Introduction to the intrinsic properties of spatial graphs

Knots and Spatial Graphs 2021 - Dongguk University

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Kookmin University

Spatial Graph Theory

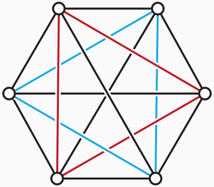
Spatial graph theory is the study of graphs embedded in $\mathbb{R}^3$.

- A *graph* $G = (V, E)$ is a set of vertices and edges.
- A *spatial graph* is an embedding of a graph in $\mathbb{R}^3$.
- A *knot* is an embedding of a circle in $\mathbb{R}^3$.
- A *link* is a collection of knots which do not intersect.

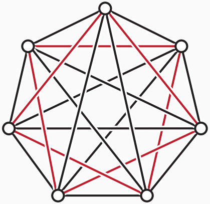
Conway and Gordon's result

Conway-Gordon [1983]

Every embedding of K_6 contains a non-split link.



Every embedding of K_7 contains a non-trivial knot.



Conway and Gordon's result

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For every embedding of K_6,

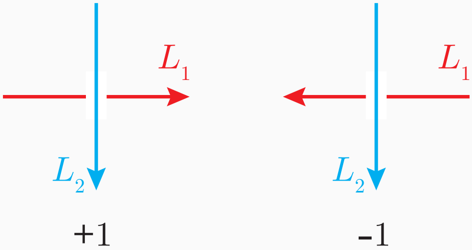
$$\sum_{(3,3)-cycles} lk(L) \equiv 1 \pmod 2.$$

For every embedding of K_7,

$$\sum_{7-cycles} \alpha(K) \equiv 1 \pmod 2.$$

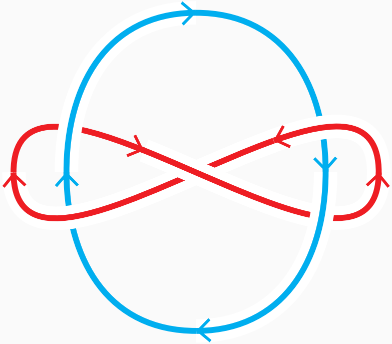
Linking number

Each point at which L_1 crosses under L_2 counts as following;



The sum of these numbers over all crossings of L_1 under L_2 is called $lk(L_1, L_2)$.

Linking number



$$lk(L_1, L_2) = -1 + 1 = 0.$$

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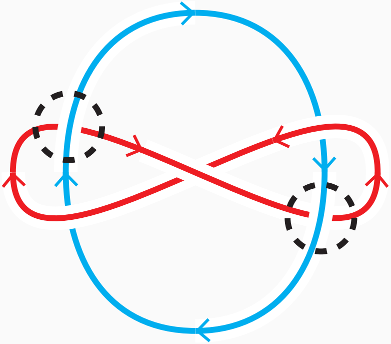

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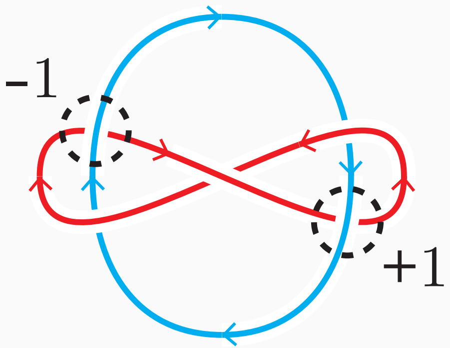

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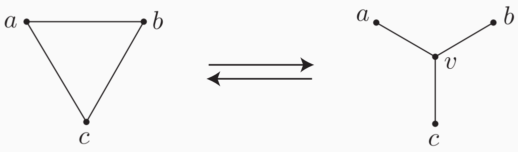
Intrinsically linked graphs

- A graph G is *intrinsically linked* (IL) if every embedding of G contains a non-splittable link.
- A graph G is intrinsically linked and has no proper minor which is intrinsically linked, G is said to be *minor minimal intrinsically linked*.

∇-Y move

- *∇-Y move* and *Y-∇ move*

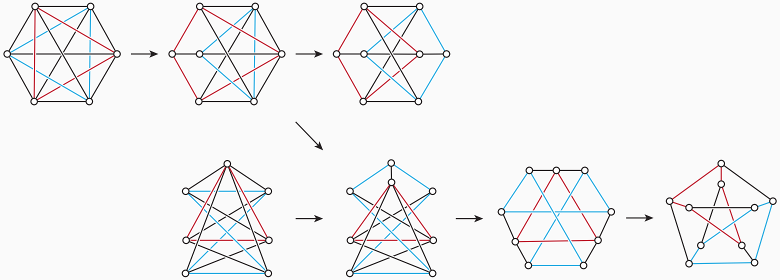


If G' is obtained from G by some ∇-Y or Y-∇ moves then G and G' are *cousin*. The set of all cousins of G is called the G *family*.

Minor minimal intrinsically linked graphs

Robertson-Seymour-Thomas [1995]

There are exactly seven minor minimal intrinsically linked graphs.



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For every embedding of K_7,

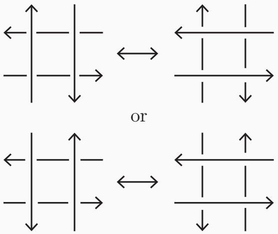
$$\sum_{7-cycles} \alpha(K) \equiv 1 \pmod 2.$$

Arf invariant and pass equivalent

Arf invariant

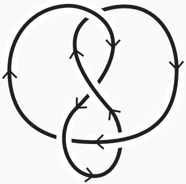
$$\alpha(K) = \begin{cases} 0 & \textit{K is pass equivalent to the unknot} \\ 1 & \textit{K is pass equivalent to the trefoil knot} \end{cases}$$

Two knots are *pass equivalent* if they are related by a finite sequence of pass-moves, which are illustrated below.



pass-move

Example of pass-move



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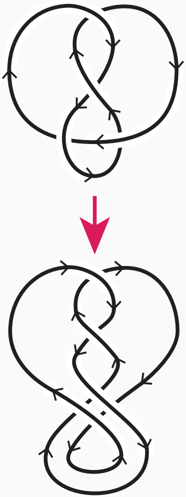

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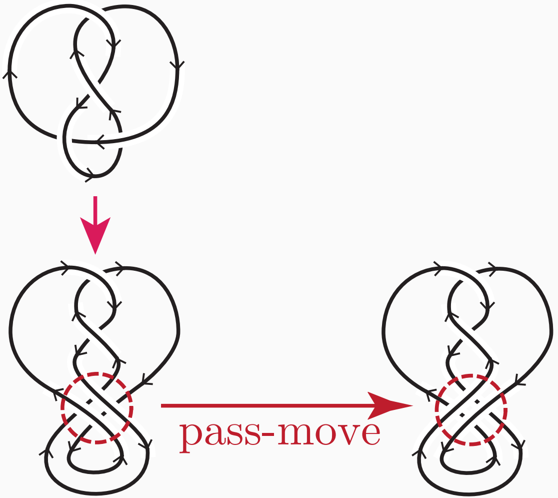

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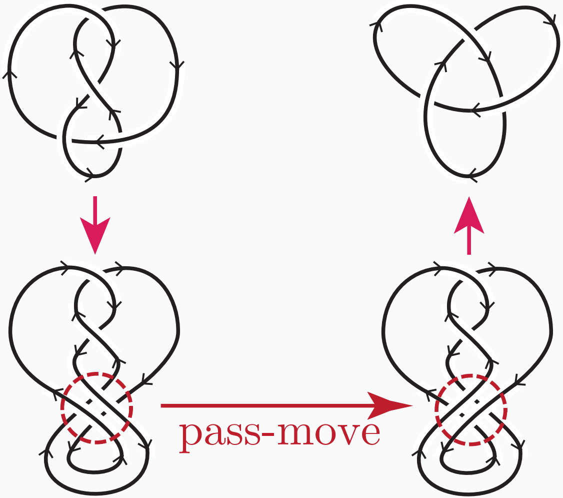

Intrinsically knotted graphs

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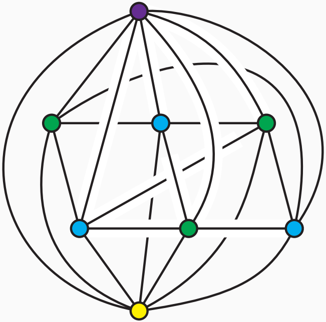
- A graph G is intrinsically knotted and has no proper minor which is intrinsically knotted, G is said to be *minor minimal intrinsically knotted*.

$K_{3,3,1,1}$



Motwani-Raghunathan-Saran [1988]

They claimed that they could prove that $K_{3,3,1,1}$ is intrinsically knotted using Conway-Gordon techniques directly

Kohara-Suzuki [1992]

Motwani, et al. are incorrect by showing a particular embedding of $K_{3,3,1,1}$ that contains exactly two trefoil knots as Hamiltonian cycles.

(conjectured that $K_{3,3,1,1}$ is IK.)

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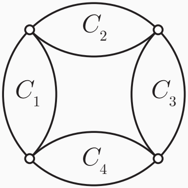
Key Lemma

Foisy's Key Lemma

Given an embedding of the graph D_4,

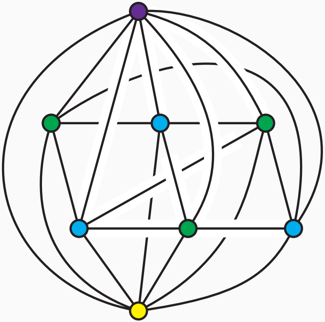
$\sigma \neq 0 \iff lk(C_1, C_3) \neq 0$ and $lk(C_2, C_4) \neq 0$.

σ : mod 2 sum of the Arf invariants of the all Hamiltonian cycles in that embedding.



Foisy showed that every embedding of $K_{3,3,1,1}$ has D_4 with $lk(C_1, C_3) \neq 0$ and $lk(C_2, C_4) \neq 0$ as minor.

$K_{3,3,1,1}$



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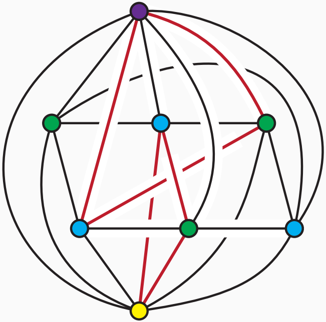

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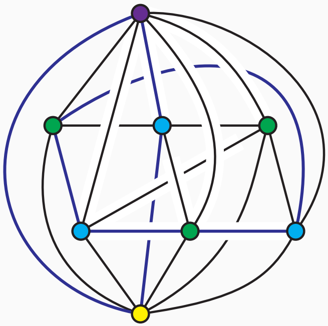

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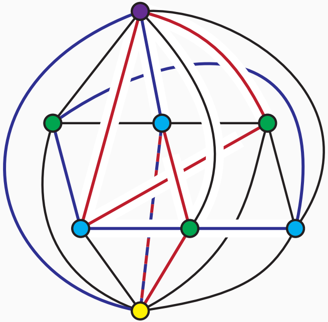

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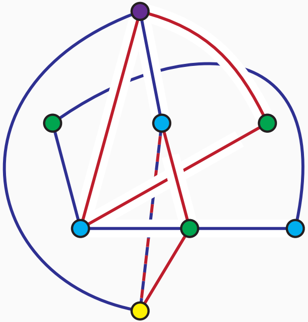

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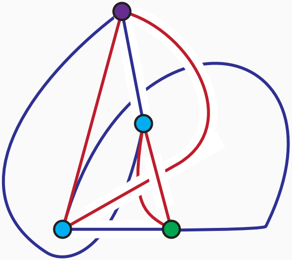

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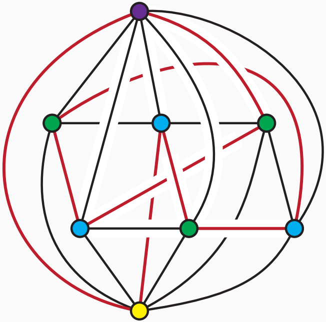

Minor minimal intrinsically knotted graphs

Robertson-Seymour [2003]

There are only finite number of minor minimal intrinsically knotted graphs.

Open Problem

Finding the complete set of minor minimal intrinsically knotted graphs.

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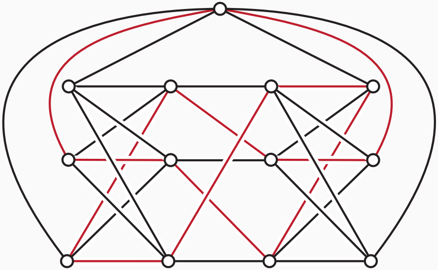
Minor minimal intrinsically knotted graphs

Foisy [2003]

H is a minor minimal intrinsically knotted graph.



∇-Y move

Motwani-Raghunathan-Saran [1988]

∇-Y move preserves intrinsic knottedness.

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Some Y-∇ moves do not preserve intrinsic knottedness.

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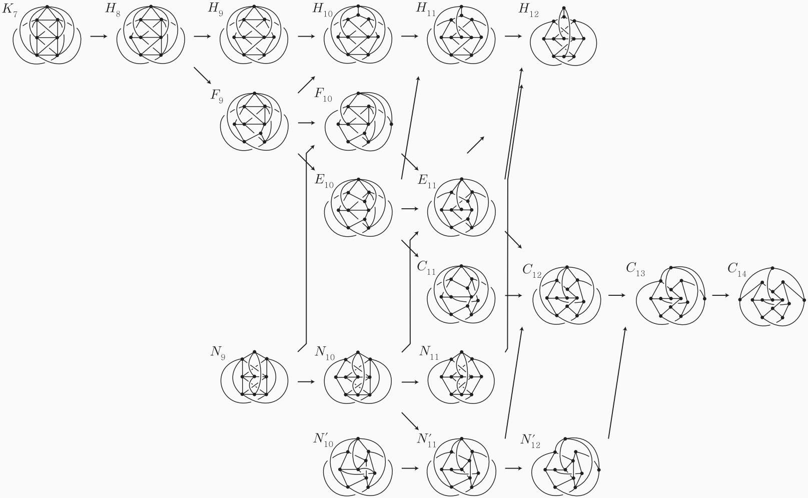
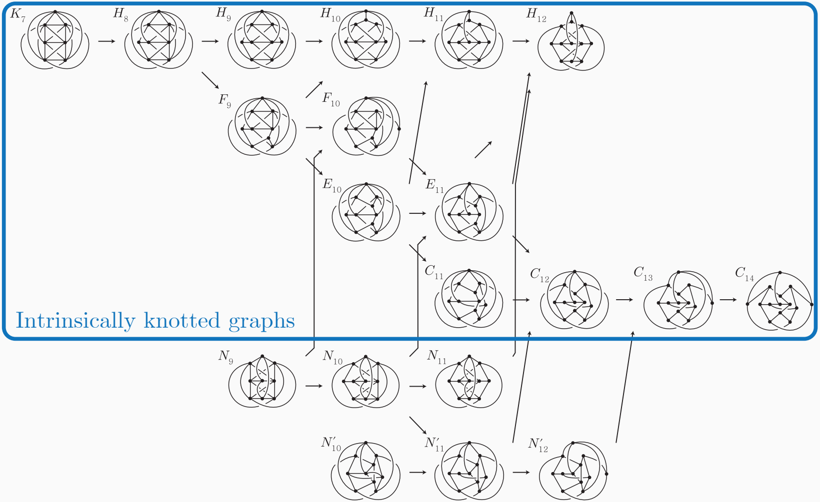
K_7 family



K_7
H_8
H_9
H_{10}
H_{11}
H_{12}
F_9
F_{10}
E_{10}
E_{11}
C_{11}
C_{12}
C_{13}
C_{14}
N_9
N_{10}
N_{11}
N'_{10}
N'_{11}
N'_{12}

K_7 family



K_7
H_8
H_9
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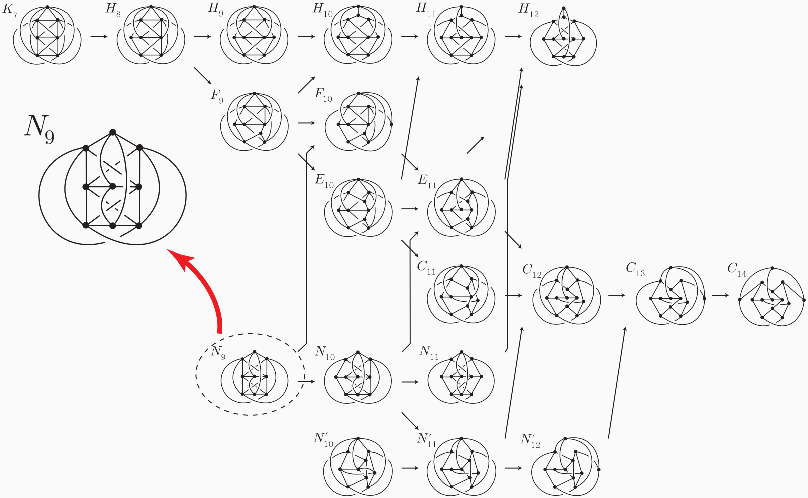
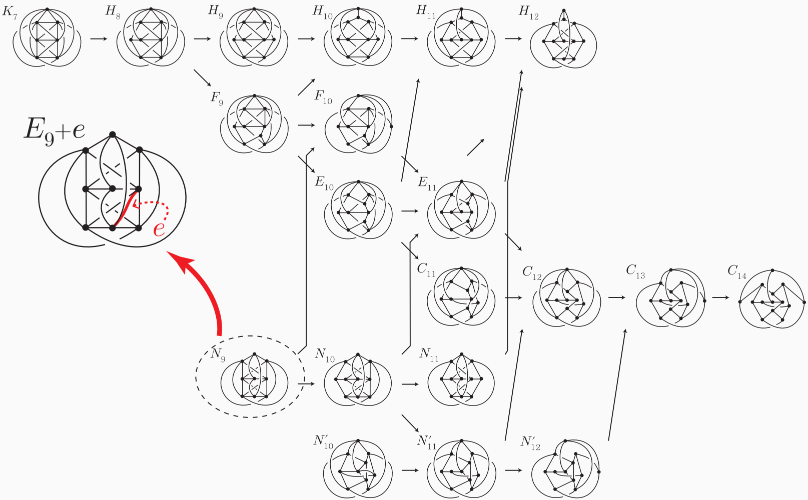
H_{11}
H_{12}
F_9
F_{10}
E_{10}
E_{11}
C_{11}
C_{12}
C_{13}
C_{14}
Intrinsically knotted graphs
N_9
N_{10}
N_{11}
N'_{10}
N'_{11}
N'_{12}

K_7 family



K7
H8
H9
H10
H11
H12
F9
F10
E10
E11
C11
C12
C13
C14
N9
N10
N11
N'10
N'11
N'12
N9

K_7 family



K_7
H_8
H_9
H_{10}
H_{11}
H_{12}
F_9
F_{10}
E_9+e
e
E_{10}
E_{11}
C_{11}
C_{12}
C_{13}
C_{14}
N_9
N_{10}
N_{11}
N'_{10}
N'_{11}
N'_{12}

Minor minimal intrinsically knotted graphs

Goldberg-Mattman-Naimi [2014]

Family	Total graphs	IK graphs	MMIK graphs
K_7	20	14	14
$K_{3,3,1,1}$	58	58	58
$E_9 + e$	110	110	33
$G_{9,28}$	1609	1609	≥ 156
$G_{14,25}$	$> 600{,}000$	unknown	≥ 1

There are at least **263** MMIK graphs(include H).

Intrinsically knotted graphs with at most 21 edges

A graph is *n-apex* if it can obtain a planar graph by removing n vertices.

Blain-Bowlin-Fleming-Foisy-Hendricks-Lacombe [2007], Ozawa-Tsutsumi [2007]

If G is a 2-apex, then G is not intrinsically knotted.

Johnson-Kidwell-Michael [2010]

Any intrinsically knotted graph consists at least 21 edges.

K-Lee-Lee-Oh [2015], Barsotti-Mattman [2016]

Exactly 14 intrinsically knotted graphs have 21 edges.

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Intrinsically knotted graphs with 22 edges

K-Lee-Lee-Mattman-Oh [2017], K-Mattman-Oh [2018]

There are exactly eight triangle-free intrinsically knotted graphs with 22 edges which has a vertex with degree 5 or more.

Conjecture for TFIK

There are exactly 19 triangle-free intrinsically knotted graphs with 22 edges.

Conjecture for MMIK

There are exactly 92 minor minimal intrinsically knotted graphs with 22 edges.

Bipartite intrinsically knotted graphs

K-Mattman-Oh [2017]

There are exactly two minor minimal bipartite intrinsically knotted graphs with at most 22 edges

K-Mattman-Oh [2021+]

There are exactly six bipartite intrinsically knotted graphs with 23 edges which satisfies $\delta(G) \geq 3$.

Chirality

Chirality is a geometric property of some molecules and ions.

A chiral molecule/ion is non-superposable on its mirror image.

Gal [2012]

The artificial sweetener aspartame has two enantiomers.
L-aspartame tastes sweet whereas D-aspartame is tasteless.

Jaffe-Altman-Merryman [1964]

D-penicillamine is used in chelation therapy and for the treatment of rheumatoid arthritis whereas L-penicillamine is toxic as it inhibits the action of pyridoxine, an essential B vitamin.

A Molecular Structure and a Spatial Graph

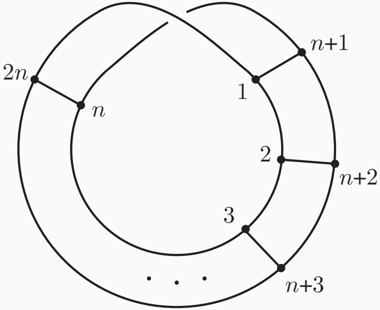
In spatial graph theory, molecular structures are interpreted as a graph embedded in S^3

- A molecule is *chemically achiral* if it can continuously change to its mirror image, otherwise it is *chemically chiral*.
- An embedding of a graph G in S^3 is *topologically achiral* if it is ambient isotopic to its mirror image, otherwise it is *topologically chiral*.
- A graph G is *intrinsically chiral* if every embedding of G in S^3 is topologically chiral, otherwise it is *achirally embeddable*.

Möbius Ladder M_n

A *Möbius ladder*, denoted by M_n, is the graph consisting of a $2n$-cycle K and n edges $\alpha_1, \ldots, \alpha_n$.

K is the *loop* of M_n and $\alpha_1, \ldots, \alpha_n$ are the *rungs* of M_n.



An standard embedding of M_n

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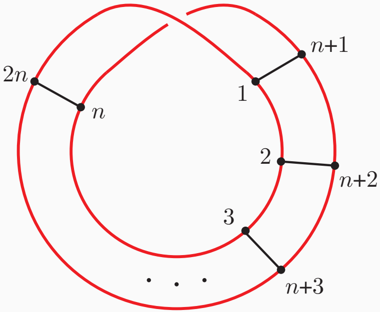

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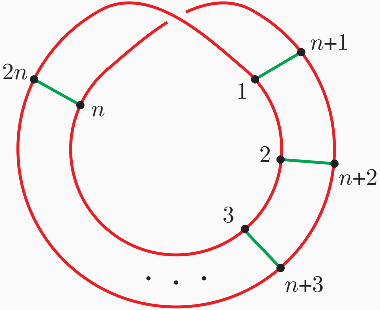


An standard embedding of M_n

Möbius Ladder

Simon [1986]

Every standard embedding of M_n, for $n \geq 4$,

there is no orientation reversing diffeomorphism h of S^3 with $h(M_n) = M_n$ and $h(K) = K$.

Flapan [1989]

Every embedding of M_n, for $n \geq 3$ be odd,

there is no orientation reversing homeomorphism h of S^3 with $h(M_n) = M_n$ and $h(K) = K$.

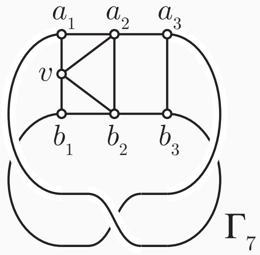
Intrinsically Chiral Graphs

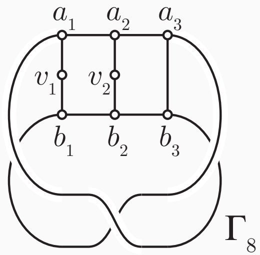
Flapan-Weaver [2013]

The complete graphs K_{4n+3} with $n \geq 1$ are intrinsically chiral, and all other complete graphs are achirally embeddable.

K-Choi-No [2021]

Γ_7 and Γ_8 are minor minimal intrinsically chiral graphs.

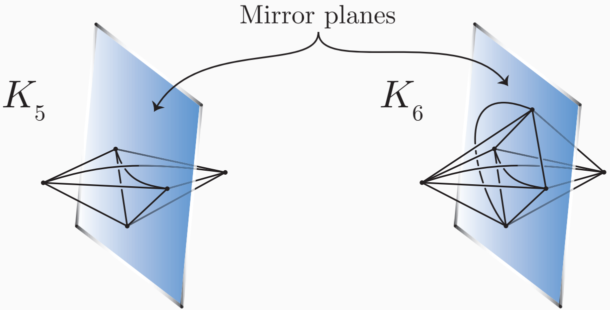




The Mirror Symmetry Embedding

An embedding of a graph G is *mirror symmetry* when it is symmetrical on the left and right with respect to a plane $\mathcal{M}$.

The mirror symmetry embedding is topologically achiral.



Flapan [1989]

Let M_n be a Möbius ladder which is embedded in S^3 with loop K, where n is an odd number. Then there is no diffeomorphism $h : S^3 \to S^3$ which is orientation reversing with $h(M_n) = M_n$ and $h(K) = K$.

Definition

A *graph automorphism* of G is a permutation ϕ on the set of vertices V that satisfies the property that $\{u_i, u_j\} \in E$ if and only if $\{\phi(u_i), \phi(u_j)\} \in E$.

Notations

Let G be a simple connected graph, v be a vertex of G, and e be an edge of G.

- $|G|$: the number of vertices of G
- $\|G\|$: the number of edges of G
- $\deg(v)$: degree of a vertex v in G
- $\delta(G)$: the minimum degree among all vertices of G
- G/e : a graph obtained from G by contracting e
- $G \setminus e$: a graph obtained from G by deleting e

Counting Lemma

Lemma

Let G be a connected graph. If G is not planar, then $\|G\| - |G| \geq 3$.

Proof.

If G is a non-planar graph, then G has $K_{3,3}$ or K_5 as minor.

Let e be an edge of G.

$\|G\| - |G| = \|G/e\| - |G/e| > \|G \setminus e\| - |G \setminus e|$

This means that if H is a minor of G, then $\|G\| - |G| \geq \|H\| - |H|$.

Since $\|K_{3,3}\| - |K_{3,3}| = 3$ and $\|K_5\| - |K_5| = 5$, $\|G\| - |G| \geq 3$. $\square$

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Counting Lemma

Lemma

Let G be a connected graph. If G is not planar, then $\|G\| - |G| \geq 3$.

Proof.

If G is a non-planar graph, then G has $K_{3,3}$ or K_5 as minor.

Let e be an edge of G.

$\|G\| - |G| = \|G/e\| - |G/e| > \|G \setminus e\| - |G \setminus e|$

This means that if H is a minor of G, then $\|G\| - |G| \geq \|H\| - |H|$.

Since $\|K_{3,3}\| - |K_{3,3}| = 3$ and $\|K_5\| - |K_5| = 5$, $\|G\| - |G| \geq 3$. □

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n-apex and intrinsic properties

For connected graphs,

- not -1-apex $\Rightarrow \|G\| - |G| \geq 1$
  $\|K_{2,3}\| - |K_{2,3}| = 6 - 5 = 1$
- not 0-apex $\Rightarrow \|G\| - |G| \geq 3$ Intrinsically Chiral
  $\|\mathit{\Gamma}_8\| - |\mathit{\Gamma}_8| = 11 - 8 = 3$
- not 1-apex $\Rightarrow \|G\| - |G| \geq 5$ Intrinsically Linked
  $\|PetersenGraph\| - |PetersenGraph| = 15 - 10 = 5$
- not 2-apex $\Rightarrow \|G\| - |G| \geq 7$ Intrinsically Knotted
  $\|HeawoodGraph\| - |HeawoodGraph| = 21 - 14 = 7$
- not 3-apex $\Rightarrow \|G\| - |G| \geq 9$ Intrinsically 3-linked?

Thanks for listening