Stick number of knots and links

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Knots and Spatial Graphs 2021

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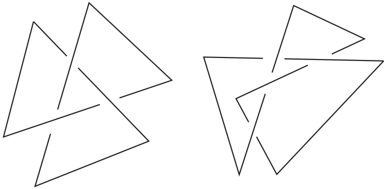
Contents

Definitions

- A *stick knot* is a simple closed curve in $\mathbb{R}^3$ which consists of finite number of straight line segments.
- A *stick number* $s(K)$ of a knot K is the minimal number of sticks required to construct this stick knot.



- $s(3_1) = 6$
- $s(4_1) = 7$.

General upper bound

Negami(1991)

$\frac{5+\sqrt{8c(K)+9}}{2} \leq s(K)$ for a link K except for the trivial knot, and $s(K) \leq 2c(K)$ for a link K which has neither the Hopf link as a connected sum factor nor a splittable trivial component.

Huh-Oh(2011)

$s(K) \leq \frac{3}{2}(c(K)+1)$ for any nontrivial knot K. In particular, $s(K) \leq \frac{3}{2}c(K)$ for any non-alternating prime knot.

Upper bound of knots and links

- Torus knot(Jin 1997) : $s(T_{p,q}) \leq 2q(p < q)$ and equlity holds if $2 \leq p < q \leq 2p$.
- Composite knot(Adams et al. 1997, Jin 1997) : $s(K_1 \sharp K_2) \leq s(K_1) + s(K_2) - 3$.
- 2-bridge(Rational) link(Huh-N.-Oh 2011) : $s(K) \leq c(K) + 2$ if $c(K) \leq 6$.
- Montesinos link(Lee-N.-Oh 2021+) :

$$s(K) \leq \begin{cases} c(K) + 1 & \text{if} \quad K \text{ is alternating,} \\ c(K) + 3 & \text{if} \quad K \text{ is non-alternating.} \end{cases}$$

if each rational tangle of the diagram has five or more index of the related Conway notation.

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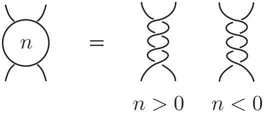
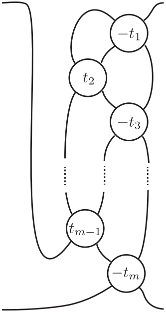
Huh-N.-Oh (2011)

$s(K) \leq c(K) + 2$ *for all 2-bridge links* K *with* $c(K) \geq 6$.

Rational tangle and 2-bridge links

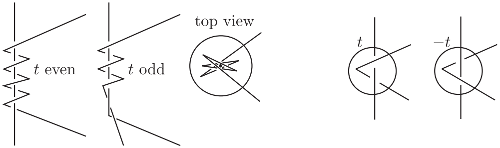
- Integer tangles and a rational tangle with Conway notation $[t_1, t_2, \cdots, t_m]$





Stick Presentation of a 2-bridge link

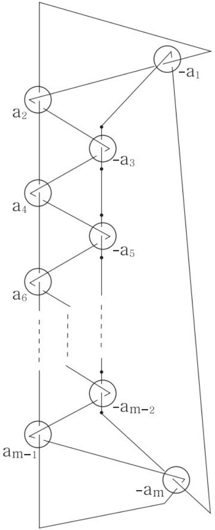
- How to construct an integer tangle by $t + 1$ sticks



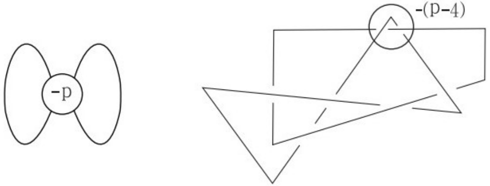
(a) (b)

Stick Presentation of a 2-bridge link

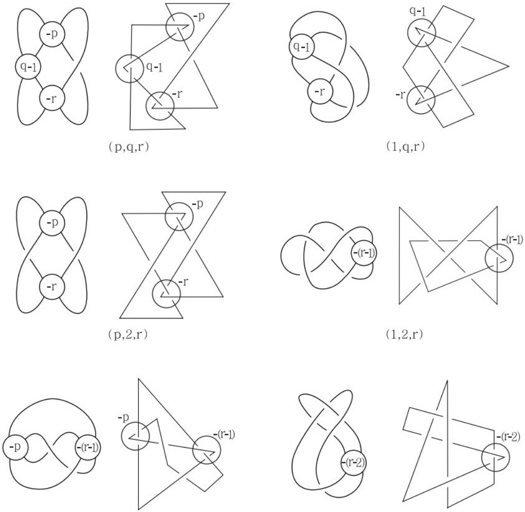
- How to construct a 2-bridge link by $c(K) + 2$ sticks



- Special case 1 : $[p]$



- Special case 2 : $[p, q, r]$



(p,q,r) (1,q,r)
(p,2,r) (1,2,r)
(p,1,r) (1,1,r)

- Knotinfo

KnotInfo: Table of Knots

Search Results

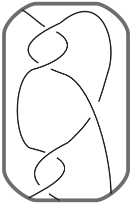
Export to CSV

Name	Stick Number
3_1	6
4_1	7
5_1	8
5_2	8
6_1	8
6_2	8
6_3	8
7_1	9
7_2	9
7_3	9
7_4	9
7_5	9
7_6	9
7_7	9
8_1	[9,10]
8_2	[9,10]
8_3	[9,10]
8_4	[9,10]
8_5	[9,10]
8_6	[9,10]

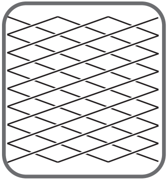
[First] [Previous] [Next] [Last] **[Show All]**

Total 249 knots found: show 20 knots starting from 1 Go

- Pillowcase form of a rational tangle



[2, 1, 2]



3/8

- $$\frac{1}{2+\frac{1}{1+\frac{1}{2}}} = \frac{3}{8}$$

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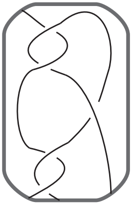

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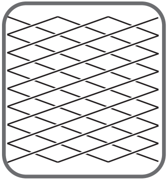

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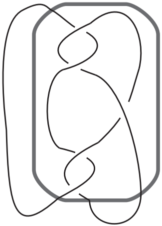

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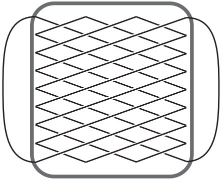

3/8

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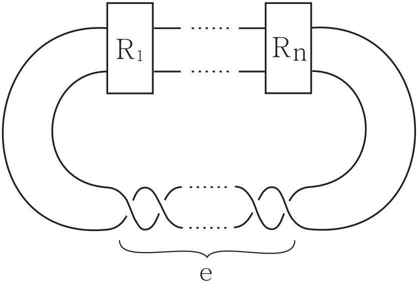
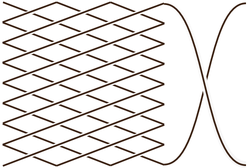
Definition

- Montesinos link



- Each R_i's are rational tangles and $n \geq 3$.



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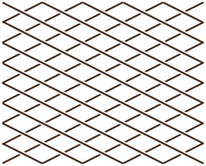

$3/8 \oplus -1$ $\qquad$ $-5/8$

- $\dfrac{3}{8} = \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}$ and $-\dfrac{5}{8} = -\cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}$.

- If $p < \dfrac{q}{2}$ and $\dfrac{p}{q} = \cfrac{1}{r + \cfrac{s}{t}}$, then $\dfrac{q-p}{q} = \cfrac{1}{1 + \cfrac{1}{(r-1) + \cfrac{s}{t}}}$.
- Thus $\dfrac{p}{q}$- and $-\dfrac{q-p}{q}$- tangles have same number of crossings.

- A reduced Montesinos diagram D is a diagram of a Montesinos link satisfying one of the following two conditions;
  (1) D is alternating, or
  (2) $e = 0$ and each R_i is an alternating rational tangle diagram, with at least two crossings, placed in D so that the two lower ends of R_i belong to arcs incident to a common crossing in R_i.
- They are can be respectively rephrased by
  (1*) $e \geq 0$ and all R_i's are positively alternating in D, or
  (2*) $e = 0$, and $R_1, \ldots, R_t$ are positively alternating in D and $R_{t+1}, \ldots, R_n$ are negatively alternating for some t.

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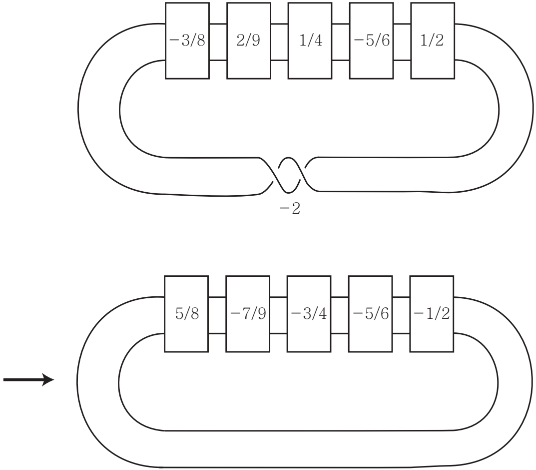
Example



Likorish-Thistlethwaite (1988)

If a link L admits a reduced Montesinos diagram having n crossings, then L has the crossing number n.

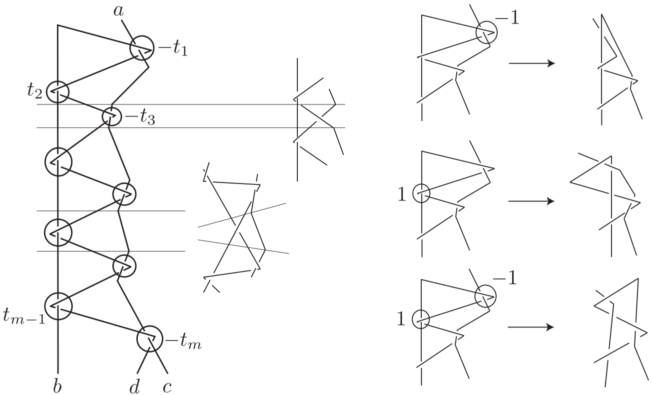
Lee-N.-Oh (2021+)

Let K be a knot or link which admits a reduced Montesinos diagram having $c(K)$ crossings. If each rational tangle of the diagram has five or more index of the related Conway notation, then

$$s(K) \leq \begin{cases} c(K)+1 & \text{if } \; K \text{ is alternating,} \\ c(K)+3 & \text{if } \; K \text{ is non-alternating.} \end{cases}$$

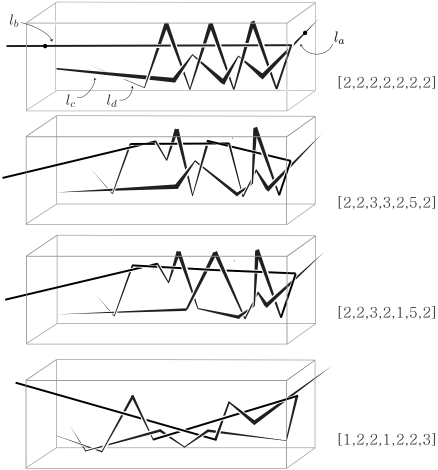
Proof of Main Theorem

- How to construct a rational tangle by $t_1 + t_2 + \cdots + t_m + 1$ sticks



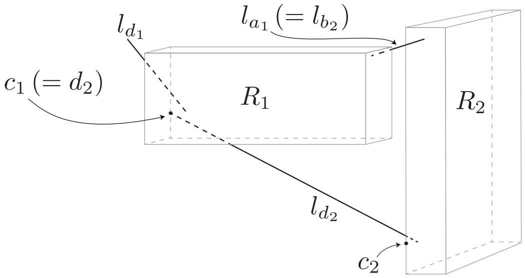
Proof of Main Theorem

- Stick rational tangles in a virtual boxes



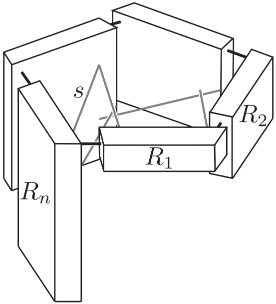
Proof of Main Theorem

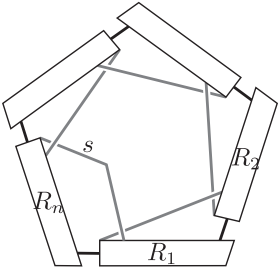
- How to combine rational tangles



Proof of Main Theorem

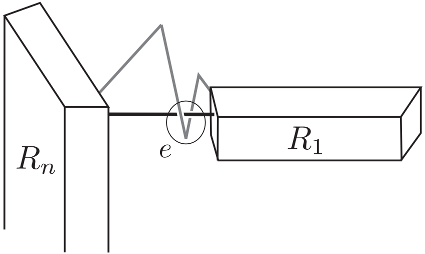
- Alternating Montesinos link





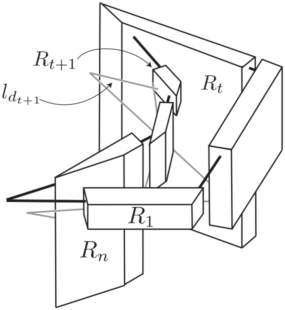
Proof of Main Theorem

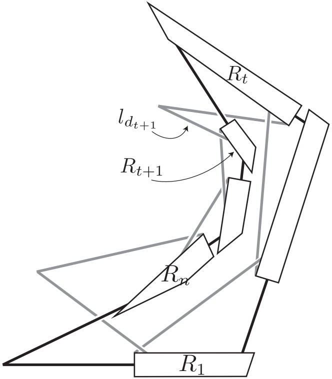
- $e \neq 0$ case



Proof of Main Theorem

- Non-alternating Montesinos link





Proof of Main Theorem

- For the case
  (1*) $e \geq 0$ and all R_i's are positively alternating in D,

we need one more stick to connect c_n and d_1. Then we can construct K by $c(K) + 1$ sticks.

- For the case
  (2*) $e = 0$, and $R_1, \ldots, R_t$ are positively alternating in D and $R_{t+1}, \ldots, R_n$ are negatively alternating for some t,

we need one more stick to connect c_t and d_{t+1}, and two more sticks to connect R_n and R_1. Therefore we need $c(K) + 3$ sticks to construct K.

Thus the proof is complete.

Questions

(1) What is the upper bound of stick number for the Montesinos link including rational tangles with Conway notation $[r_1]$ or $[r_1, r_2, r_3]$?

(2) In general, non-alternating links have better upper bound for stick numbers. How about for Montesinos links?

Thank you