Yamada polynomial of brunnian θ-curves

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KNOTS AND LINKS IN LINEAR EMBEDDINGS OF K_6

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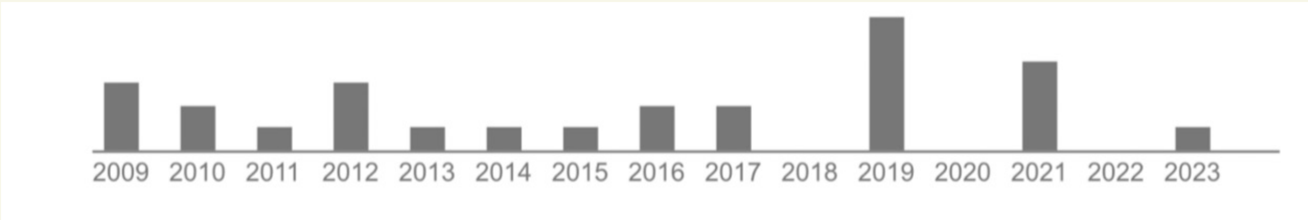
ABSTRACT. We investigate the number of knots and links in linear embeddings of K_6, the complete graph with 6 vertices. Concretely, we show that any linear embedding of K_6 contains either only one Hopf link, or three Hopf links and one trefoil knot.

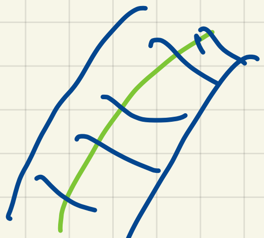


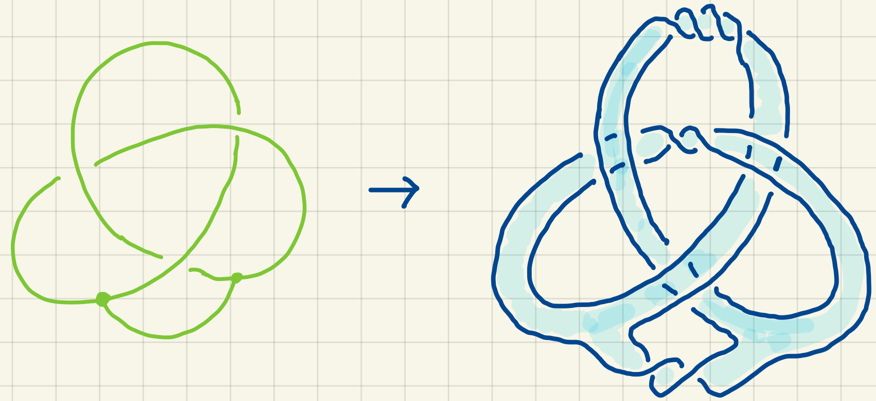
- For a θ-curve Θ,

  S_Θ = a two-puntured disk in $\mathbb{R}^3$

  s.t. $\begin{cases} \Theta \text{ is the spine of } S_\Theta \\ \text{the Seifert form of } S_\Theta \text{ is zero.} \end{cases}$

  Then $L_\Theta = \partial S_\Theta$ is called the associated link of Θ.





- [Kau, Simon, Wol, Zhao]

  The assc. lk is an amb. iso. inv. of θ-curves.

$\therefore$ Invariants of assc. lks are invariants of θ-curves.

In this talk,

Yamada polynomial of θ-curves

$\updownarrow$ relation? $\leftarrow$ by using Jaeger's 2-variable polys.

Jones polynomial of associated lk of θ-curves

Yamada Polynomial

① For an abstract graph G,

$$h(G;x,y) = \sum_{F\subset E(G)} (-x)^{-|F|}\, x^{\mu(G-F)}\, y^{\beta(G-F)}.$$

# of compts

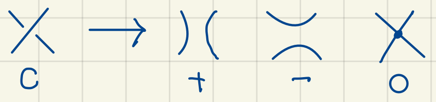
1st Betti # $= E-V+\mu$

$h(\phi) = 1$.

For a spatial-graph diagram D,

a state S is a ft $S:\{\text{crossings of } D\} \to \{+,-,0\}$.

② D_S is a planar graph obtained by changing each crossing c according $S(c)$:



③ For D, the Yamada Polynomial is defined to be

$$Y(D;A) = \sum_{S\in\mathcal{S}} A^{|S^{-1}(+)|-|S^{-1}(-)|}\, h(D_S;\ x=-1;\ y=-A-2-A^{-1})$$

$\mathcal{S}$ ↳ the set of all states of D.

Then,

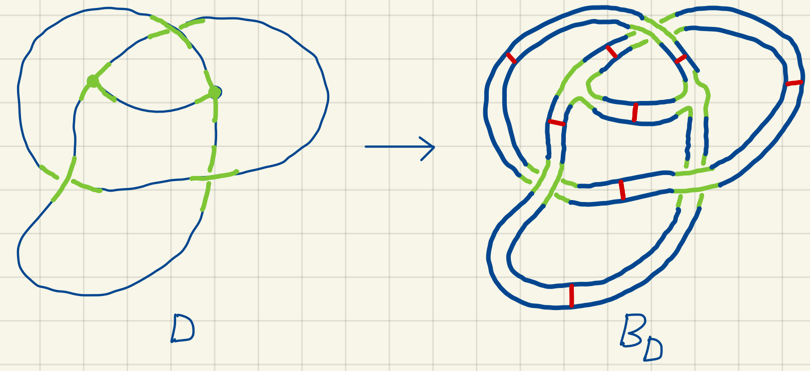
$Y(A)$ is an RV-isotopy invariant of spatial graphs upto powers of $(-A)$.

$$Y(\times) = A\,Y()() + A^{-1}\,Y(\asymp) + Y(\times)$$

Jaeger Poly: 2-variable poly of spatial graphs.

D: diagram of spatial graph.

① B_D: band diagram with bars obtained from D.



② The Jaeger Polynomial is defined through a repetition of bar-reduction:

- $R(\ ;a,t) = \dfrac{1}{t+t^{-1}}\left(R(\ ) + t^{-1}R(\ ) + \dfrac{t-t^{-1}}{1-at}R(\ )\right)$
- For a bar diagram B with no bar,

$R(B) = \mathcal{D}(L_B;\ a,\ z=t-t^{-1})$

↳ link diagram s.t. $L_B = \partial B$.

↳ Dubrovnik version of Kauffman Poly.

- $\mathbf{J(D,a,t) := R(B_D)}$

③ Jaeger's version of Yamada Poly:

$$\mathbf{J(D;A) = J(D,\ a=-A^3,\ t=A)}$$

(*) $\mathcal{D}(a=-A^3,\ t=A) = B(A)$

↳ bracket poly of links.

Then,

$J(A)$ is an RV-inv upto $-A^4$

Setting $a=-A^3$ & $t=A$, $R(\ ) = R(\ ) + \dfrac{1}{A^2+A^{-2}}R(\ )$

$$J(\times) = A^4\,J()() + A^{-4}\,J(\asymp) + (A^2+A^{-2})\,J(\times)$$

Proposition 1.

(G : a planar graph
(D : a diagram of a spatial emb. of G

$\Rightarrow$ $Y(D; A^4) = -(A^2 + A^{-2})^{E(G)-V(G)+1} \mathcal{J}(D; A)$

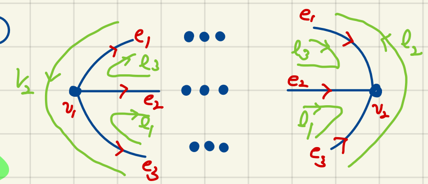
Proof :

- Induction on # of crossings of D
- If D has no crossing,

$$Y(D; A^4) = (-1)^{|E|-|V|} \underbrace{F(G; A^2+2+A^{-4})}_{\text{flow poly for abstract graphs.}}$$

$$= -(A^2 + A^{-2})^{E-V+1} \mathcal{J}(D; A)$$

Associated Links of Θ-curves

① 

D_Θ : diagram of a Θ-curve Θ.

$l_1 \sim e_2 - e_3$, $l_2 \sim e_3 - e_1$, $l_3 \sim e_1 - e_2$

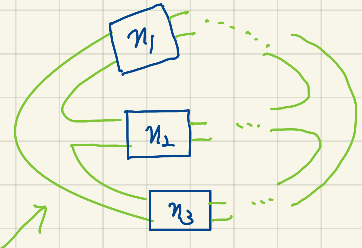
$L = l_1 \cup l_2 \cup l_3$.

ω_{ij} = sum of crossing signs btw e_i & e_j.

$n_i = -\omega_{ii} + \dfrac{\omega_{ij} + \omega_{ik} - \omega_{jk}}{2}$, $\{i, j, k\} = \{1, 2, 3\}$

Adding n_i full-twists to L,

We have an associated lk $L(n_1, n_2, n_3)$ of Θ



② For a link diagram D, $\langle\ \rangle$: bracket poly.

the Jones poly $V(D; A) = (-A^3)^{-\omega} \langle D \rangle$, ω : sum of crossing signs.

⊛ $\omega(L(n_1, n_2, n_3)) = -2(n_1 + n_2 + n_3)$.

Prop 2.

$$V(L(n_1, n_2, n_3))$$

$$= A^{8(n_1+n_2+n_3)} \left\{ \langle L \rangle + \sum_{i=1}^{3} \frac{1-(A^{-8})^{n_i}}{\varphi} \langle l_i^{(2)} \rangle + \frac{1}{\varphi^2}\left(2 - \sum_{i=1}^{3} A^{-8n_i} + A^{-8(n_1+n_2+n_3)}\right) \right\},$$

($\varphi = A^2 + A^{-2}$
($l_i^{(2)}$ = the 2-parallel of l_i

pf) Reduce n_1, n_2, n_3 by using the identities

$\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$

$\langle \text{kink} \rangle = -A^3 \langle \frown \rangle$, $\langle \text{kink} \rangle = (-A)^{-3} \langle \frown \rangle$

$\langle D \sqcup O \rangle = -\varphi \langle D \rangle$

$\langle O \rangle = 1$

The normalized Yamada poly $\tilde{Y}(D_\Theta) = (-A)^{2(n_1+n_2+n_3)} Y(D_\Theta)$

is an ambient isotopy inv.

Hence, by prop 1, the corresponding normalized Jaeger poly

$\tilde{\mathcal{J}}(D_\Theta) = (-A^4)^{2(n_1+n_2+n_3)} \mathcal{J}(D_\Theta)$ is also an amb. isot. inv.

Prop 3. $\tilde{\mathcal{J}}(D_\Theta) = (-A^4)^{2(n_1+n_2+n_3)} \left\{ \langle L \rangle + \dfrac{1}{\varphi} \sum_{i=1}^{3} \langle l_i^{(2)} \rangle + \dfrac{2}{\varphi^2} \right\}$

pf) By the identities, $R(\cup) = 0$, $R(\text{bar}) = R(\text{bar})$, $R(\text{bar}) = R(\text{bar})$

B_{D_Θ} can be assumed to be L with one bar for each edge.

Using $R(\text{bar}; a, t) = \dfrac{1}{t + t^{-1}} \left(R(\asymp) + t^{-1} R(\|) + \dfrac{t - t^{-1}}{1 - at} R(\asymp) \right)$,

resolve the three bars.

$\boxed{\text{Thm.}}$ $\tilde{\mathcal{J}}(\Theta) - V(\mathcal{L}) = \frac{1}{\varphi}\sum_{i=1}^{3}\tilde{\mathcal{J}}(K_i) - \frac{1}{\varphi^2}$,

$\left(\begin{array}{l}\mathcal{L}:\ \text{assoc. lk of } \Theta \\ \{K_1, K_2, K_3\}:\ \text{subknots of } \Theta.\end{array}\right.$

- For a diagram D_K of a knot K,

$$\mathcal{J}(D_K) = R(\underbrace{K^{(2)}[1]}_{\text{2-parallel with one bar}}) = R(K^{(2)}) + \frac{1}{\varphi}R(O) = \langle K^{(2)}\rangle + \frac{1}{\varphi}$$

$\mathcal{J}$ is inv under moves (II) & (III)
$\mathcal{J}(\ell) = A^8\mathcal{J}(\Omega)$, $\mathcal{J}(\ell') = A^{-8}\mathcal{J}(\Omega)$ $\Big\} \Rightarrow$ $\tilde{\mathcal{J}}(K) = (A^8)^{-\omega(D_K)}\mathcal{J}(D_K)$ is an amb. iso. inv of K.

- From Prop 2 & 3,

$$\tilde{\mathcal{J}}(\Theta) - V(\mathcal{L}) = \frac{1}{\varphi}\Big\{ A^{8(n_2+n_3)}\big(\langle L_1^{(2)}\rangle + \frac{1}{\varphi}\big) + A^{8(n_1+n_3)}\big(\langle L_2^{(2)}\rangle + \frac{1}{\varphi}\big) + A^{8(n_1+n_2)}\big(\langle L_3^{(2)}\rangle + \frac{1}{\varphi}\big)\Big\} - \frac{1}{\varphi^2}.$$

For the knot diagram L_i, $\omega(L_i) = \omega_{jj} + \omega_{kk} - \omega_{jk} = -(n_j + n_k)$.

$\therefore$ the difference $= \frac{1}{\varphi}\sum_{i=1}^{3}\tilde{\mathcal{J}}(L_i) - \frac{1}{\varphi^2} = \frac{1}{\varphi}\sum_{i=1}^{3}\tilde{\mathcal{J}}(K_i) - \frac{1}{\varphi^2}$. //

Cor. If Θ is brunnian,
$\tilde{\mathcal{J}}(\Theta) - V(\mathcal{L}) = -3 + \frac{2}{\varphi^2}$.

($\therefore$ The two polynomials are equivalent for brunnian θ-curves)

pf) Each K_i is trivial.
$\tilde{\mathcal{J}}(K_i) = \mathcal{J}(O) = -\varphi + \frac{1}{\varphi}$. ///

Thank you!