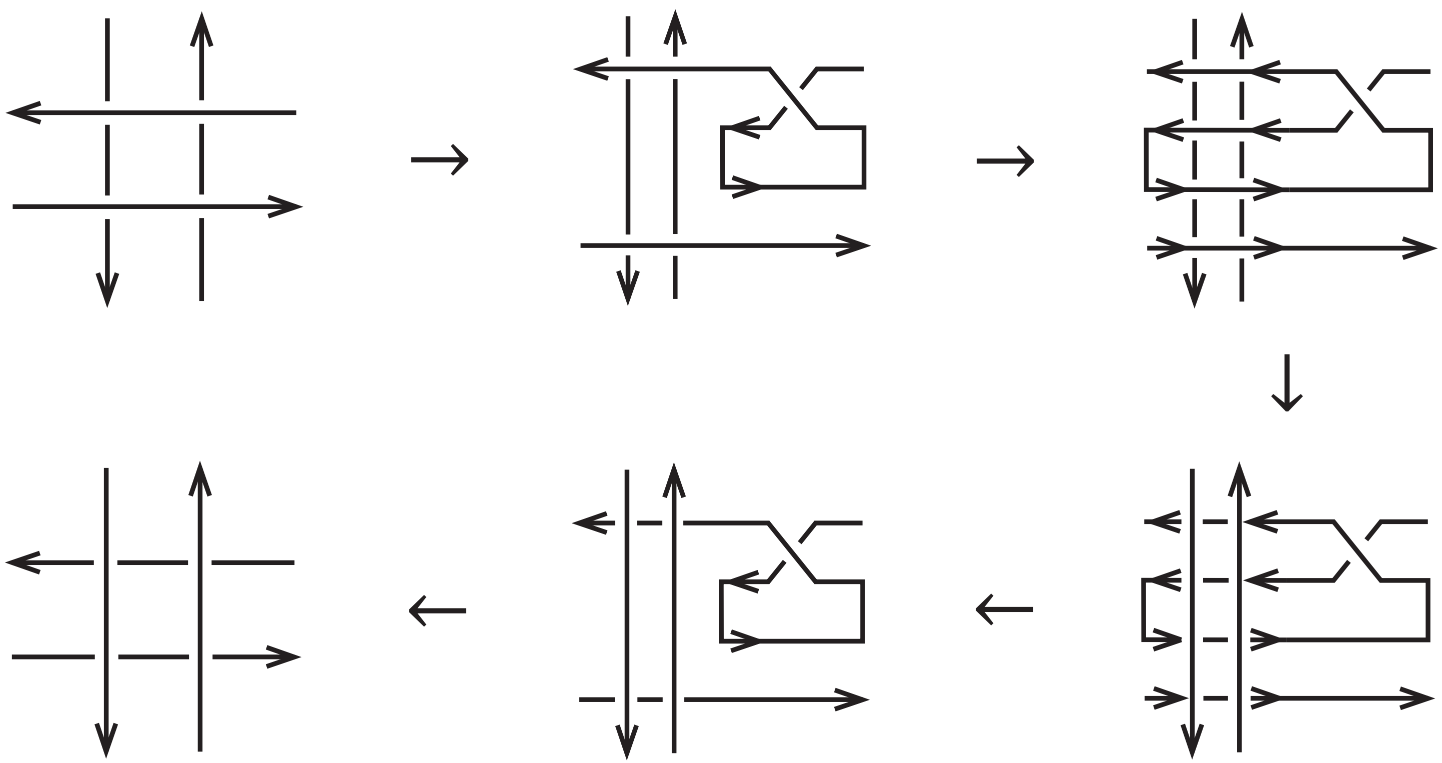
- $p(L)$ **and** $nt(L)$

L: a proper link with Arf invariant 0

pass-move number $p(L)$: the minimal number of pass-moves required for L to be an unknot or an unlink.

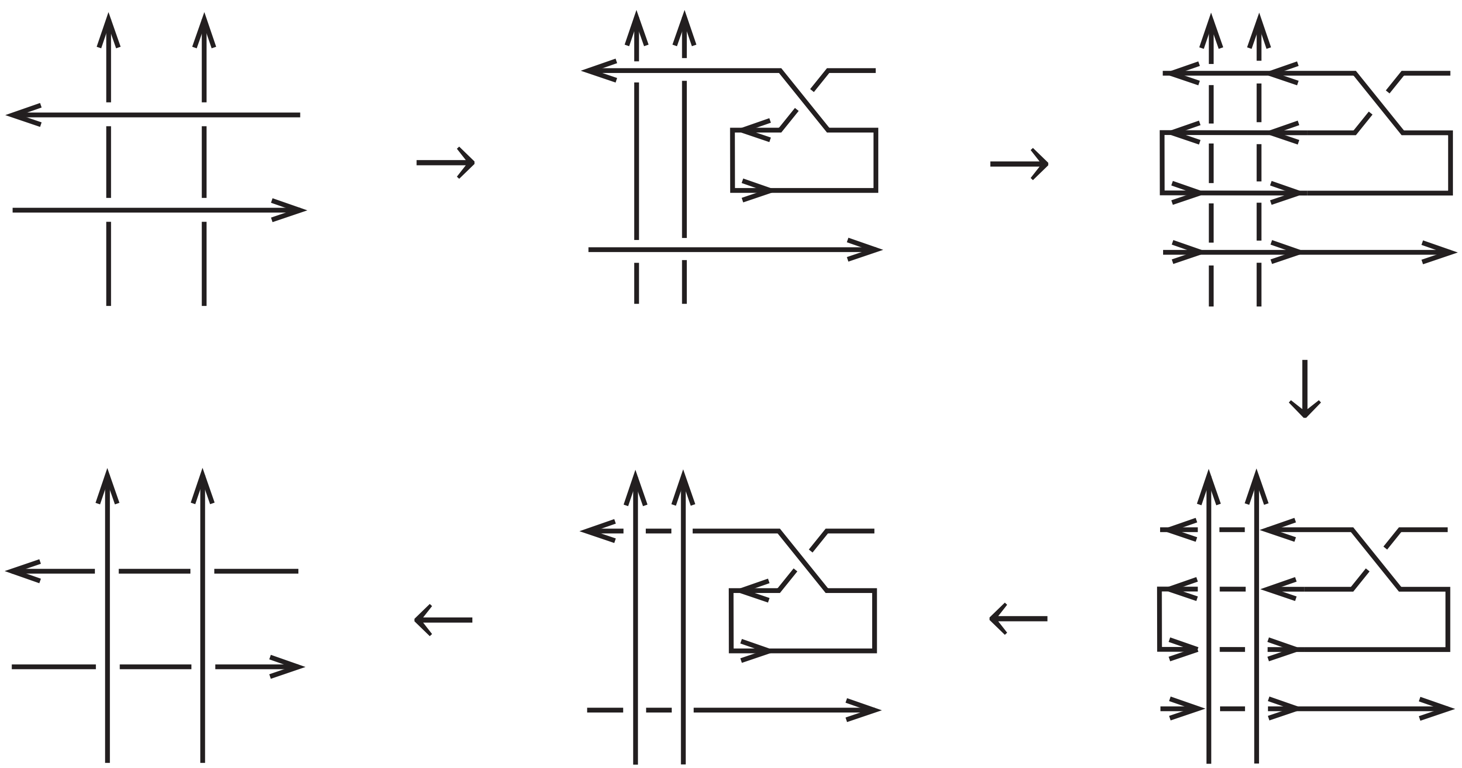
1-2-*move number* $nt(L)$: the minimal number of 1-2-moves required for L to be an unknot or an unlink

Proposition 2. A pass-move is realized by applying a 1-2-move twice.



Corollary 1. $nt(L) \leq 2p(L)$.

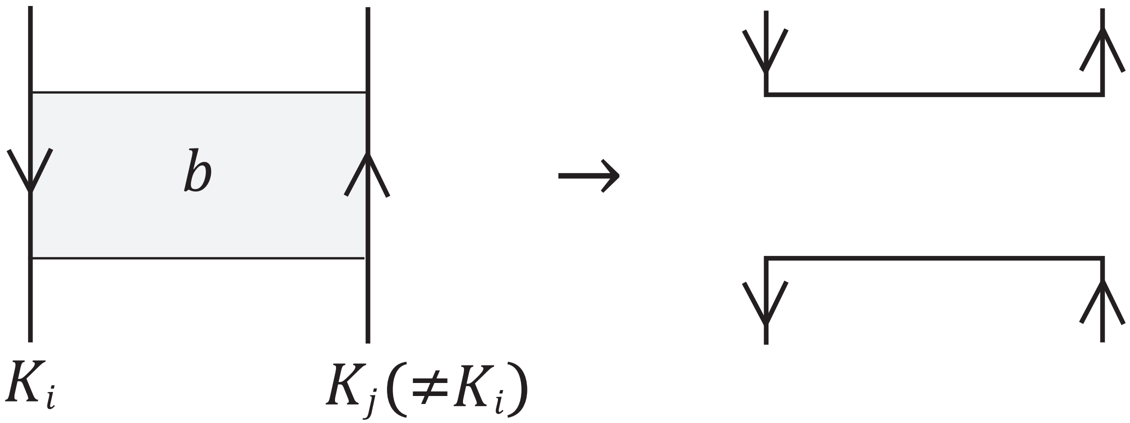
Proposition 3. A 1-2-move is realized by applying a #-move twice.



∴ A pass-move is realized by applying a #-move four times.

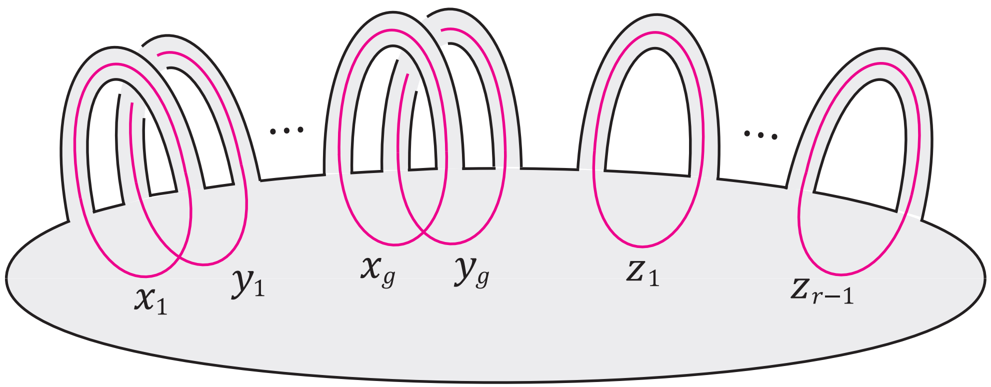
- **Fusion**

A *fusion* is a band sum along different components of a link.



Let F be a connected genus-g Seifert surface of an r-component link L.

Let $\mathcal{B} = \{x_i, y_i, z_k \,|\, i = 1, \ldots, g \text{ and } k = 1, \ldots, r-1\}$ be a basis of $H_1(F; \mathbb{Z}_2)$ represented by loops in F such that $|x_i \cap y_j| = \delta_{ij}$ (the Kronecker delta) and z_k is a k-th component of L.



• Arf invariant

For a loop l in F, let $q(l) = \mathrm{lk}(l^+, l) \pmod 2$, where l^+ is a loop obtained by slightly pushing l to the positive direction of F.

For the basis $\mathcal{B}$, let

$$\mathrm{Arf}(F, \mathcal{B}) = \sum_{i=1}^{g} q(x_i) q(y_i) \pmod 2.$$

This value is called the *Arf invariant* of F with respect to $\mathcal{B}$.

An Arf invariant depends on the choice of F and $\mathcal{B}$.
But for proper links, it is an invariant of a link.

• Known results

Proposition 4. Suppose that L_1 and L_2 are proper links. Then a split union $L_1 \sqcup L_2$ is also a proper link, and $\mathrm{Arf}(L_1 \sqcup L_2) = \mathrm{Arf}(L_1) + \mathrm{Arf}(L_2) \pmod 2$.

Proposition 5. Suppose that L is a proper link. If L' is a link obtained from L by a fusion, then L' is also a proper link and $\mathrm{Arf}(L') = \mathrm{Arf}(L)$.

Proposition 1. Suppose that L_1 and L_2 are proper links. Then L_1 and L_2 are pass-move equivalent if and only if $\mathrm{Arf}(L_1) = \mathrm{Arf}(L_2)$.

Theorem 1. Suppose that L_1 and L_2 are proper links. Then L_1 and L_2 are 1-2-move equivalent if and only if $\text{Arf}(L_1) = \text{Arf}(L_2)$.

Proof) $\Longleftarrow$) Suppose that $\text{Arf}(L_1) = \text{Arf}(L_2)$.
Then L_1 and L_2 are pass-move equivalent by **Proposition 1**.
Then by **Proposition 2**, L_1 and L_2 are 1-2-move equivalent.

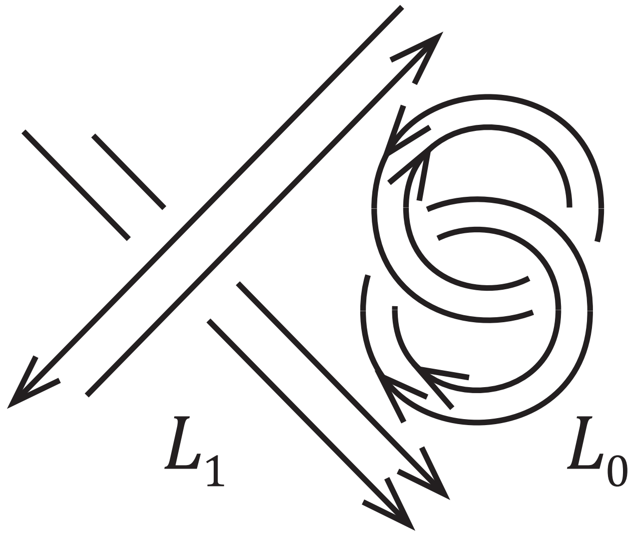
$\Longrightarrow$) Suppose that L_1 and L_2 are 1-2-move equivalent.

Let L_0 be an untwisted 2-cable of a Hopf link as in the figure. The link L_0 is a proper link.

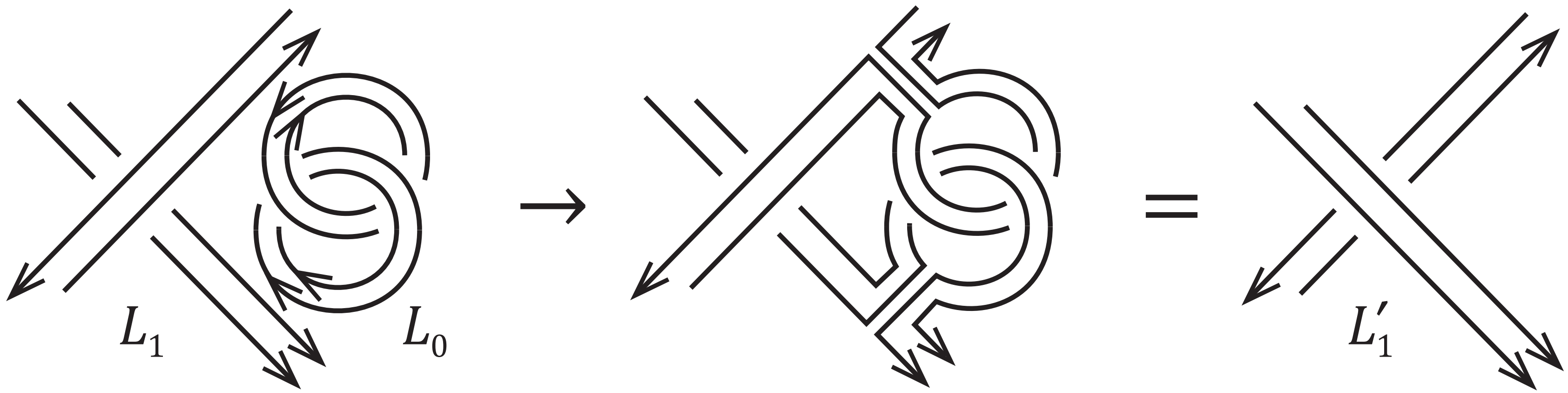
L_0

Since we can obtain an unlink by banding the two anti-parallel components of L_0 and Arf(an unlink) $= 0$,
Arf$(L_0) = 0$ by **Proposition 5**.

By **Proposition 4**, $L_1 \sqcup L_0$ is a proper link and $\mathrm{Arf}(L_1 \sqcup L_0) = \mathrm{Arf}(L_1)$.

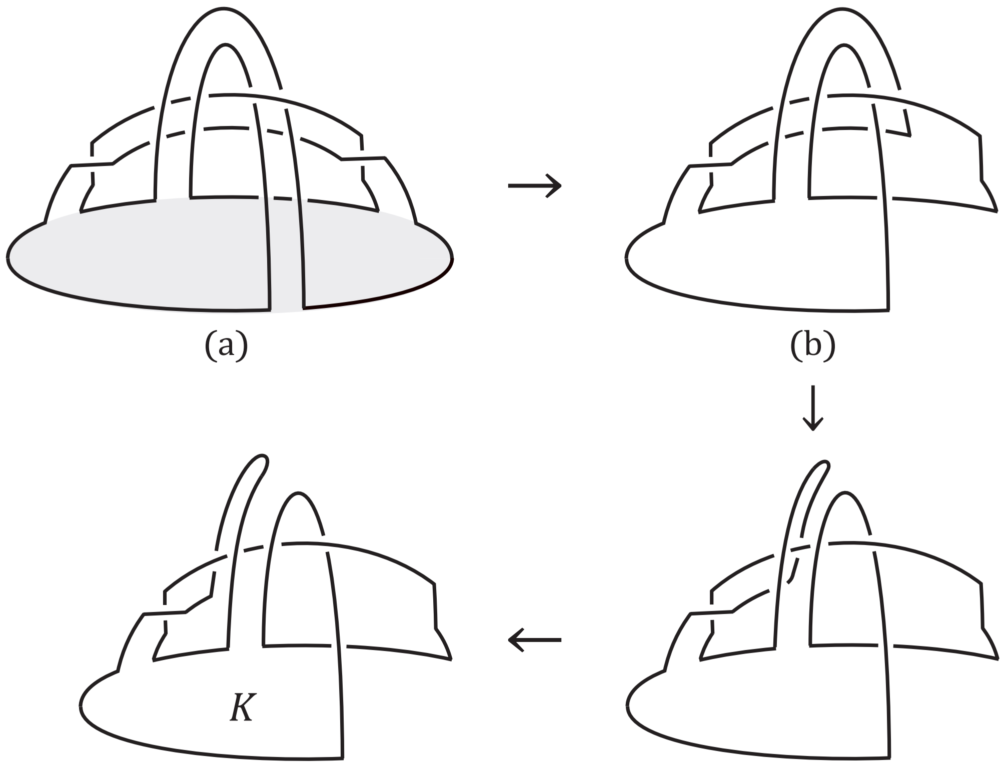


Performing a fusion operation four times to $L_1 \sqcup L_0$ has the same effect as a 1-2-move on L_1.
Let L_1' be a link obtained from L_1 by a single 1-2-move.
Then $\mathrm{Arf}(L_1') = \mathrm{Arf}(L_1 \sqcup L_0) = \mathrm{Arf}(L_1)$.



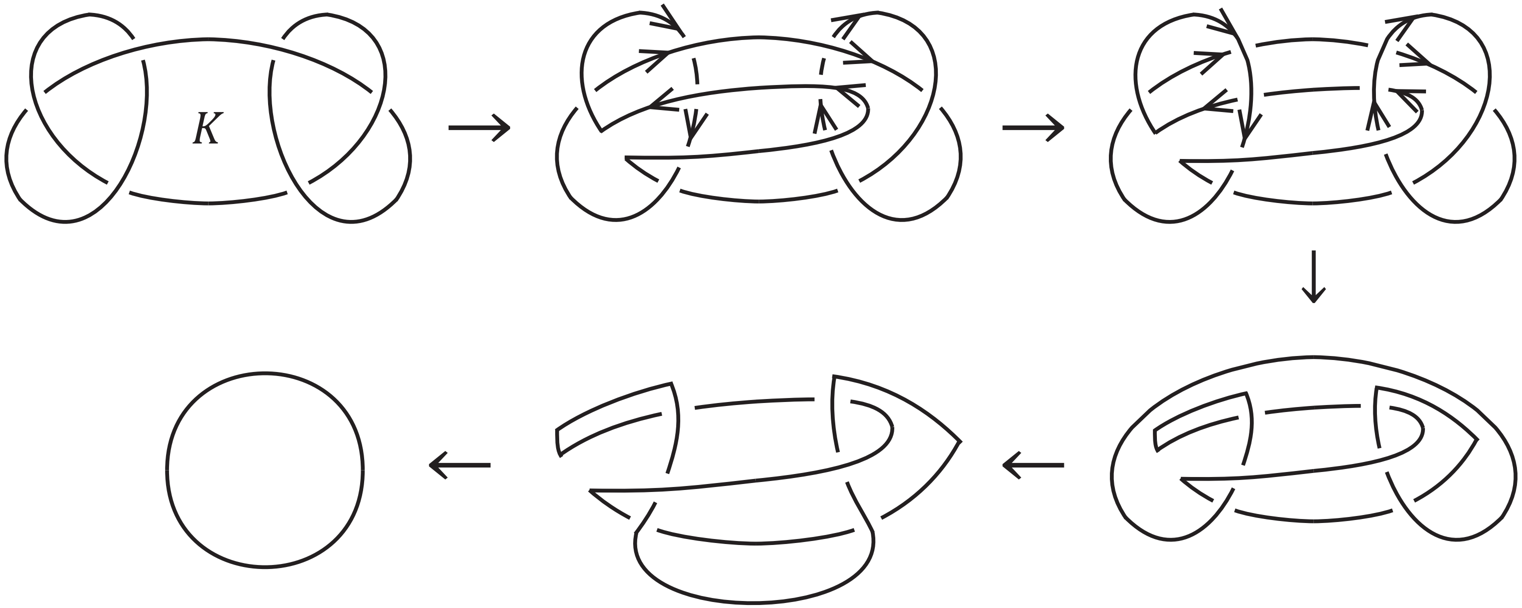
By applying the above argument finitely many times, we conclude that $\mathrm{Arf}(L_1) = \mathrm{Arf}(L_2)$.

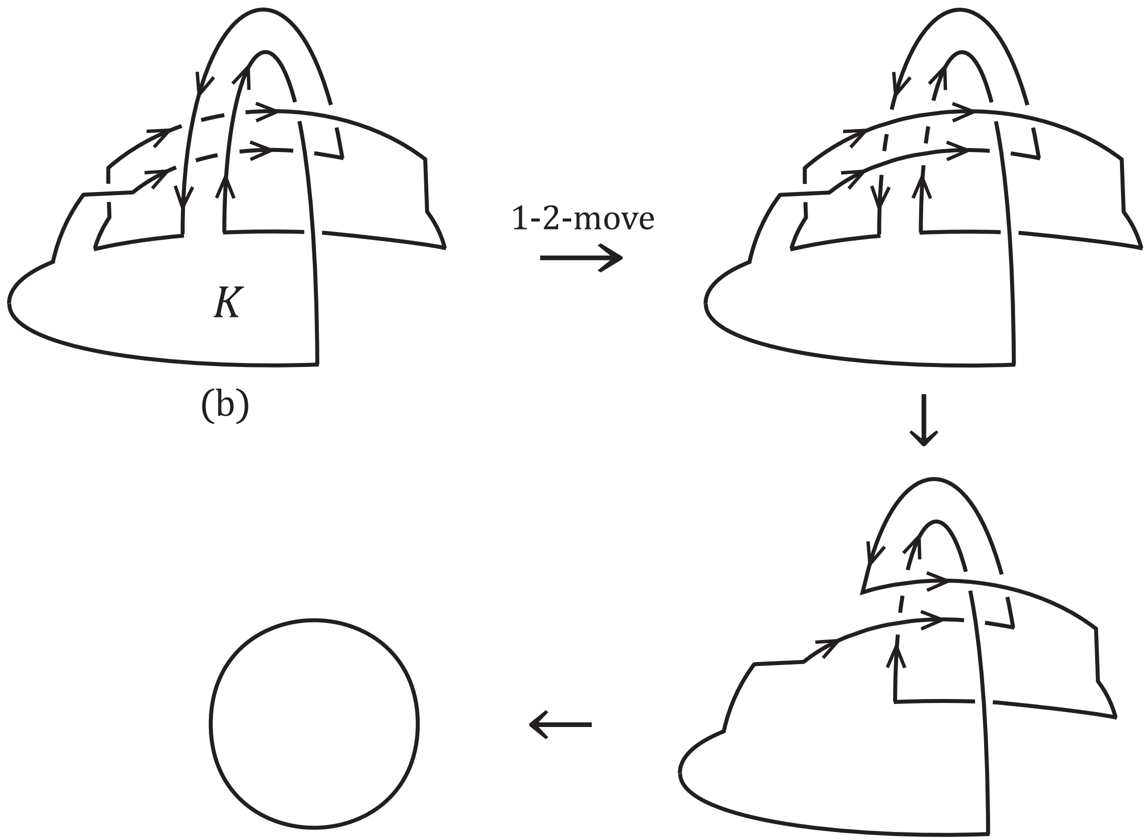
The knot in (a) is obtained from a disk by banding operations and taking the boundary.



It is isotoped to $K =$ (a left-hand trefoil) # (a right-hand trefoil).

It is well known that $p(K) = 1$.

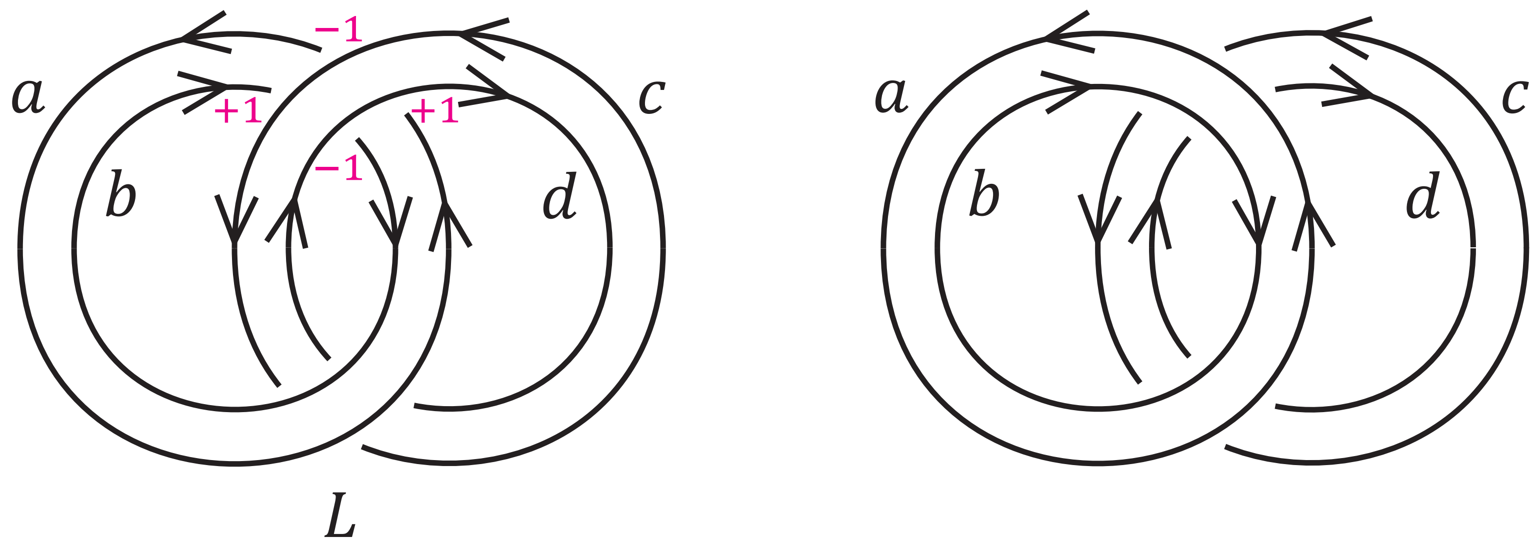




$nt(K) = 1.$

$\therefore\ p(K) = nt(K) = 1.$

L: an untwisted 2-cable of a Hopf link as in the figure



$p(L) = 1$.

For L,
$\mathsf{lk}(a,c) = -1$, $\mathsf{lk}(a,d) = 1$, $\mathsf{lk}(b,c) = 1$, $\mathsf{lk}(b,d) = -1$,
$\mathsf{lk}(a,b) = \mathsf{lk}(c,d) = 0$,

For the unlink,
$\mathsf{lk}(a,c) = \mathsf{lk}(a,d) = \mathsf{lk}(b,c) = \mathsf{lk}(b,d) = \mathsf{lk}(a,b) = \mathsf{lk}(c,d) = 0$.

Claim. $nt(L) = 2$.

Sketch of proof) Suppose that $nt(L) \neq 2$.

Since $nt(L) \leq 2p(L) = 2$ by **Corollary 1**, $nt(L) = 1$.

Consider a diagram D of L such that a single 1-2-move on D yields a diagram D_0 of an unlink.

Since by only a single 1-2-move
all linking numbers $\mathrm{lk}(a, c)$, $\mathrm{lk}(a, d)$, $\mathrm{lk}(b, c)$, $\mathrm{lk}(b, d)$ change to 0, the four components a, b, c, d should be involved in the four strands of the 1-2-move on D.

By investigating linking numbers, we get a contradiction.

Question.

Is there a knot K such that $p(K) = 1$ and $nt(K) = 2$?

Thank you for your attention.