Aztec Bipyramid and Dicube Tiling

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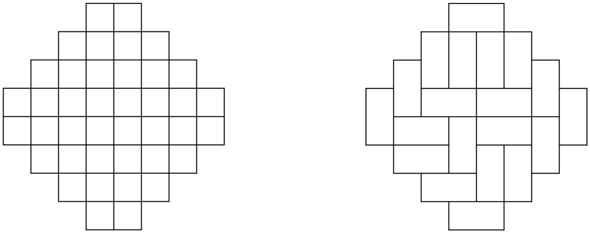
Contents

1. Motivation & Introduction

Aztec Diamond and Domino Tiling

- The Aztec diamond of order n is a quadrilaterally symmetric region composed of $2n$ rows of unit squares, where the rows have lengths $2, 4, \ldots, 2n-2, 2n, 2n, 2n-2, \ldots, 4, 2$.



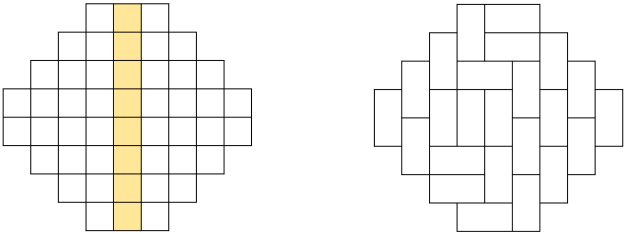
Aztec diamond of order 4 and its domino tiling.

Theorem (Elkies-Kuperberg-Largen-Propp, 1992)

The number of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$.

Augmented Aztec Diamond and Domino Tiling

- The augmented Aztec diamond of order n is obtained from the Aztec diamond of order n by replacing the two longest columns in the middle with three columns.

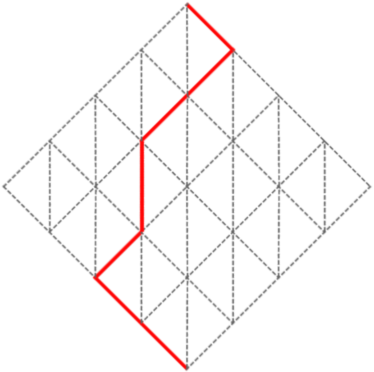


Augmented Aztec diamond of order 4 and its domino tiling.

Theorem (Sachs-Zernits, 1994)

The number of domino tilings of the augmented Aztec diamond of order n is $\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}$, *known as the nth central Delannoy number* $D(n)$.

Delannoy Paths and Delannoy numbers



A Delannoy path on a 4×4 square grid.

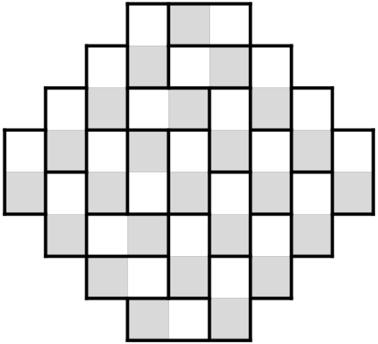
A Delannoy path consists of finite steps in

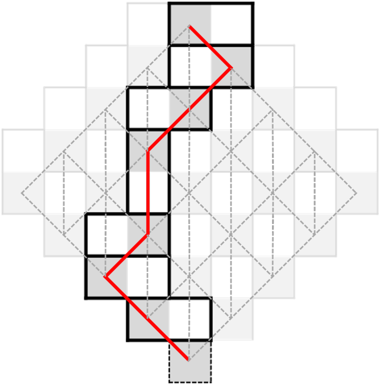
$$\{(0,-2),\ (1,-1),\ (-1,-1)\}.$$

The nth central Delannoy number is the number of Delannoy paths on an $n \times n$ square grid.

$$D(n) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}.$$

Connection between Delannoy Paths and Domino Tilings (Kim-Lee-Oh, 2017)

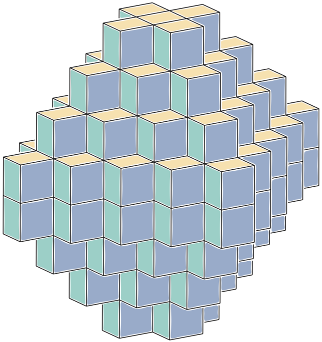




Aztec Bipyramid and Dicube Tilings

A three-dimensional extension of an augmented Aztec diamond and domino tilings.

- A dicube is a $2 \times 1 \times 1$ cuboid.
- The Aztec bipyramid $\mathcal{P}_n$ is a 3D-extension of an augmented Aztec diamond.
  - ∗ The figure is the Aztec bipyramid of order 4.
  - ∗ The largest cross section of $\mathcal{P}_4$ is the augmented Aztec diamond of order 4.
- $\mathcal{P}_n$ admits a dicube tiling because each vertical column has even number of unit cubes.



Aztec Bipyramid and Dicube Tilings

Theorem (Choi-Lee-Oh, 2022)

The number of dicube tilings of the Aztec bipyramid $\mathcal{P}_n$ of order n is given by

$$\sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k}^2 .$$

2. Replacing Dicube Tilings by Delannoy Paths

Settings for Dicube Tilings of General Polycubes

A Polycube is a union of a finite number of unit cubes in $\mathbb{R}^3$ with a connected interior.

We place polycubes in $\mathbb{R}^3$ as follows:

- the unit cubes in the polycubes have centers at integer lattice points;
- their faces are parallel to the $xy-$, yz- or zx-planes.

A polycube is said to be *dicube-tilable* if it has a dicube tiling.

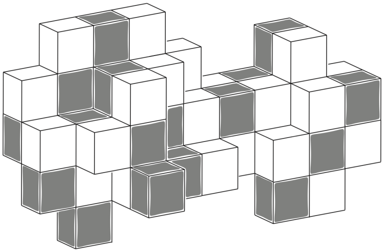
Settings for Dicube Tilings of General Polycubes

The cubes in C are alternately colored black and white like a 3D-checkerboard coloring.

Given a colored polycube C,

- a black cube is called a *pivot cube* if it is in C and has its top face in ∂C;
- a unit cube is called a *ghost cube* if it is not in C but adjacent to a white cube in C along its top face.

Naturally, we color the ghost cubes black.



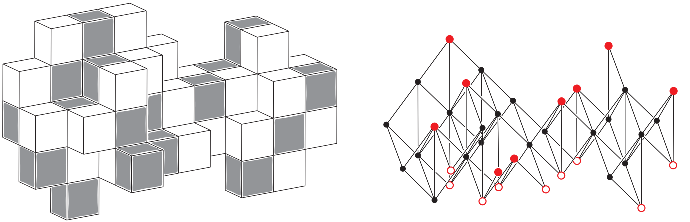
Dual Lattice Graph of a polycube

We associate C with a graph $\Gamma_C = (V_C, E_C)$, called the *dual lattice graph* of C.

- V_C contains all the center points of black cubes in C and the ghost cubes of C.

We call the center points of pivot / ghost cubes the *pivot / ghost vertices*.

- For each non-ghost vertex (x, y, z) in V_C, if C contains a white cube centered at $(x+1, y, z)$, then (x, y, z) and $(x+1, y, z-1)$ are joined by an edge in E_C.
  (Likewise, for possible white cubes at $(x-1, y, z), (x, y+1, z), (x, y-1, z), (x, y, z-1)$.)



Polycube C and its dual lattice graph Γ_C.

A Necessary Condition for a Polycube to be Dicube-Tilable

Proposition

If a polycube C is dicube-tilable,
then C must have the same number of pivot cubes and ghost cubes.
(hence the dual lattice graph Γ_C has the same number of pivot vertices and ghost vertices).

Idea of Proof: "Dicube-tilable polycubes have the same number of black and white cubes."

Attach the following cubes to C, then we obtain another polycube C':

- a white cube on the top of each pivot cube of C. (p more white cubes)
- all ghost cubes of C. (g more black cubes)

Then C' is dicube-tilable because all the topmost / bottommost cubes are white / black.

$\therefore p$ (# of pivot cubes) $= g$ (# of ghost cubes).

Three-dimensional Delannoy paths

A three-dimensional *Delannoy path* is a lattice path in $\mathbb{Z}^3$ of finite length with steps in

$$\{(0,1,-1),(0,-1,-1),(1,0,-1),(-1,0,-1),(0,0,-2)\}.$$

Given a dicube-tilable polycube C,
a *Delannoy path system* in Γ_C is a non-intersecting family of Delannoy paths satisfying

- each path runs from a pivot vertex to a ghost vertex in Γ_C;
- every pivot vertex of Γ_C is joined to a ghost vertex of Γ_C by a path in the system.

Note that every ghost vertex is an endpoint of a Delannoy path by the previous proposition.

Theorem (Choi-Lee-Oh, 2022)

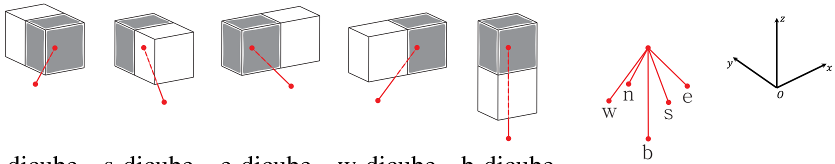
If a polycube C is dicube-tilable, then there is a bijection between the set of dicube tilings of C and the set of Delannoy path systems in Γ_C.

Sketch of proof

- $\mathbb{T}_C$: the set of dicube tilings of C
- $\mathbb{S}_C$: the set of Delannoy path systems in Γ_C

Construct a map $\Phi : \mathbb{T}_C \to \mathbb{S}_C$ as follows:

For a dicube tiling $T \in \mathbb{T}_C$, Φ replaces D-dicubes with steps in Γ_C, ignoring t-dicubes;



D-dicubes $\longrightarrow$

n-dicube	s-dicube	e-dicube	w-dicube	b-dicube
$(0, 1, -1)$	$(0, -1, -1)$	$(1, 0, -1)$	$(-1, 0, -1)$	$(0, 1, -1)$

Sketch of proof

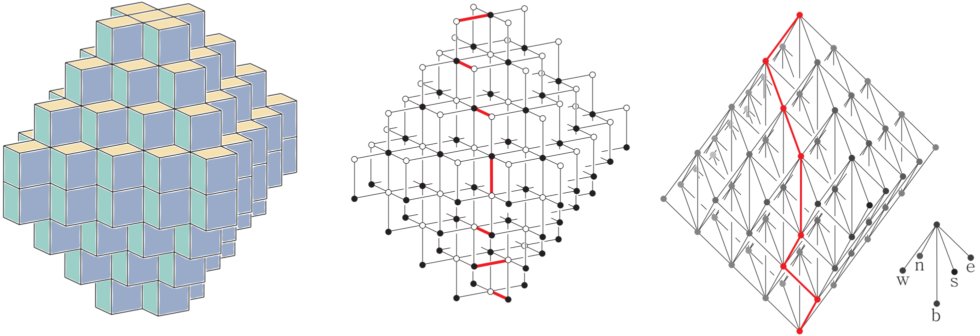
1. Φ is well defined.
   - For each pivot vertex, $\Phi(T)$ contains a Delannoy path joining it to some ghost vertex of Γ_C.
   - Any two Delannoy paths in $\Phi(T)$ do not intersect.
   - $\Phi(T) \in \mathbb{S}_C$.
2. Φ is injective. It follows immediately from the definition of Φ.
3. Φ is surjective.
   - Given a Delannoy path system in $\mathbb{S}_C$,
     replace each step in the system with a D-dicube by reversing the construction of Φ.
   - Let Q be the union of the D-dicubes, which is the sub-polycube of C.
     * The complementary polycube Q^c of Q in C can be tiled only by t-dicubes.
     * Any white cube W in Q^c has a black cube in Q^c adjacent to the bottom face of W.

3. The Number of Dicube Tilings of Aztec Bipyramid

Aztec Bipyramid $\mathcal{P}_n$ and Dicube Tiling

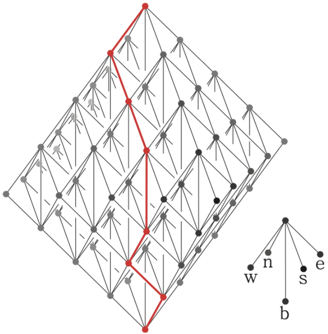
The number of dicube tilings of $\mathcal{P}_n$ = the number of Delannoy path systems in $\Gamma_{\mathcal{P}_n} = \Gamma_n$.

- $\mathcal{P}_n$ has a unique pivot cube centered at $(0, 0, n)$ and ghost cube centered at $(0, 0, -n)$.
- That is, Γ_n contains one Delannoy path.



The Number of Delannoy Paths in Γ_n

A Delannoy path in Γ_n is a path from $(0, 0, n)$ to $(0, 0, -n)$.



- Given $(a, b, c) \in \mathbb{Z}^3_{\geq 0}$ such that $a + b + c = n$,

  we construct a Delannoy path in Γ_n by arranging, in a row,

  - a steps of $(0, 0, -2)$,
  - b steps of $(1, 0, -1)$, b steps of $(-1, 0, -1)$,
  - c steps of $(0, 1, -1)$ and c steps of $(0, -1, -1)$.

- A Delannoy path in Γ_n corresponds to a triple $(a, b, c) \in \mathbb{Z}^3_{\geq 0}$ such that $a + b + c = n$.

$$\therefore \ \text{\# of Delannoy paths with } a+b+c=n \ = \ \text{\# of arrangements of steps} \ = \ \binom{a + 2b + 2c}{a, b, b, c, c} = \binom{b + c + n}{a, b, b, c, c}$$

The Number of Dicube Tilings of the Aztec bipyramid

Theorem (Choi-Lee-Oh, 2022)

The number of dicube tilings of the Aztec bipyramid $\mathcal{P}_n$ of order n is given by

$$\sum_{\substack{a+b+c=n\\ a,b,c\geq 0}} \binom{b+c+n}{a,b,b,c,c} = \sum_{k=0}^{n} \binom{n+k}{n-k}\binom{2k}{k}^2.$$

$$\sum_{a+b+c=n} \binom{b+c+n}{a,b,b,c,c} = \sum_{a+b+c=n} \frac{(b+c+n)!}{a!(b!\,c!)^2} = \sum_{k=0}^{n}\sum_{b=0}^{k} \frac{(n+k)!}{(n-k)!\,(b!\,(k-b)!)^2} \qquad (\text{let } b+c=k)$$

$$= \sum_{k=0}^{n}\sum_{b=0}^{k} \frac{(n+k)!}{(n-k)!\,(2k)!}\,\frac{(2k)!}{k!\,k!}\,\frac{(k!)^2}{(b!\,(k-b)!)^2}$$

$$= \sum_{k=0}^{n}\binom{n+k}{n-k}\binom{2k}{k}\sum_{b=0}^{k}\binom{k}{b}\binom{k}{k-b} = \sum_{k=0}^{n}\binom{n+k}{n-k}\binom{2k}{k}^2$$

Thank You!