Comparison of Ascending Number and Genus of Knots

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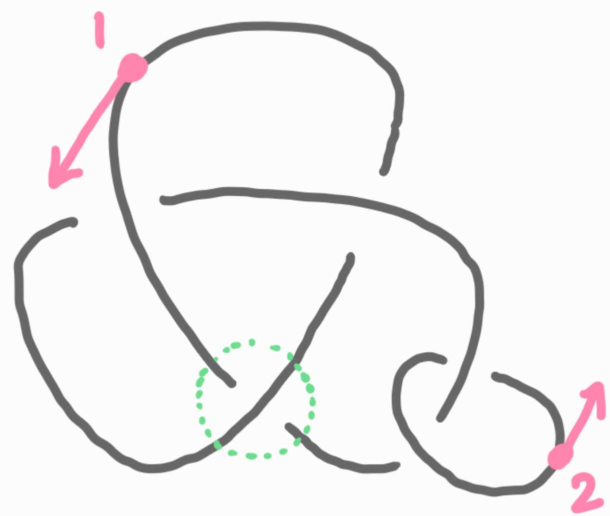
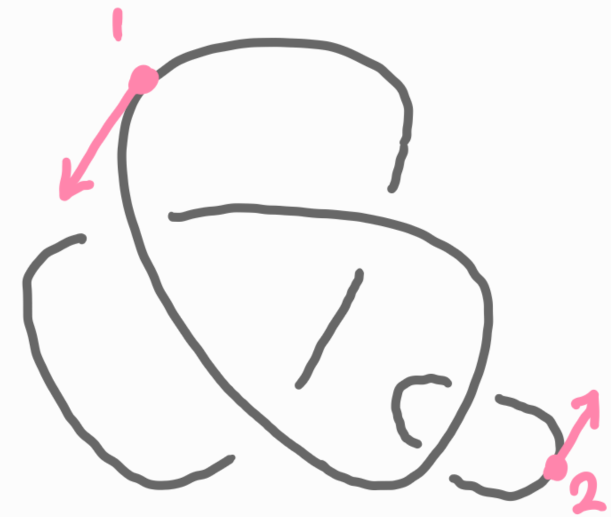
Knots and Spatial Graphs 2021



D : a based ordered oriented link diagram of link L



$d(D)$: the descending diagram of $\tilde{L}$

Definition (Ozawa, 2010)

- L : link,
- D : based ordered oriented link diagram of L

$\rightsquigarrow$ $a(D)$ = # different crossings between D and $d(D)$

$$a(L) = \min_{D \in [L]} a(D)$$

↳ ascending number of L

* Note: For any knot K, $u(K) \leq a(K)$

Theorem (Ozawa)

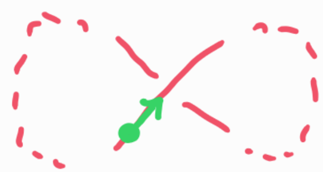
- $a(K) \leqq \left[\dfrac{c(K) - 1}{2} \right]$ if K : knot
- $a(L) \leqq \left[\dfrac{c(L)}{2} \right]$ if L : link

<sketch of proof>

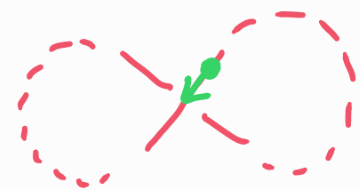
① arbitrary L

$$\frac{c(L)}{2} \geqq \min\{a(D_L), a(-D_L)\}$$
$$\geqq a(L)$$

② $L = K$



D_K



$-D'_K$

$$\frac{c-1}{2} \geqq \min\{a(D_K), a(-D'_K)\}$$
$$\geqq a(K)$$

$\blacksquare$

Theorem (Ozawa)

For n component link L,

$$a(L) \geq br(L) - n.$$

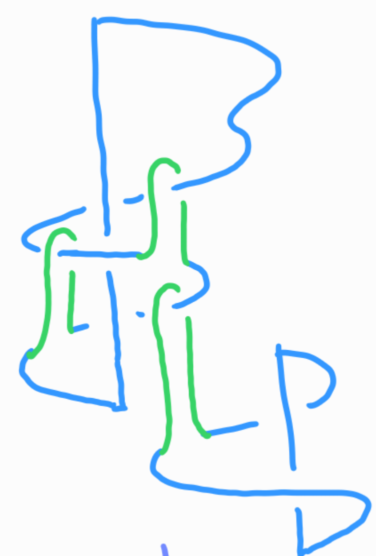
<sketch of proof>

D: based oriented diagram with $a(D) = a(L)$ $\rightsquigarrow$

$\downarrow$

$d(D)$

$\rightsquigarrow$



$\downarrow$

D

$\blacksquare$

Proposition (Ozawa) $a(K_1 \# K_2) \leqq a(K_1) + a(K_2)$.

Conjecture (Ozawa) $a(K_1 \# K_2) = a(K_1) + a(K_2)$?

Corollary If K_i satisfies $a(K_i) = b_r(K_i) - 1 \quad (i = 1, 2)$,

then $a(K_1 \# K_2) = a(K_1) + a(K_2)$.

<proof>

$$b_r(K_1 \# K_2) - 1 \leqq a(K_1 \# K_2) \leqq a(K_1) + a(K_2)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$(b_r(K_1) - 1) + (b_r(K_2) - 1)$$

$\blacksquare$

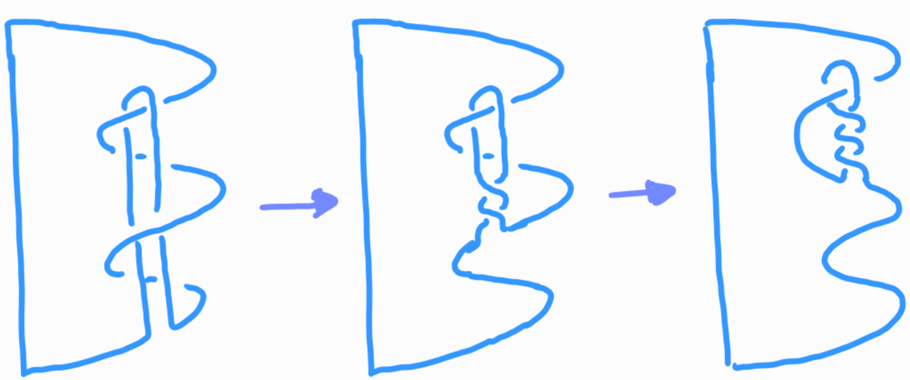
Theorem (Ozawa)

trivial link

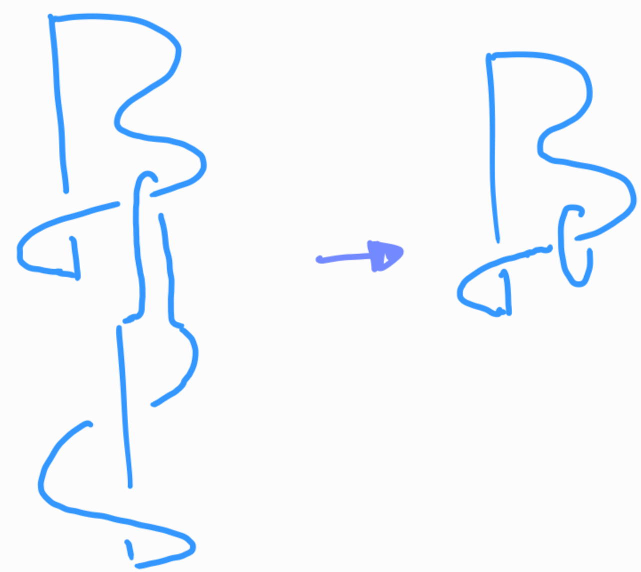
$$a(L) = 1 \iff L : \text{twist knot} \circ O^{n-1} \quad \text{or} \quad \text{Hopf link} \circ O^{n-1}$$

<sketch of proof>

①



②



K: prime knot with $c(K) \leq 10$

- $c(K) \leq 7$ $\rightsquigarrow$ Ozawa

- $K = 8_1, 8_3, 8_6, 8_8, 8_{11}, 8_{12}, 8_{14}, 8_{15}, 8_{18}, 8_{19}, 8_{21}, 9_1, 9_2, 10_1$ $\rightsquigarrow$ Ozawa

- $K = 8_4, 9_3, 9_4, 9_6, 9_7, 9_{47}, 9_{48}$ $\rightsquigarrow$ Okuda (1998)

- $K = 8_{13}, 8_{20}$ $\rightsquigarrow$ Fujimura (1998)

- $K = 8_2, 8_5, 9_{11}, 9_{20}, 9_{36}, 9_{43}, 10_2, 10_8, 10_{14}, 10_{21}, 10_{25}, 10_{39}, 10_{46}, 10_{50}, 10_{56}, 10_{61}, 10_{72}, 10_{76}, 10_{85}, 10_{92}, 10_{98}, 10_{100}, 10_{111}, 10_{127}, 10_{149}, 10_{150}$ — Higa (2019)

  ↳ Using Conway polynomial $\nabla_K(z)$.

- $K = 8_7, 8_9, 8_{10}, 9_{22}, 9_{24}, 9_{26}, 9_{27}, 9_{28}, 9_{29}, 9_{30}, 9_{41}, 10_{11}, 10_{12}, 10_{15}, 10_{16}, 10_{19}, 10_{22}, 10_{23}, 10_{26}, 10_{27}, 10_{28}, 10_{29}, 10_{30}, 10_{32}, 10_{40}, 10_{41}, 10_{42}, 10_{44}, 10_{54}, 10_{59}, 10_{60}, 10_{65}, 10_{68}, 10_{69}, 10_{70}, 10_{71}, 10_{73}, 10_{75}, 10_{77}, 10_{78}, 10_{84}, 10_{93}, 10_{94}, 10_{96}, 10_{97}, 10_{110}, 10_{112}, 10_{113}, 10_{117}, 10_{122}, 10_{123}, 10_{125}, 10_{126}, 10_{131}, 10_{138}, 10_{153}$ — Higa (2020)

  ↳ Using Γ- polynomial $\Gamma(K)$

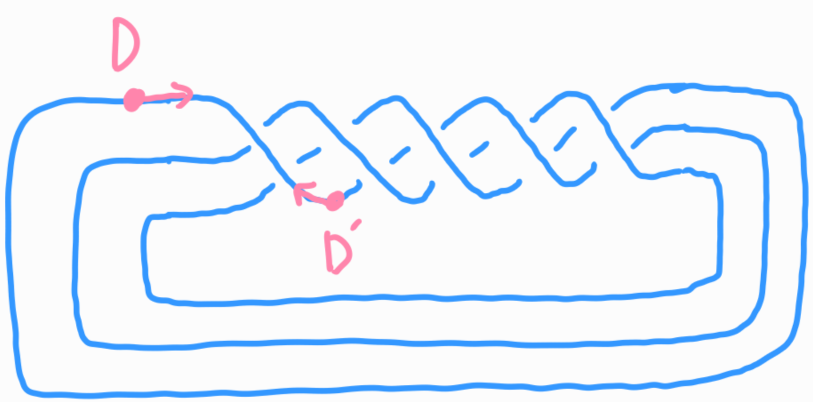
Theorem (Ozawa)

$$a(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

<sketch of proof>

$$\frac{(p-1)(q-1)}{2} = u(T_{p,q}) \leq a(T_{p,q}) \leq \frac{(p-1)(q-1)}{2}$$

$$a(D) + a(D') = (p-1)(q-1)$$



$\square$

- $K =$ twist or torus $\Rightarrow a(K) = g(K)$
- Of the 119 prime knots whose $a(K)$ is known, 102 satisfy $a(K) = g(K)$.

Question

What is the relationship between $a(K)$ and $g(K)$?

Let D be a based oriented knot diagram with base point b.

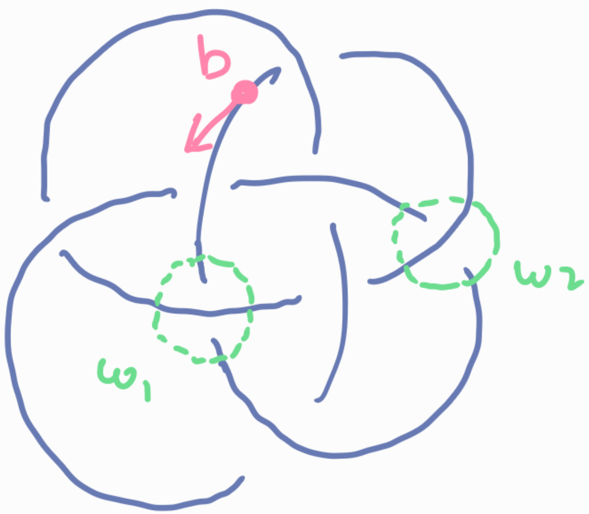
Then $\exists$ continuous function $f: [0,1] \longrightarrow \mathbb{R}^2$
s.t. $f([0,1]) = D$, $f(0) = f(1) = b$, $f|_{(0,1)}$: immersion

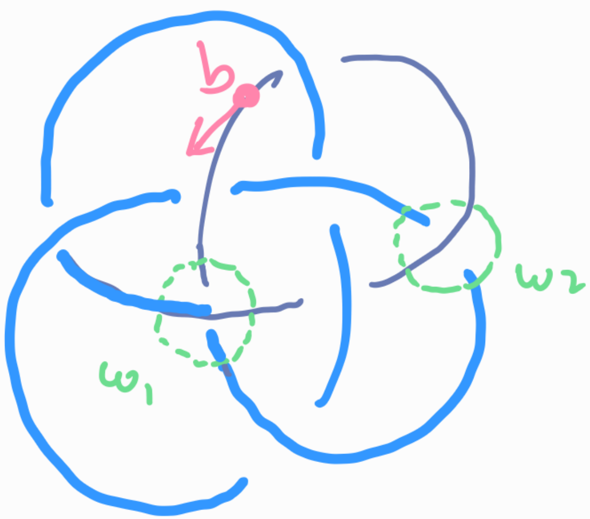
- Let c be a crossing in D,
  and let $t_0, t_1 \in [0,1]$ s.t. $f(t_0) = f(t_1) = c$.
  $\rightsquigarrow$ $D_c := f([t_0, t_1])$. (* Note : $D_c \not\ni b$)

- For any crossing c in D

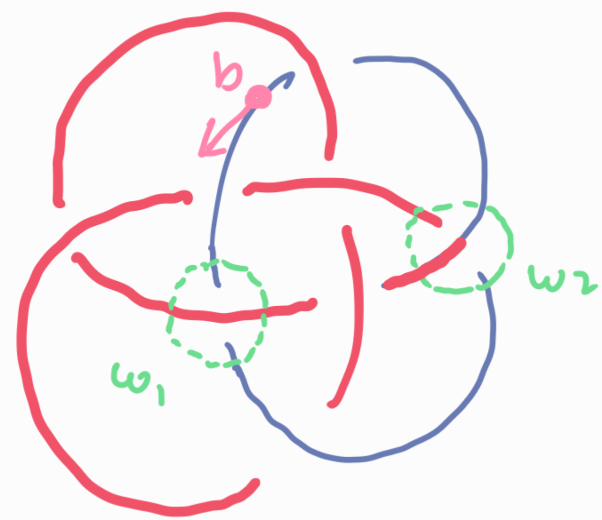
$$\varphi_b(c) = |D_c \cap \overrightarrow{bc}| - |D_c \cap \overline{bc}|.$$

Example





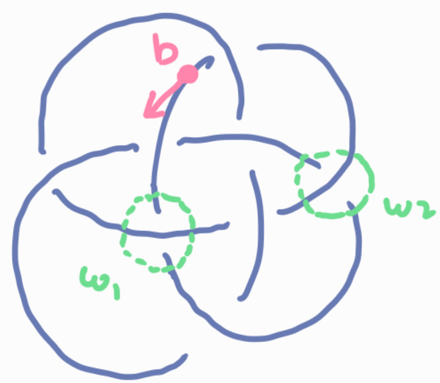
$\varphi_b(w_1) = 1$



$\varphi_b(w_2) = 0$

• warping crossing : crossing w in D
(by Kawauchi) s.t. $w \neq d(w)$

$\rightsquigarrow$ $\exists$ canonical order along orientation and base point b

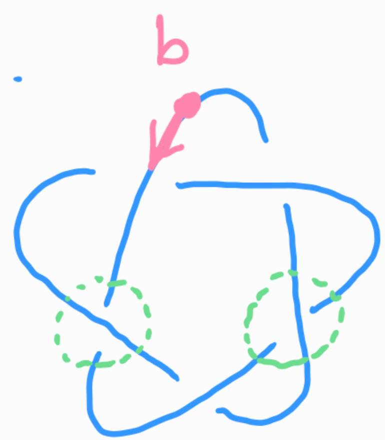


Definition

For $\forall i = 1, \cdots, a(D)$, $\varphi_b(w_i) = 0$

$\Rightarrow$ D is called a loose diagram

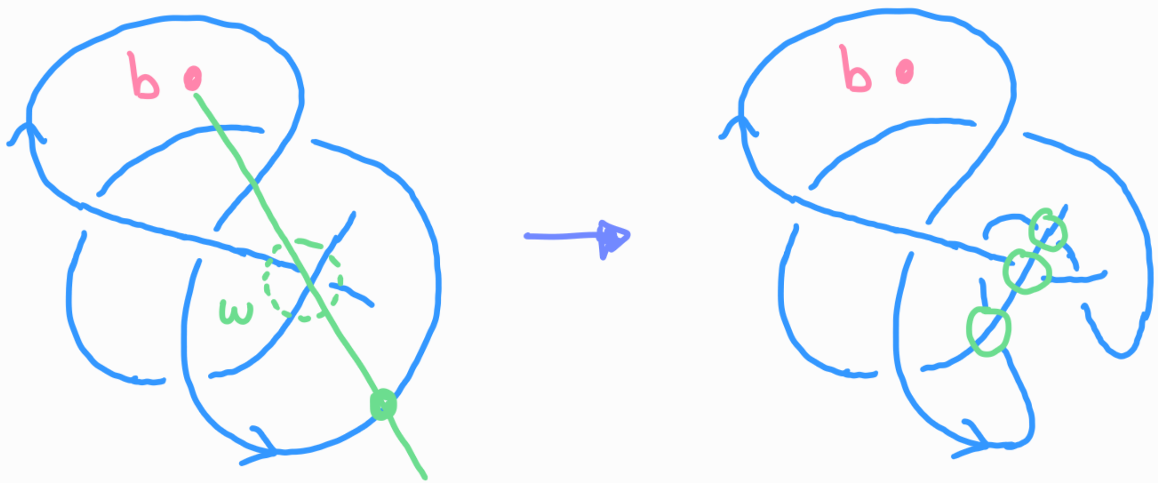
e.g.



Proposition

For any knot K, there is a loose diagram of K.

<proof>



D_c $\qquad\qquad$ D_c'

∎

Theorem

Let K be a knot and let D be a diagram of K. Then

$$g(K) \leqslant a(D) + \sum_{i=1}^{a(D)} \varphi_b(w_i).$$

Corollary

If K has a loose diagram with $a(D) = a(K)$, then

$$g(K) \leqslant a(K).$$

<proof of Theorem>

Let D: a based oriented diagram of K
and let $f: [0,1] \longrightarrow \mathbb{R}^2$ be a continuous function
s.t. $f([0,1]) = D$, $f(0) = f(1) = b$, $f|_{(0,1)}$: immersion.

Then

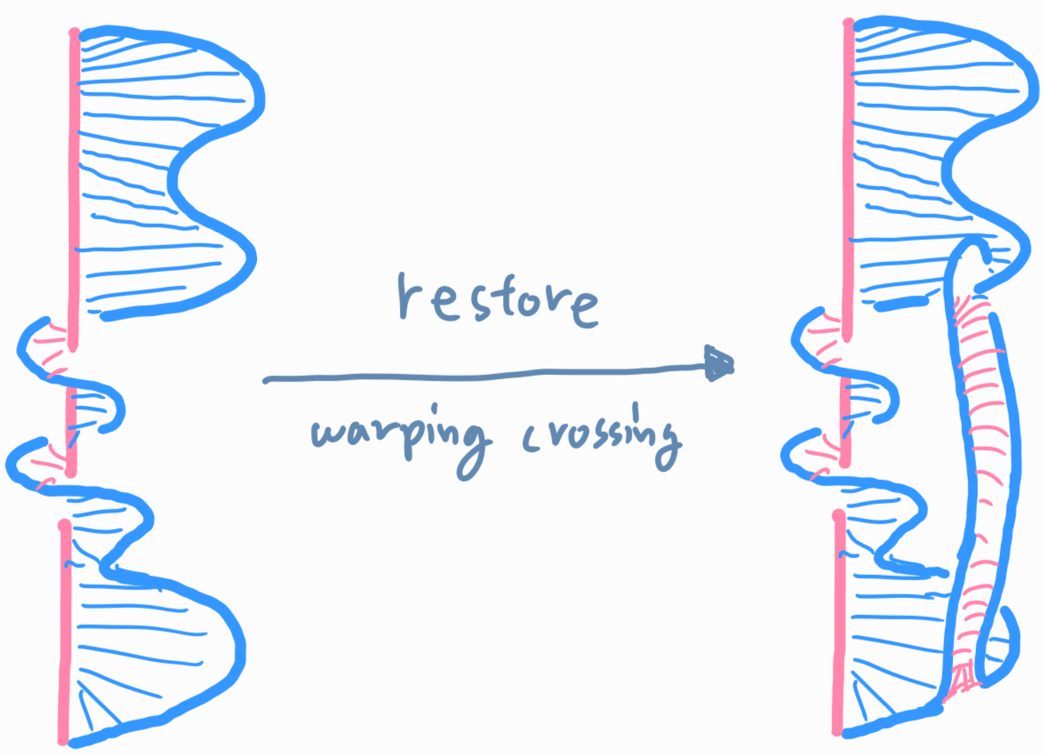
$$U_D = \{(f(t), t) \mid t \in [0,1]\} \cup \{b\} \times [0,1]$$

is the unknot in $\mathbb{R}^3$
corresponding to the descending diagram $d(D)$.

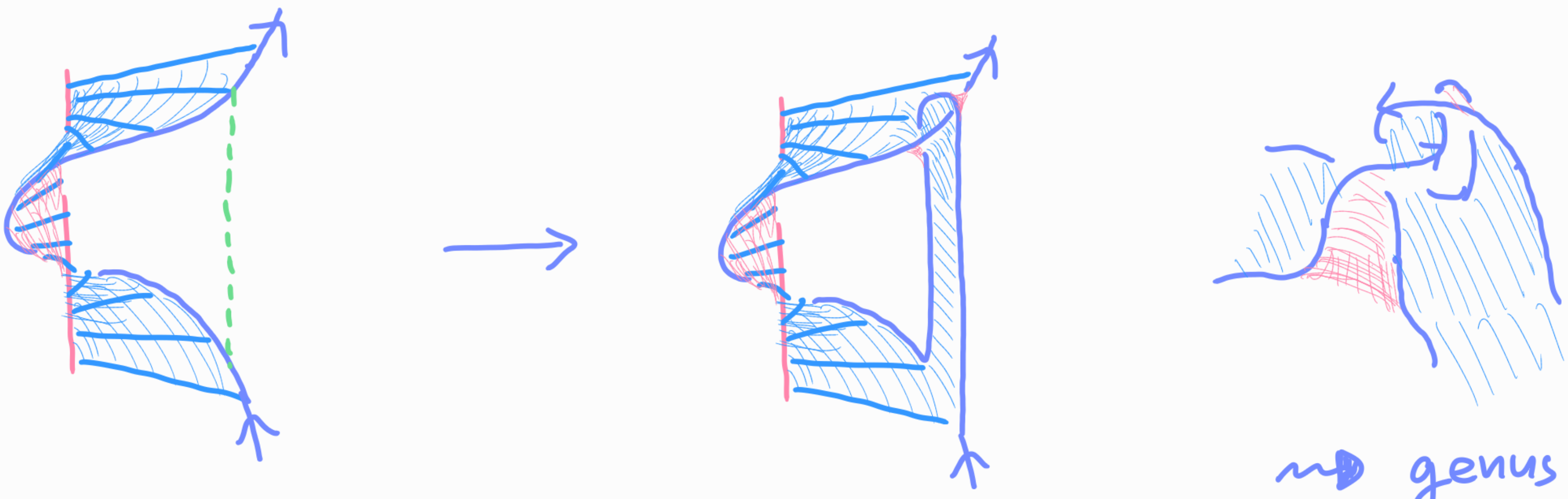
$$E_D = \{ (s \cdot b + (1-s) f(t), t) \mid s, t \in [0,1] \}$$

is a disk with $\partial E_D = U_D$.

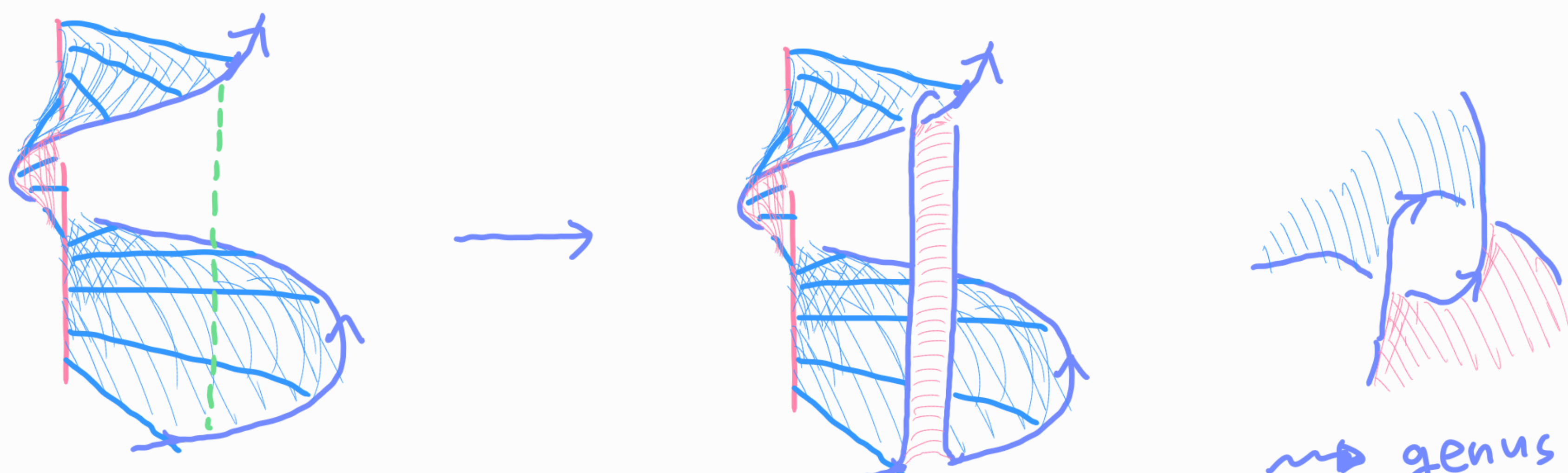
<u>Goal</u>: Make a Seifert surface of K from E_D



Let w_i ($i = 1, \cdots, a(D)$) : warping crossing of D

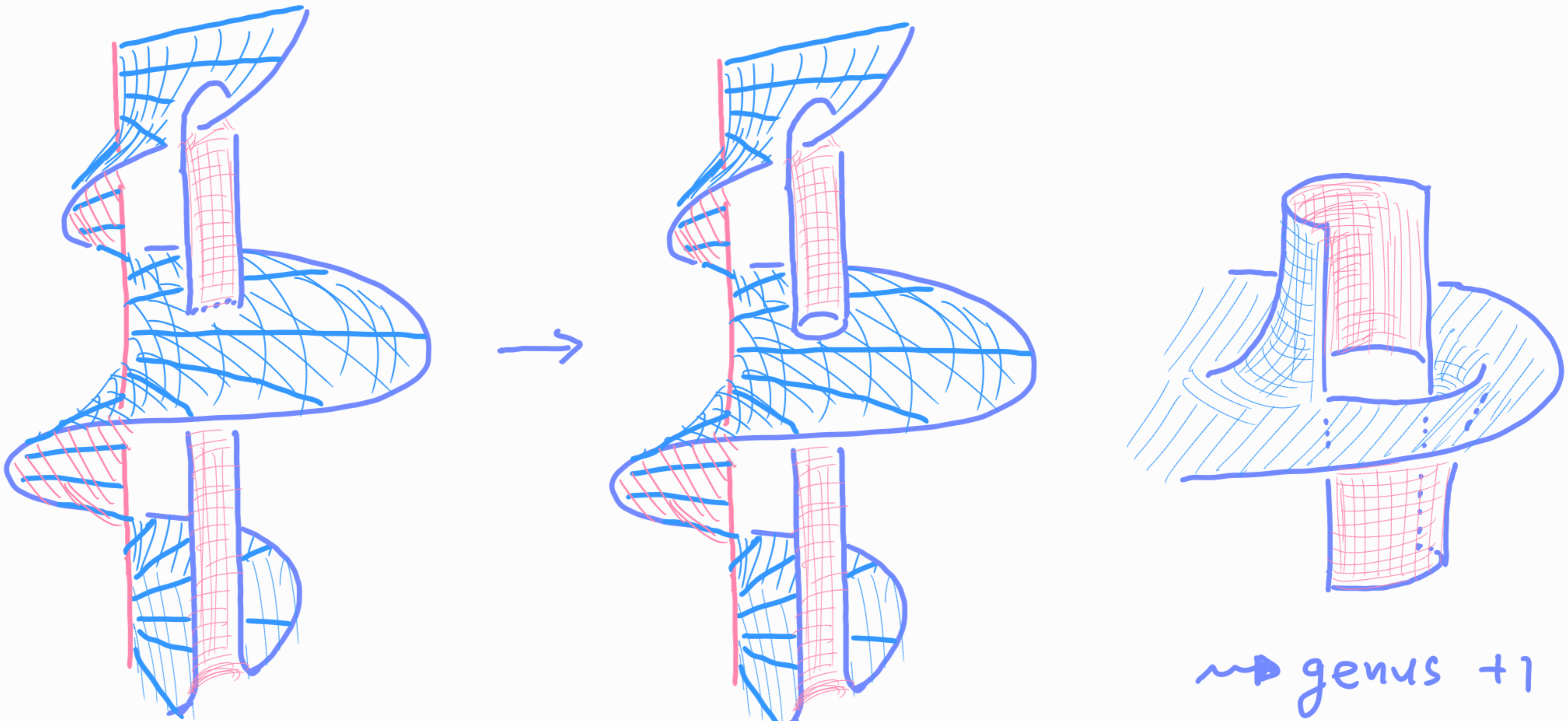


$\rightsquigarrow$ genus $+1$



$\rightsquigarrow$ genus $+1$

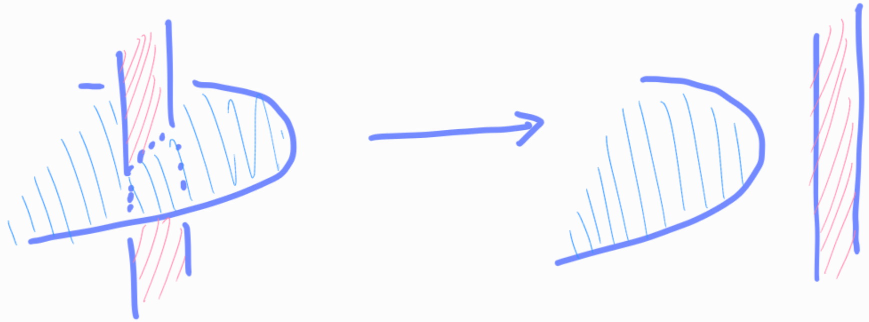
If $\exists w_i$ s.t. $\varphi_b(w_i) \geq 1$, then



$$\therefore \; g(K) \leq a(D) + \sum_{i=1}^{a(D)} \varphi_b(w_i)$$

Corollary $g_s(K) \leq a(K)$

<proof>

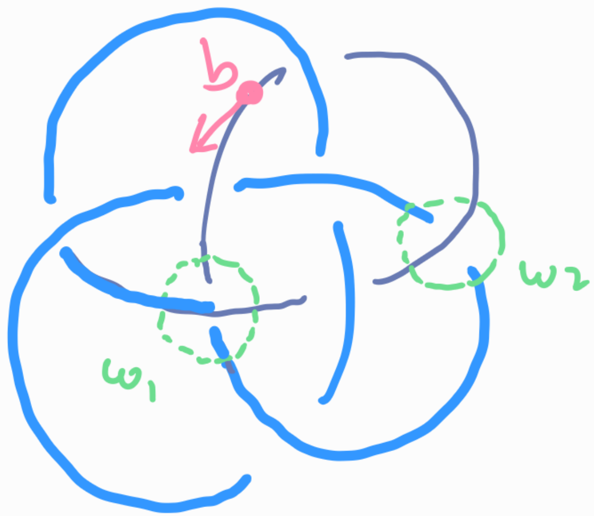


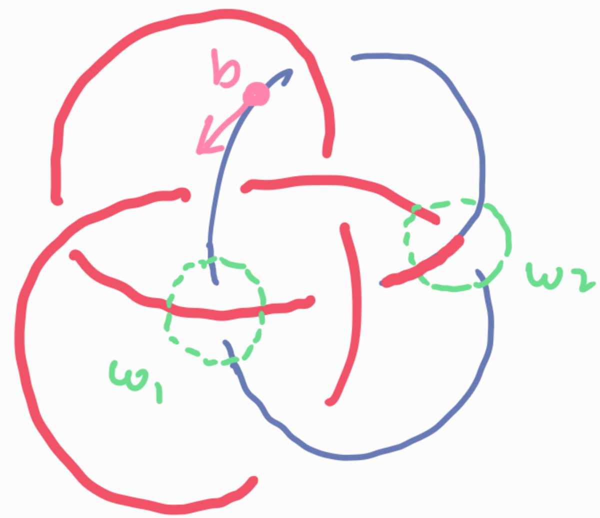
in $\mathbb{R}^4$

$\blacksquare$

$*$ Note: $g_s(K) \leq u(K) \leq a(K)$

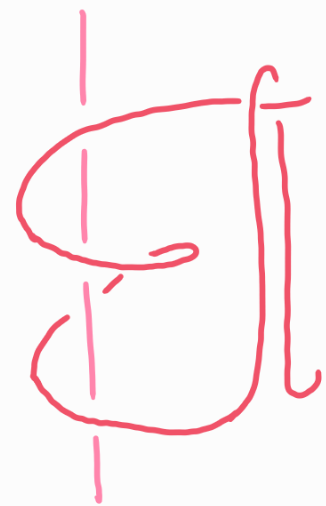
Example





$\varphi_b(w_1) = 1$

$\varphi_b(w_2) = 0$



$$\rightsquigarrow g(8_{18}) = 3 \leq a(D) + \sum_{i=1}^{2} \varphi_b(w_i) = 3$$

D : loose diagram $\Rightarrow g(K) \leq a(D)$

Question $\exists$ loose diagram with $a(D) = g(K)$

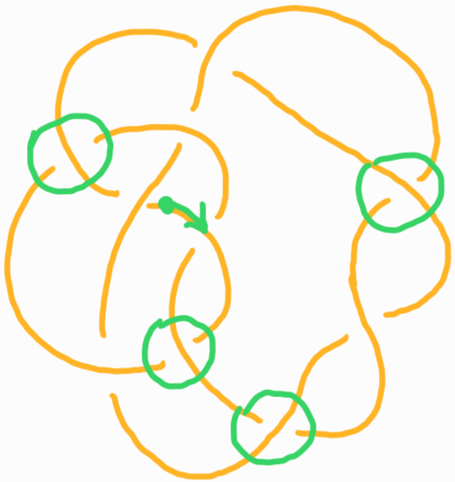
$\Rightarrow g(K) = a(K)$?

↳ $g(K) = 1$: Yes

$g(K) = 2$: Yes

$\rightsquigarrow a(K) = 1 \Leftrightarrow K$: twist (Ozawa)

$g(K) \geq 4$: No $\rightsquigarrow$ Counter example : 10_{85}



$g(10_{85}) = 4$

$a(10_{85}) = 3$

(Higa)

Conjecture K : genus 3 knot

If $\exists$ loose diagram D of K s.t. $a(D) = 3$,
then $a(K) = 3$.

↳ seems to be true ...

<idea>

Suppose that $a(K) = 2$.

⇝ $\exists$ based oriented diagram D with base point b
s.t. $a(D) = 2$.

$\Rightarrow$ either $\varphi_b(w_1) = 1$ or $\varphi_b(w_2) = 1$

Want : $\nexists$ loose diagram D' s.t. $a(D') = 3$.

Case 1 : $D_{w_1} \not\ni w_2$ or $D_{w_2} \not\ni w_2$

$\rightsquigarrow K = K_1 \# K_2$ where K_1, K_2 : twist knots

$\Rightarrow g(K) = 2$ $(\rightarrow\leftarrow)$

Case 2 : $D_{w_1} \ni w_2$ and $D_{w_2} \ni w_1$

$D \xrightarrow{RI} D' \Rightarrow |a(D') - a(D)| \leq 1$

$D \xrightarrow[RIII+b]{RII, RIII} D' \Rightarrow a(D') = a(D) \pm 1$

$\Big\} \Rightarrow$?

+

Shimizu's work

Proposition

For any nonnegative integer n,

$\exists$ a knot K s.t. $g(K) - a(K) = n$.

<proof>

① By Ozawa, torus knots & twist knots satisfy that $g(K) = a(K)$.

② $\exists K$ s.t. $g(K) - a(K) = 1$ (e.g. $8_{18}, 10_{85}, \cdots$)

$$\Rightarrow a(K_1 \# K_2) \leq a(K_1) + a(K_2) \leq g(K_1) - 1 + g(K_2) - 1$$

$$= g(K_1 \# K_2) - 2.$$

$\blacksquare$

How about $g(K) < a(K)$?

↳ e.g. $7_4, 8_3, 9_{41}, 10_{11}, 10_{16}, 10_{28}, 10_{30}, 10_{68}, 10_{97}, 10_{131}$

First step : $g(K)=1$

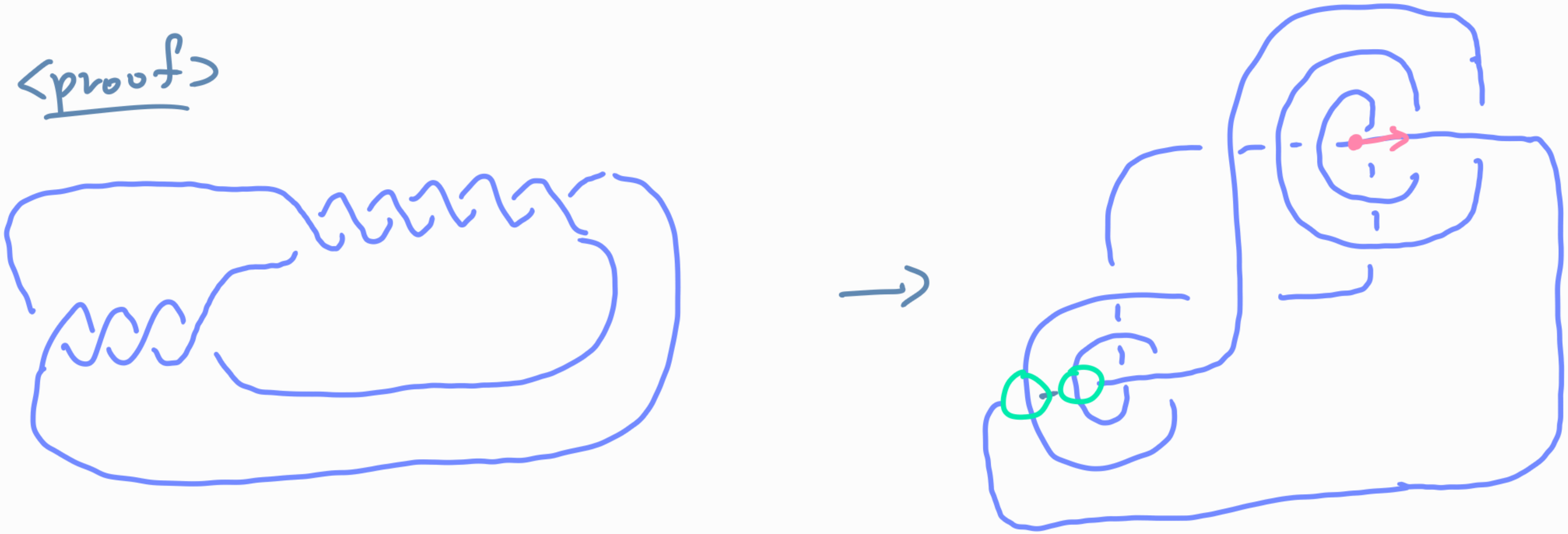
⇝ two bridge knot : $C(2n, 2m)$

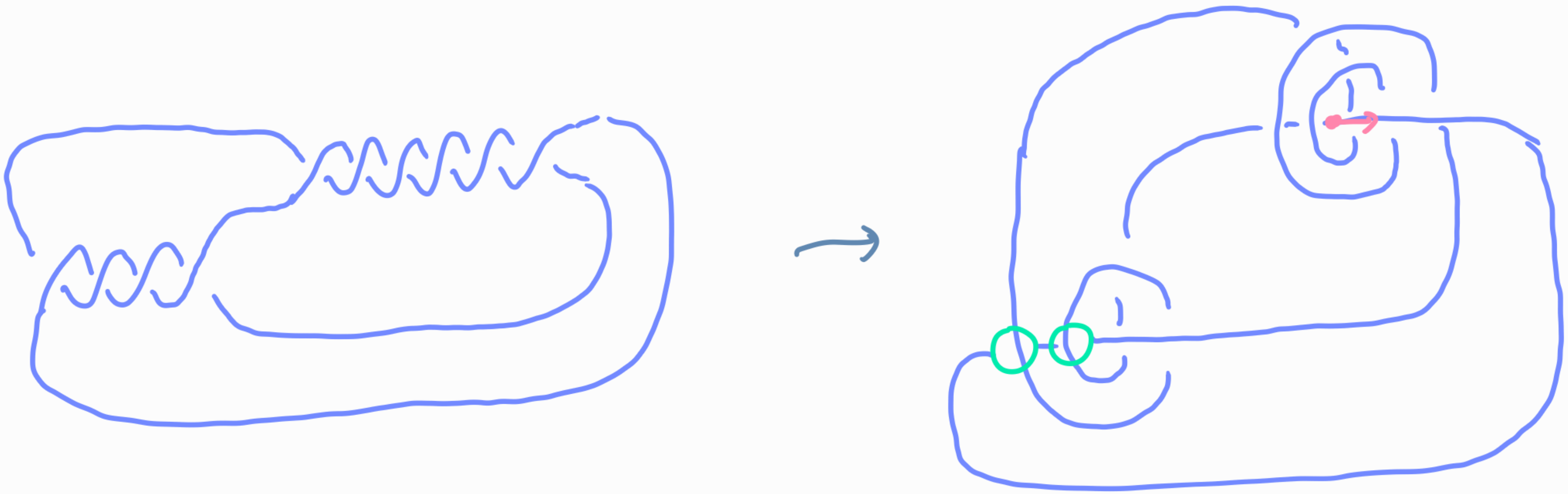
three bridge knot : $P(2n+1, 2m+1, 2l+1)$

⋮ (Fukuhama - Ozawa - Teragaito)

Theorem $a(C(2n,2m)) = \min\{|n|,|m|\}$.

<proof>





Corollary For any $|m| \geq 2$, $a(C(4, 2m)) = 2$

<proof>

$C(4, 2m)$: Not twist knot.

$\blacksquare$

Question $a(P(2n+1, 2m+1, 2l+1))$?

Thank you

for

your attention!