Petal number of torus knots of type $(r, r+2)$

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Acknowledgements

This work is submitted to the Special Issue of JKTR in Memory of Vaughan Jones.

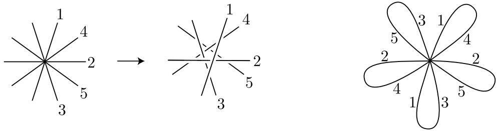
H. Kim, S. No, and H. Yoo proved

$$p(T_{r,s}) = 2s - 1, \ (1 < r < s, \ r \equiv 1 \mod s - r)$$

in the article,
Petal number of torus knots using superbridge indices,
JKTR vol.31, No.13, 225006(2022).

Petal projection and petal number

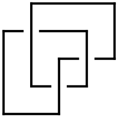
A **petal projection** of a knot K is a projection of K with a single multicrossing such that there are no nesting loops.



A petal projection with an n-multicrossing is called an ***n*-petal**.
The **petal number**, $p(K)$, is the minimum number of loops among all petal projections of K, or equivalently, the minimum number of strands passing through the single multicrossing.
Suppose we have a petal projection with n loops. We label the strands passing through the n-multicrossing with $1, 2, \ldots, n$ from top to bottom. From one end of the top strand we read the labels clockwise half way around the multicrossing. This sequence of labels is called the **petal permutation** of the petal projection.
The figure on the right shows a 5-petal of the left-handed trefoil knot with the petal permutation $(1, 4, 2, 5, 3)$.

Grid diagram and petal grid diagram

A **grid diagram** is a knot diagram which is composed of finitely many horizontal edges and the same number of vertical edges such that vertical edges always cross over horizontal edges.
Every knot admits a grid diagram.

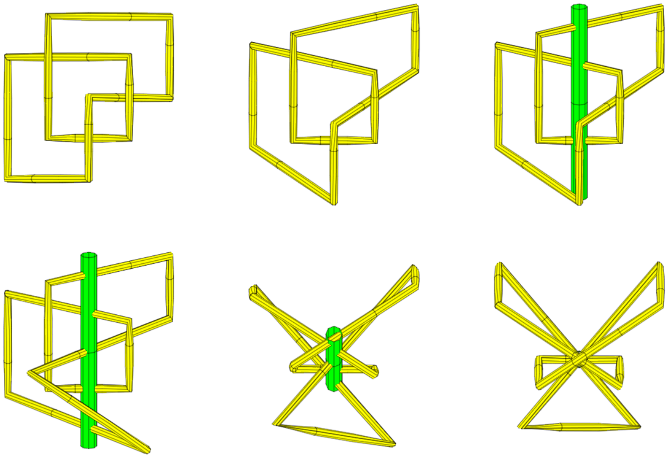


According to Adams et al. [2015], a p-petal of a knot has an associated grid diagram with p vertical edges satisfying the following properties:

1. There is exactly one vertical edge whose adjacent horizontal edges point in opposite directions — one points to the left of the vertical edge, and the other points to the right. We call this edge the **inflection edge**, and denote it I. The horizontal edges adjacent to I have length $\frac{p-1}{2}$.
2. Each remaining vertical edge's adjacent horizontal edges have length $\frac{p+1}{2}$ and $\frac{p-1}{2}$.

Such a grid diagram will be called a **petal grid diagram**.

5-petal of a trefoil knot from its petal grid diagram



Main theorem

Let $\alpha(K)$ denote the **arc index** of K which is the minimum number of vertical edges among all grid diagrams of K.

Proposition 1. (Adams et al.)

Let K be a nontrivial knot. Then

$$p(K) \geq \begin{cases} \alpha(K) & \text{if } \alpha(K) \text{ is odd,} \\ \alpha(K)+1 & \text{if } \alpha(K) \text{ is even.} \end{cases}$$

Proposition 2. (Etnyre and Honda)

Let r and s be relatively prime integers such that $2 \leq r < s$. Then

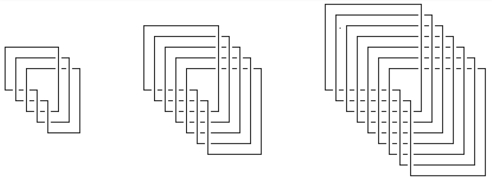
$$\alpha\left(T_{r,s}\right) = r + s.$$

Theorem

Let r be an odd integer and $r \geq 3$. Then $p\left(T_{r,r+2}\right) = 2r+3$.

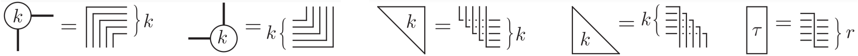
Proof of main theorem

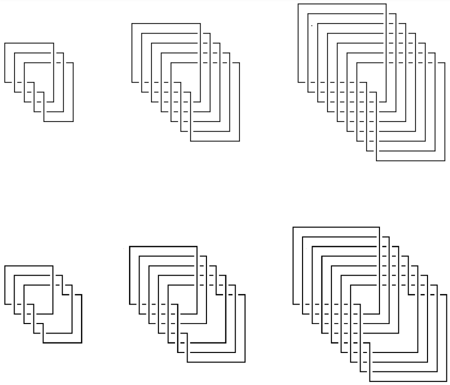
By the propositions, we have $p(T_{r,r+2}) \geq 2r+3$ for any odd number $r \geq 3$.
To prove the theorem, we show that $p(T_{r,r+2}) \leq 2r+3$ by constructing a petal grid diagram of $T_{r,r+2}$ having $2r+3$ vertical edges.

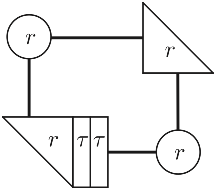


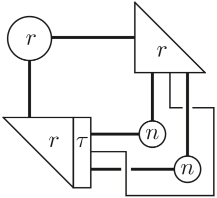
The figure shows minimal grid diagrams of $T_{3,5}$, $T_{5,7}$, and $T_{7,9}$. They are closed braids. We will deform such grid diagrams by braid conjugations and then obtain petal grid diagrams using grid diagram moves.

Deformation of grid diagrams by braid conjugation









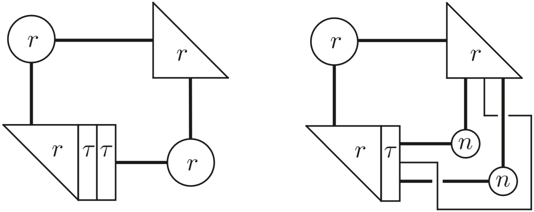
$T_{r,r+2}$ as closures of two conjugate braids

Let $r = 2n+1$ and let σ_i denote the standard i-th generator of the braid group B_r of r strings.

Let Δ denote the positive half-twist of r strings and let $\tau = \prod_{i=1}^{2n} \sigma_i = \sigma_1\sigma_2\cdots\sigma_{2n}$.

Lemma

$\Delta^2\tau^2$ and $\Delta^2\tau(\sigma_{n+1}\sigma_{n+2}\cdots\sigma_{2n})(\sigma_{2n}\sigma_{2n-1}\cdots\sigma_{n+1})$ are conjugates.



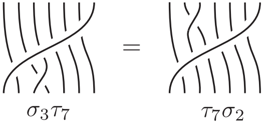
Proof of lemma

We use the following braid relations:

$$\begin{aligned} \sigma_i\sigma_j\sigma_i^{\varepsilon} &= \sigma_j^{\varepsilon}\sigma_i\sigma_j \quad |i-j|=1,\ \varepsilon=\pm1,\ 1\le i,j\le 2n \\ \sigma_i\sigma_j^{\varepsilon} &= \sigma_j^{\varepsilon}\sigma_i \quad\ \ |i-j|>1,\ \varepsilon=\pm1,\ 1\le i,j\le 2n \end{aligned} \tag{1}$$

$$\sigma_i^{\varepsilon}\Delta^2 = \Delta^2\sigma_i^{\varepsilon} \qquad \varepsilon=\pm1,\ 1\le i\le 2n \tag{2}$$

$$\sigma_i^{\varepsilon}\tau = \tau\sigma_{i-1}^{\varepsilon} \qquad \varepsilon=\pm1,\ 2\le i\le 2n \tag{3}$$



Cases $n = 1$ and $n = 2$

When $n = 1$, $\tau = \sigma_1\sigma_2$ and the following shows that $\Delta^2\tau(\sigma_2)(\sigma_2)$ is a conjugate of $\Delta^2\tau^2$:

$$\begin{aligned}(\sigma_2^{-1})\Delta^2\tau^2(\sigma_2) &= \Delta^2\tau\sigma_1^{-1}\tau(\sigma_2)\\ &= \Delta^2\tau\sigma_1^{-1}(\sigma_1\sigma_2)(\sigma_2)\\ &= \Delta^2\tau(\sigma_2)(\sigma_2)\end{aligned}$$

When $n = 2$, $\tau = \sigma_1\sigma_2\sigma_3$ and the following shows that $\Delta^2\tau(\sigma_3\sigma_4)(\sigma_4\sigma_3)$ is a conjugate of $\Delta^2\tau^2$:

$$\begin{aligned}(\sigma_3^{-1}\sigma_4^{-1}\sigma_2^{-1})\Delta^2\tau^2(\sigma_2\sigma_4\sigma_3) &= \Delta^2\tau(\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1})\tau(\sigma_2\sigma_4\sigma_3)\\ &= \Delta^2\tau(\sigma_2^{-1}\sigma_1^{-1})\tau\sigma_2^{-1}(\sigma_2\sigma_4\sigma_3)\\ &= \Delta^2\tau(\sigma_2^{-1}\sigma_1^{-1})(\sigma_1\sigma_2\sigma_3\sigma_4)\sigma_2^{-1}(\sigma_2\sigma_4\sigma_3)\\ &= \Delta^2\tau(\sigma_3\sigma_4)(\sigma_4\sigma_3)\end{aligned}$$

Cases $n \geq 3$

Let $n \geq 3$ and let

$$\begin{aligned} c_k &= \sigma_{k+1}\sigma_{k+3}\cdots\sigma_{2n-k+1}, \quad k = 1, \ldots, n \\ \beta_0 &= \Delta^2\tau^2 \\ \beta_1 &= c_1^{-1}\beta_0 c_1 \\ \beta_k &= c_k^{-1}\beta_{k-1}c_k, \quad k = 2, \ldots, n. \end{aligned}$$

Then $\beta_1, \beta_2, \ldots, \beta_n$ are all conjugates of $\beta_0 = \Delta^2\tau^2$. We compute them inductively.

$$\begin{aligned} \beta_1 &= (\sigma_{2n}^{-1}\sigma_{2n-2}^{-1}\cdots\sigma_2^{-1})\Delta^2\tau^2(\sigma_2\sigma_4\cdots\sigma_{2n}) \\ &= \Delta^2\tau(\sigma_{2n-1}^{-1}\sigma_{2n-3}^{-1}\cdots\sigma_1^{-1})\tau(\sigma_2\sigma_4\cdots\sigma_{2n}) \\ &= \Delta^2\tau\sigma_1^{-1}(\sigma_{2n-1}^{-1}\cdots\sigma_3^{-1})\tau(\sigma_2\sigma_4\cdots\sigma_{2n}) \\ &= \Delta^2\tau\sigma_1^{-1}\tau(\sigma_{2n-2}^{-1}\cdots\sigma_2^{-1})(\sigma_2\sigma_4\cdots\sigma_{2n}) \\ &= \Delta^2\tau(\sigma_1^{-1})\tau(\sigma_{2n}) \end{aligned}$$

$$\begin{aligned}
\beta_1 &= \Delta^2 \tau(\sigma_1^{-1})\tau(\sigma_{2n}) \\
\beta_2 &= c_2^{-1}\beta_1 c_2 \\
&= (\sigma_{2n-1}^{-1}\sigma_{2n-3}^{-1}\cdots\sigma_3^{-1})\Delta^2\tau(\sigma_1^{-1})\tau(\sigma_{2n})(\sigma_3\sigma_5\cdots\sigma_{2n-1}) \\
&= \Delta^2\tau(\sigma_{2n-2}^{-1}\sigma_{2n-4}^{-1}\cdots\sigma_2^{-1})\sigma_1^{-1}\tau(\sigma_{2n})(\sigma_3\sigma_5\cdots\sigma_{2n-1}) \\
&= \Delta^2\tau(\sigma_2^{-1}\sigma_1^{-1})\tau(\sigma_{2n})(\sigma_{2n-3}^{-1}\cdots\sigma_3^{-1})(\sigma_3\sigma_5\cdots\sigma_{2n-1}) \\
&= \Delta^2\tau(\sigma_2^{-1}\sigma_1^{-1})\tau(\sigma_{2n}\sigma_{2n-1})
\end{aligned}$$

Suppose that

$$\beta_k = \Delta^2\tau(\sigma_k^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1})\tau(\sigma_{2n}\sigma_{2n-1}\cdots\sigma_{2n-k+1}),$$

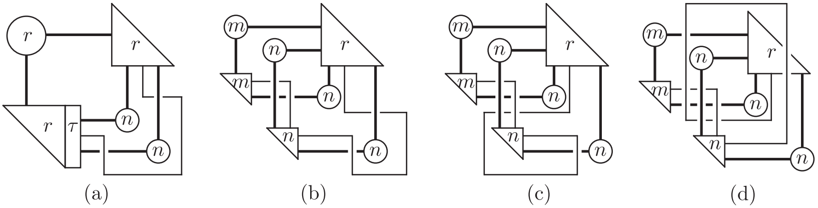
for $k < n$. We proceed by an induction on k.

$\beta_n = \Delta^2\tau(\sigma_{n+1}\sigma_{n+2}\cdots\sigma_{2n})(\sigma_{2n}\sigma_{2n-1}\cdots\sigma_{n+1})$

$$\begin{aligned}
\beta_{k+1} &= (\sigma_{2n-k}^{-1}\sigma_{2n-k-2}^{-1}\cdots\sigma_{k+2}^{-1})\beta_k(\sigma_{k+2}\sigma_{k+4}\cdots\sigma_{2n-k}) \\
&= (\sigma_{2n-k}^{-1}\sigma_{2n-k-2}^{-1}\cdots\sigma_{k+2}^{-1})\Delta^2\tau(\sigma_k^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1})\tau \\
&\qquad (\sigma_{2n}\sigma_{2n-1}\cdots\sigma_{2n-k+1})(\sigma_{k+2}\sigma_{k+4}\cdots\sigma_{2n-k}) \\
&= \Delta^2\tau(\sigma_{2n-k-1}^{-1}\sigma_{2n-k-3}^{-1}\cdots\sigma_{k+1}^{-1})(\sigma_k^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1})\tau \\
&\qquad (\sigma_{2n}\sigma_{2n-1}\cdots\sigma_{2n-k+1})(\sigma_{k+2}\sigma_{k+4}\cdots\sigma_{2n-k}) \\
&= \Delta^2\tau(\sigma_{k+1}^{-1}\sigma_k^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1})\tau(\sigma_{2n-k-2}^{-1}\sigma_{2n-k-4}^{-1}\cdots\sigma_{k+2}^{-1}) \\
&\qquad (\sigma_{2n}\sigma_{2n-1}\cdots\sigma_{2n-k+1})(\sigma_{k+2}\sigma_{k+4}\cdots\sigma_{2n-k}) \\
&= \Delta^2\tau(\sigma_{k+1}^{-1}\sigma_k^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1})\tau(\sigma_{2n}\sigma_{2n-1}\cdots\sigma_{2n-k+1}\sigma_{2n-k}) \\
&\vdots \\
\beta_n &= \Delta^2\tau(\sigma_n^{-1}\sigma_{n-1}^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1})\tau(\sigma_{2n}\sigma_{2n-1}\cdots\sigma_{n+1}) \\
&= \Delta^2\tau(\sigma_n^{-1}\sigma_{n-1}^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1})(\sigma_1\sigma_2\cdots\sigma_{2n})(\sigma_{2n}\sigma_{2n-1}\cdots\sigma_{n+1}) \\
&= \Delta^2\tau(\sigma_{n+1}\sigma_{n+2}\cdots\sigma_{2n})(\sigma_{2n}\sigma_{2n-1}\cdots\sigma_{n+1})
\end{aligned}$$

This proves the lemma.

Proof of main theorem

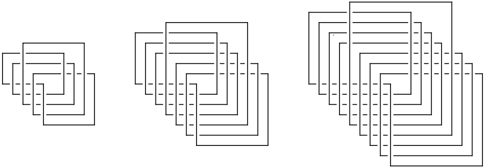


In the diagrams above, $m = n + 1$. The part (a) is the grid diagram we described in the lemma and the left half of the part (b) shows local details of that of (a). Notice that the vertical edge on the left of the bottom horizontal edge of (a) is moved to the $(n+2)$nd position from the right. The part (c) is obtained by moving the rightmost vertical edge of (b) through the back of the diagram to the $(n+2)$nd position from the left. The part (d) is obtained by moving the bottom horizontal edge of (c) to the top through the front of the diagram. The part (d) has a single inflection edge and satisfies the conditions of a petal grid diagram. This proves the main theorem.

Petal permutations

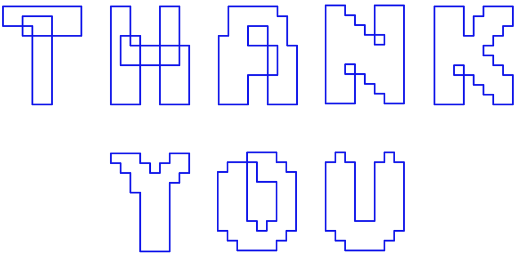
From the petal grid diagram (d), we can read the petal permutation of the associated $(2r+3)$-petal of $T_{r,r+2}$:

$$([1,3n+4],[n+2,3n+3],[n+1,3n+2],\ldots,[2,2n+3],\\ [4n+5,2n+2],[4n+4,2n+1],\ldots,[3n+6,n+3],3n+5)$$



The above are petal grid diagrams of $T_{3,5}$, $T_{5,7}$, $T_{7,9}$ with petal permutations:

$$T_{3,5} : (1,7,3,6,2,5,9,4,8)$$
$$T_{5,7} : (1,10,4,9,3,8,2,7,13,6,12,5,11)$$
$$T_{7,9} : (1,13,5,12,4,11,3,10,2,9,17,8,16,7,15,6,14)$$


THANK
YOU

A short remembrance of Vaughan

It was my final year of graduate study at Brandeis when I first met Vaughan. He came to explain about his polynomial at the Physics Department of Harvard to a small audience. As he explained what a knot is, one asked,

"What is the equivalence relation for knots?"

Then Vaughan grabbed an imaginary knot in his hand, juggled with it around his body, and showed his hand saying

"This is the relation."

It was a very physical and impressive explanation.

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