# Quasi-Alternating Links, Examples and Obstructions

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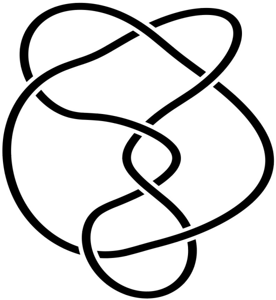
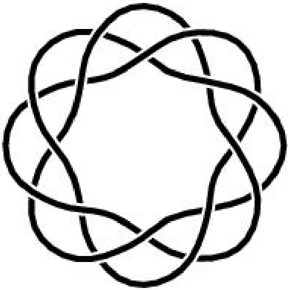
# Outline

# Alternating links

- A link diagram is alternating if the overpass and the underpass alternate as one travels along the diagram.
- A link is said to be alternating if it can be represented by an alternating diagram. Otherwise, the link is non-alternating.

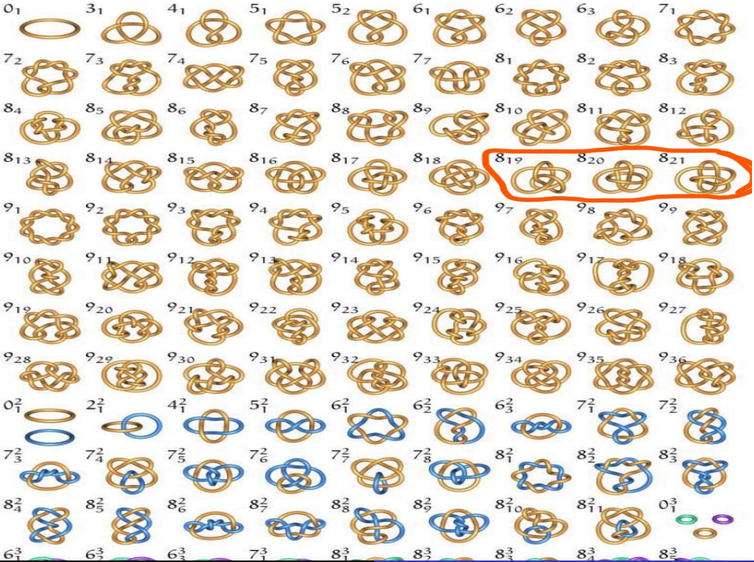


Alternating knot diagram



Non-alternating knot diagram

# Alternating links



# Alternating links

Suppose that $K$ is an alternating knot, $\Delta_K(t)$ its Alexander polynomial and $V_K(t)$ its Jones polynomial.

1. The genus of the knot $g(K)$ is the degree of its "symmetric" Alexander polynomial.
2. The coefficients of $\Delta_K(t)$ alternate in sign and have no gaps.
3. The span of $V_K(t)$ is equal to the crossing number of $K$.
4. The coefficients of $V_K(t)$ alternate in sign and have no gaps (if $K$ is prime and not a torus knot).

# Alternating links

Given an alternating link $L$ and $\Sigma_L$ the branched double-cover of $L$.

1. $\Sigma_L$ is an $L$-space (i.e. Heegard Floer Homology of $\Sigma_L$ is determined by $det(L)$), [Ozsváth and Szabó];
2. $\Sigma_L$ bounds a negative definite 4-manifold $W$ with $H_1(W) = 0$, [Ozsváth and Szabó];
3. The reduced ordinary Khovanov homology group of $L$ is thin, $KH^{i,j}(L)$ is trivial whenever $i - j \neq \frac{\sigma(L)}{2}$, [Lee];
4. The $\mathbb{Z}_2$ link Floer homology group of $L$ is thin, [Rasmussen, Ozsváth- Szabó];
5. The reduced odd Khovanov homology group of $L$ is thin, [Ozsváth, Rasmussen and Szabó];

| j \ i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | χ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 25 | | | | | | | | | | | | 1 | -1 |
| 23 | | | | | | | | | | | | | 0 |
| 21 | | | | | | | | | | 3 | 1 | | -2 |
| 19 | | | | | | | | | 1 | | | | 1 |
| 17 | | | | | | | | 3 | 3 | | | | 0 |
| 15 | | | | | | | 3 | 1 | | | | | 2 |
| 13 | | | | | | 2 | 3 | | | | | | 1 |
| 11 | | | | | 3 | 3 | | | | | | | 0 |
| 9 | | | | 1 | 2 | | | | | | | | 1 |
| 7 | | | 2 | 3 | | | | | | | | | -1 |
| 5 | | | 1 | | | | | | | | | | 1 |
| 3 | 1 | 2 | | | | | | | | | | | -1 |
| 1 | 1 | | | | | | | | | | | | 1 |

**A Khovanov Homology Thin Knot**

| j \ i | 0 | 1 | 2 | 3 | 4 | 5 | χ |
|---|---|---|---|---|---|---|---|
| 17 | | | | | | 1 | -1 |
| 15 | | | | | | 1 | -1 |
| 13 | | | | 1 | 1 | | 0 |
| 11 | | | | | 1 | | 1 |
| 9 | | | 1 | | | | 1 |
| 7 | 1 | | | | | | 1 |
| 5 | 1 | | | | | | 1 |

**Khovanov Homology Thick Knot**

# Topological Characterization of Alternating links

A topological characterization of alternating link is obtained by Greene and Howie.

**Greene, Howie**

A link is alternating if and only if it simultaneously bounds both positive and negative definite spanning surfaces.

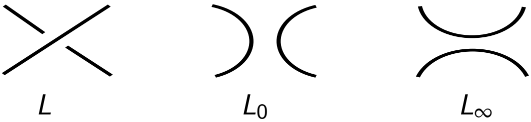
# Quasi-alternating links

## Definition

The set $\mathcal{Q}$ of quasi-alternating links is the smallest set such that:

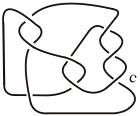
- The unknot belongs to $\mathcal{Q}$.
- If $L$ is a link with a diagram $D$ having a crossing $c$ such that
  1. Both smoothing of $D$ at $c$, $L_0$ and $L_\infty$ are in $\mathcal{Q}$,
  2. $\det(L_0), \det(L_\infty) \geq 1$,
  3. $\det(L) = \det(L_0) + \det(L_\infty)$; then $L \in \mathcal{Q}$

     We say that $L$ is quasi-alternating at the crossing $c$ with quasi-alternating diagram $D$.

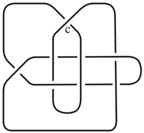


# Quasi-alternating links

1. Any non-split alternating link is QA at any crossing of any reduced alternating diagram [Ozsváth and Szabó].
2. First examples of non-alternating QA knots are $8_{20}$ and $8_{21}$.
3. QA diagrams of knots with $\leq 9$ crossings are given by Manolescu.




$8_{20}$




$8_{21}$

# From the definition

- The determinant of a QA link is positive and it is equal to 1 if and only if $L$ is the unknot;
- If $K_1$ and $K_2$ are two QA knots, then so is their connected sum $K_1 \sharp K_2$, [Champanerkar and Kofman];
- If $L$ is quasi-alternating, then so is its mirror image $L!$;

**Problem**

Decide whether a given knot or link is QA or not.

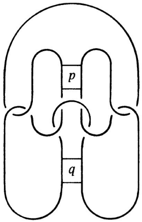
# Obstruction for QA links

1. The branched double-cover of a QA link L, $\Sigma_L$ is an $L$-space (i.e. Heegard Floer Homology of $\Sigma_L$ is determined by $det(L)$) [Ozsv*á*th and Szab*ó*];
2. The branched double-cover of a QA link bounds a negative definite 4-manifold $W$ with $H_1(W) = 0$, [Ozsv*á*th and Szab*ó*];
3. The reduced ordinary Khovanov homology group of a QA link is thin; $KH^{i,j}(L)$ is trivial whenever $i - j \neq \frac{\sigma(L)}{2}$; [Manolescu and Ozsv*á*th];
4. The $\mathbb{Z}_2$ link Floer homology group of a QA link is thin [Manolescu and Ozsv*á*th];
5. The reduced odd Khovanov homology group of a QA link is thin, [Ozsv*á*th, Rasmussen and Szab*ó*];

# Homologically thin non QA links

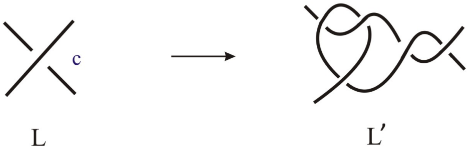
- Torus links $T(p, q)$, $p, q \geq 3$ are Khovanov homologically thick so not QA.
- The first homologically thin non QA knot: $11n_{50}$; [Greene]
- An infinite family of homologically thin, hyperbolic non QA knots: Kanenobu knots of type $K(-10n, 10n + 3)$ [Greene-Watson].

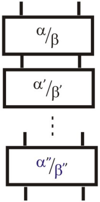
The $K(p, q)$ Kanenobu knot.



# Examples of QA links

Champanerkar-Kofman: Given a link $L$ with QA diagram $D$ at a crossing $c$. Then any link diagram obtained from $D$ by replacing $c$ by an alternating rational tangle of the same type is QA.




$$\frac{8}{3} = 2 + \frac{1}{1 + \frac{1}{2}}$$

This twisting technique has been applied to classify all QA Montesinos links.

# QA Montesinos Links

This theorem was first conjectured by Qazaqzeh-Qublan-C and Champanerkar-Ording, then recently proved by Issa.

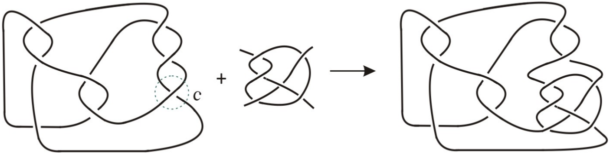
## Theorem

Let $L = M(e; t_1, ..., t_p)$ be a Montesinos link in standard form, that is, where $t_i = \frac{\alpha_i}{\beta_i} > 1$ and $\alpha_i, \beta_i > 0$ are coprime for all i. Then L is quasi-alternating if and only if
(1) $e < 1$, or
(2) $e = 1$ and $\frac{\alpha_i}{\alpha_i - \beta_i} > \frac{\alpha_j}{\beta_j}$ for some $i, j$ with $i \neq j$, or
(3) $e > p - 1$, or
(4) $e = p - 1$ and $\frac{\alpha_i}{\alpha_i - \beta_i} > \frac{\alpha_j}{\beta_j}$ for some $i, j$ with $i \neq j$.

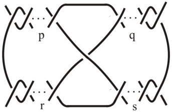
# More Examples

## Kaur-C

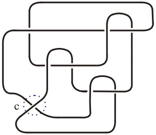
Let $L$ be a QA link diagram and $c$ be a QA crossing. If $L'$ is a link obtained from $L$ by replacing crossing $c$ by an alternating tangle $T$ (algebraic or non-algebraic) of same type, then $L'$ is QA at every crossing of $T$ in $L'$.



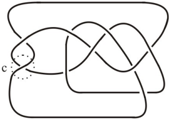
## More Examples of non-alternating QA knots




$T_{pqrs}$




$8_{21}$




$9_{47}$

$K(T)$: Diagram obtained by replacing crossing $c$ in knot diagram $K$ by a tangle $T$

$\widehat{T}$ : Clock-wise 90° rotation of $T$ followed by switching crossings.

| | |
|---|---|
| 13n329 : $8_{21}(\widehat{T}_{1112})$ | 13n357 : $8_{21}(T_{2111})$ |
| 13n351 : $8_{21}(\widehat{T}_{2111})$ | 13n520 : $8_{21}(T_{1121})$ |
| 13n513 : $8_{21}(\widehat{T}_{1121})$ | 13n530 : $8_{21}(\widehat{T}_{1211})$ |
| 13n525 : $8_{21}(\widehat{T}_{1211})$ | 14n20087 : $9_{47}(T_{1211})$ |
| 13n344 : $8_{21}(T_{1112})$ | 14n20090 : $9_{47}(T_{1121})$ |

In addition to the classification of QA pretzel and Montesinos links,

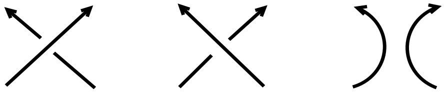
- QA links with braid index 3 are completely characterized; [Greene]
- QA links with determinants $\leq 7$ have been characterized; [Lidman-Sivek]
- QA links with at most 11 crossings have been determined;

# The Jones polynomial

The Jones polynomial is an isotopy invariant of oriented links defined by:

$$V_{\bigcirc}(t) = 1$$
$$tV_{L_+}(t) - t^{-1}V_{L_-}(t) = (\sqrt{t} + \frac{1}{\sqrt{t}})V_{L_0}(t),$$

where $L_+$, $L_-$ and $L_0$ are 3 links as pictured below:



We can always write the Jones polynomial of any link $L$ as follows:

$$V_L(t) = t^r \sum_{i=0}^{m} a_i t^i, \text{where } m \geq 0, a_0 \neq 0 \text{ and } a_m \neq 0.$$

$m := span(V_L)$

Let $L$ be a QA link.

- The coefficients of $V_L(t)$ alternate in sign. This is a consequence of its thin Khovanov homology.
- $\mathrm{KH}^{i,j}(L)$ has no gaps of length larger than 1 in the differential grading, thus $V_L(t)$ has no gap of length larger than one [Qazaqzeh-C].

| $j,i$ | -3 | -2 | -1 | 0 |
|---|---|---|---|---|
| -1 | | | | 1 |
| -3 | | | | 1 |
| -5 | | 1 | | |
| -7 | | | | |
| -9 | 1 | | | |

Table: The Khovanov homology of the trefoil knot.

# Colored Jones Polynomials

Let $J_{N,L}(q)$ be the Colored Jones polynomial of $L$.
If $L$ is alternating $J_{N,L}(q)$ satisfies the Head-tail property:
The first and the last $k$ coefficients of the polynomial do not depend on $N$ as long as $N > k$. For example, for $J_{N,6_3}(q)$ we have:

$$N=2:\quad 1-2q+2q^2-2q^3+2q^4-q^5+q^6$$
$$N=3:\quad 1-2q+4q^3-5q^4+6q^6+\cdots-q^{14}+3q^{15}-q^{16}-q^{17}+q^{18}$$
$$N=4:\quad 1-2q+2q^3+q^4-4q^5-2q^6+\cdots-2q^{29}-3q^{30}+3q^{32}-q^{34}-q^{35}+q^{36}$$
$$N=5:\quad 1-2q+2q^3-q^4+2q^5-6q^6+\cdots-2q^{53}-q^{54}+4q^{55}-q^{58}-q^{59}+q^{60}$$
$$N=6:\quad 1-2q+2q^3-q^4-2q^7+q^8+\cdots-3q^{82}+3q^{84}+q^{85}-q^{88}-q^{89}+q^{90}$$
$$N=7:\quad 1-2q+2q^3-q^4-2q^6+4q^7+\cdots+4q^{119}+q^{121}-q^{124}-q^{125}+q^{126}$$

**Question.** Head-tail property is not true QA links. Does $J_{N,L}(q)$ satisfy any similar property?

# Alexander Polynomial

Recall that if $K$ is an alternating knot and $\Delta_K(t) = \sum_{-m}^{m} \alpha_i t^i$, then:

1. $g(K) = m$.
2. $\alpha_i \alpha_{i+1} < 0$, for all $-m \leq i \leq m-1$.

Condition 1 is true for QA knots and knots which are thin in Floer Homology in general, [Ozsváth-Szabó].

**Question.** Does condition 2 extend to quasi-alternating links?

**Remark.** Fox Trapezoid Conjecture does not hold for quasi-alternating links.

# Open Problem

**Problem.** Give a topological characterization of QA links.