Linearly Free Graphs

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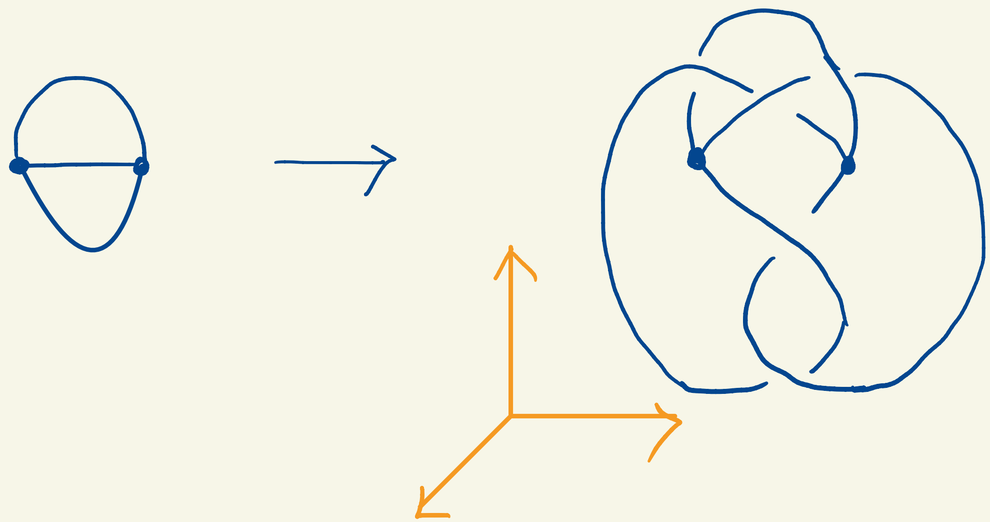
동국대에서...

Spatial graphs: graphs embedded in $\mathbb{R}^3$ or S^3.

$$f : G \hookrightarrow \mathbb{R}^3 \ (S^3)$$



Spatial Graphs : ?

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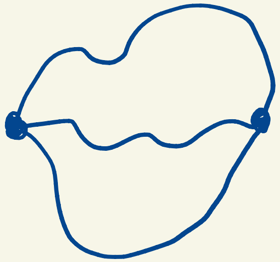
- distinguish types of embedding of a graph

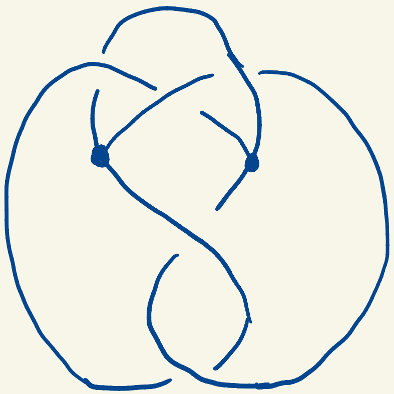
θ_1



$$R(\theta_1) = -A^2 - A - 2 - A^{-1} - A^{-2} \qquad \text{width} = 4$$

$$\neq$$

θ_2



$$R(\theta_2) = A^9 - A^8 - 2A^7 + A^6 - A^5 \qquad \text{width} = 17$$

$$+ 2A^3 + A^2 + 2A + A^{-1} - A^{-3} + A^{-4}$$

$$+ A^{-5} - A^{-6} + A^{7} + A^{-8}$$

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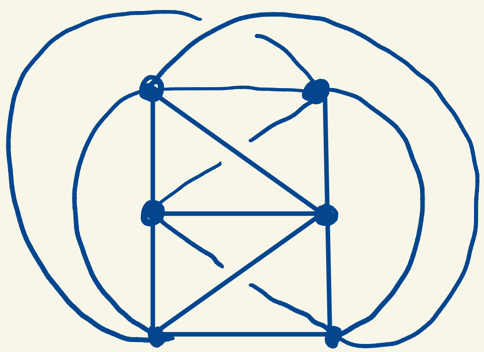

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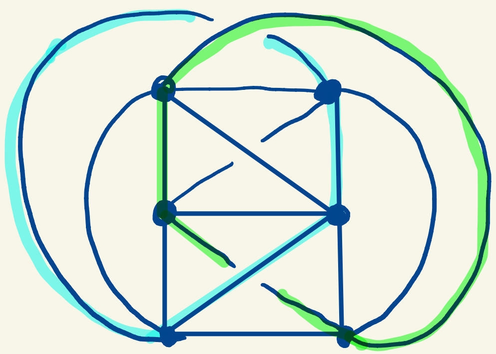
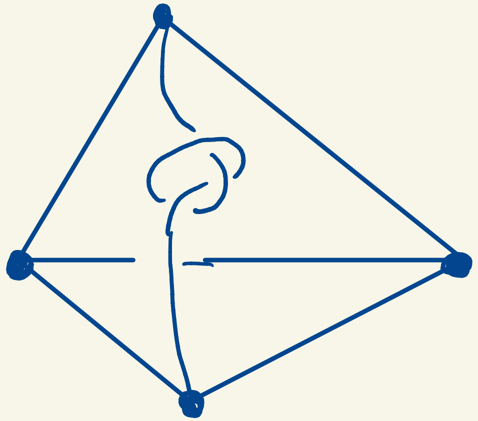
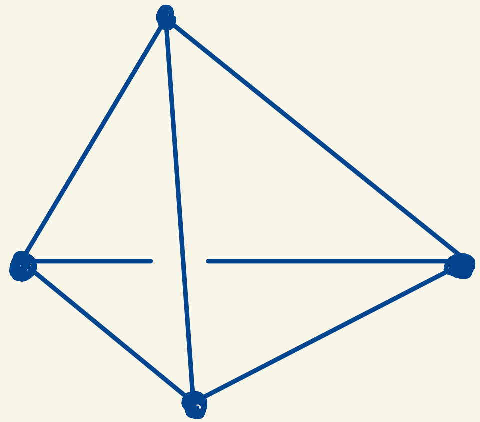


- [R-S-T] successfully characterized intrinsically linked graphs.

Motivation



$f: K_4 \to \mathbb{R}^3$



$g: K_4 \to \mathbb{R}^3$

"linear": every edge is a line segment.

(local flexibility is excluded)

$\pi_1(\mathbb{R}^3 - f(K_4))$ Not free

$\pi_1(\mathbb{R}^3 - g(K_4))$ free

Defs

① An embedding of a graph into $\mathbb{R}^3$ is linear,
if every edge is a line segment.

② An emb $f: G \to \mathbb{R}^3$ is free, if $\pi_1(\mathbb{R}^3 - f(G))$ is a free gp.

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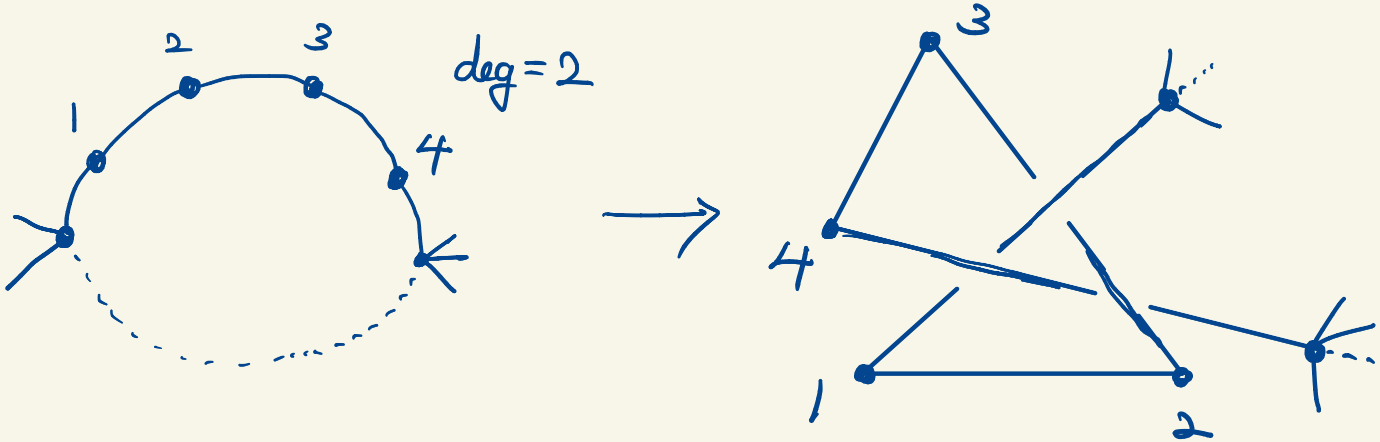
⊛



To avoid such a local flexibility, we are focused on graphs with deg at each vertex ≥ 3

[Nicholson : 1988]

Every complete graph is linearly free.

[H-L : 2015]

(1) For any $n \geq 1$, $\exists$ ∞-ly many simple conn. graphs with $\underbrace{\text{Min. deg} = n}$ which are $\underbrace{\text{Not linearly free}}$

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For (1), $\#V \geq 6(n+1)$

For (2), $\#V \geq 12n$

But K_n : $(\#V, \text{Min deg}, \text{Connectivity}) = (n, n-1, n)$

From Nicholson's Thm, we may ask:

Q. Starting with a complete graph,
How much can we enlarge it
so that (the linear-freeness is Not broken / the clique # does Not increase) ?

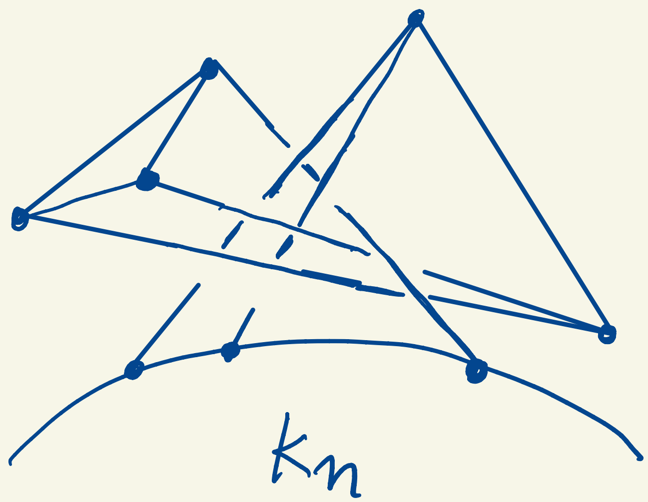
$\circledast$ Clique # of $G := \text{Max } N$ s.t. K_N is a subgraph of G.

A partial answer :

$\boxed{\text{Thm 1}}$ G : simple conn. graph s.t. $K_n \subset G$

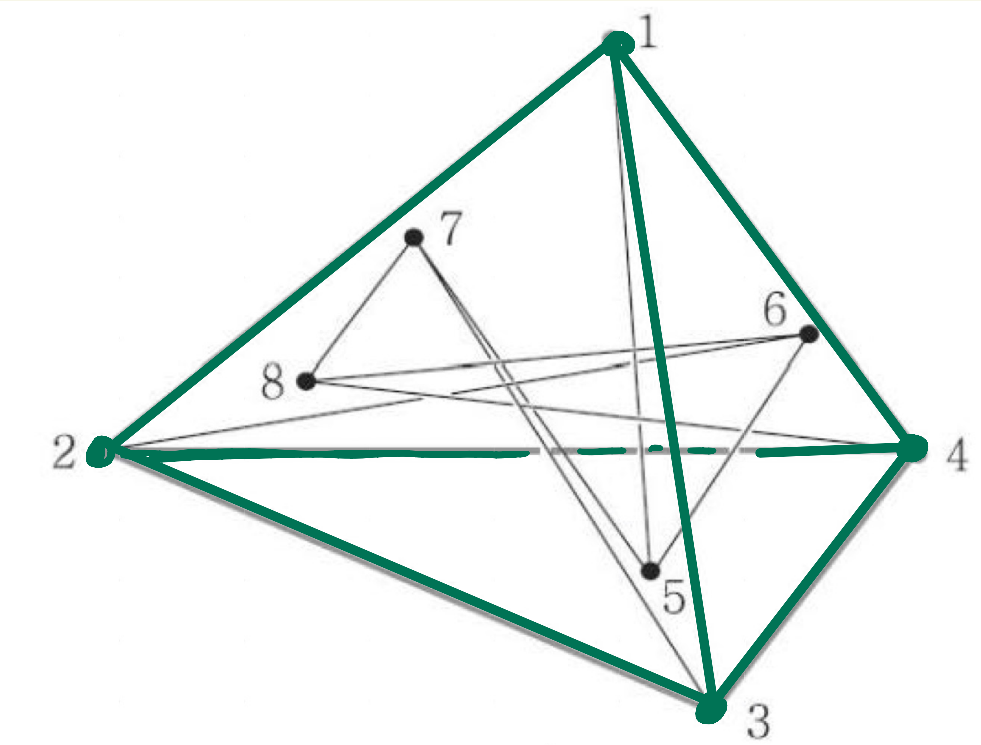
If $\# V \leq n+3$, then G is linearly free.

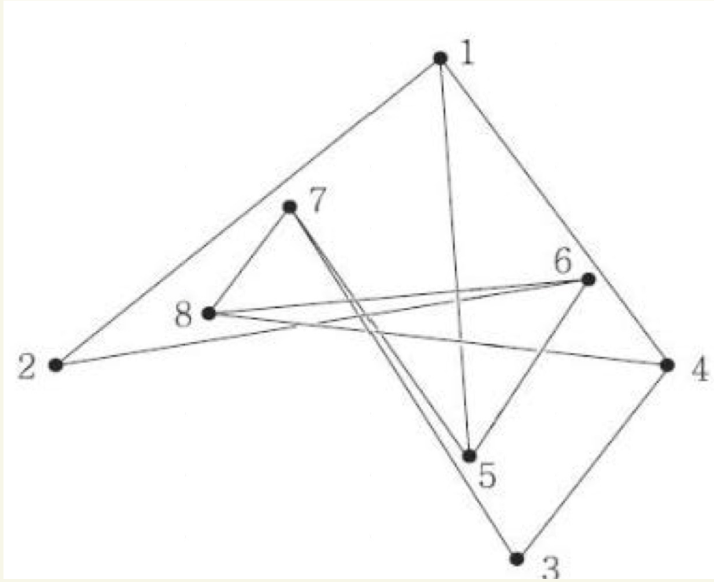
? $\# V \geq n+5$



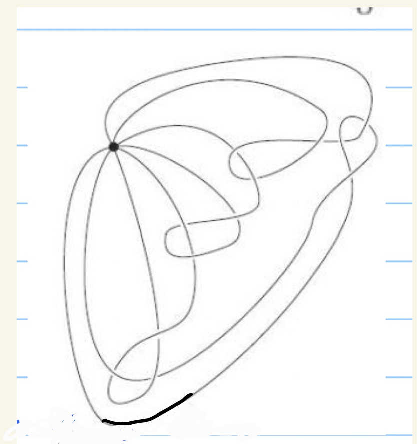
? $\#V = n+4$ ... a specific counter example for $n=4$.

G : a graph obtained by adding 4-vertices to K_4.





edge contraction



π_1 is Not free

$\pi_1 = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * ///$

$\therefore$ Not free

Q. Linearly free graphs with small # of vertices

$\boxed{\text{Thm 2}}$ G : simple conn graphs with Min. Deg ≥ 3

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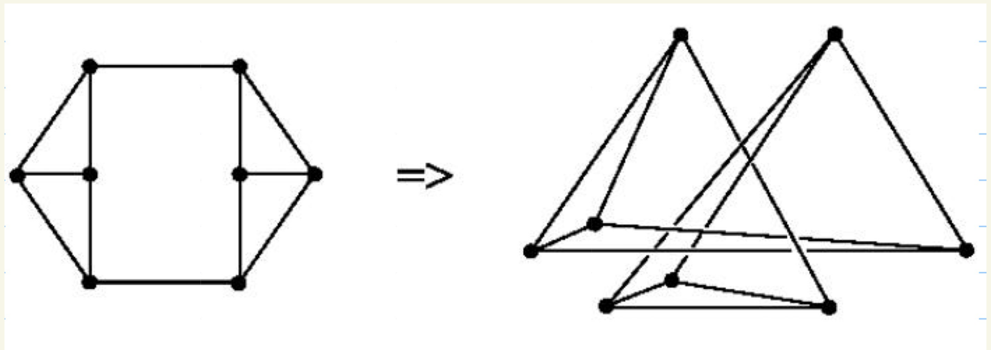
$\# V \geq 8$:



$\boxed{\text{Thm 3}}$ For $n, m \leq 6$,

the complete bipartite graph $K_{n,m}$ is linearly free.

(For every n, m ? ... still working on)

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