Twisted torus knots $T(mn+m+1, mn+1, mn+m+2, -1)$ and $T(n+1, n, 2n-1, -1)$ are torus knots

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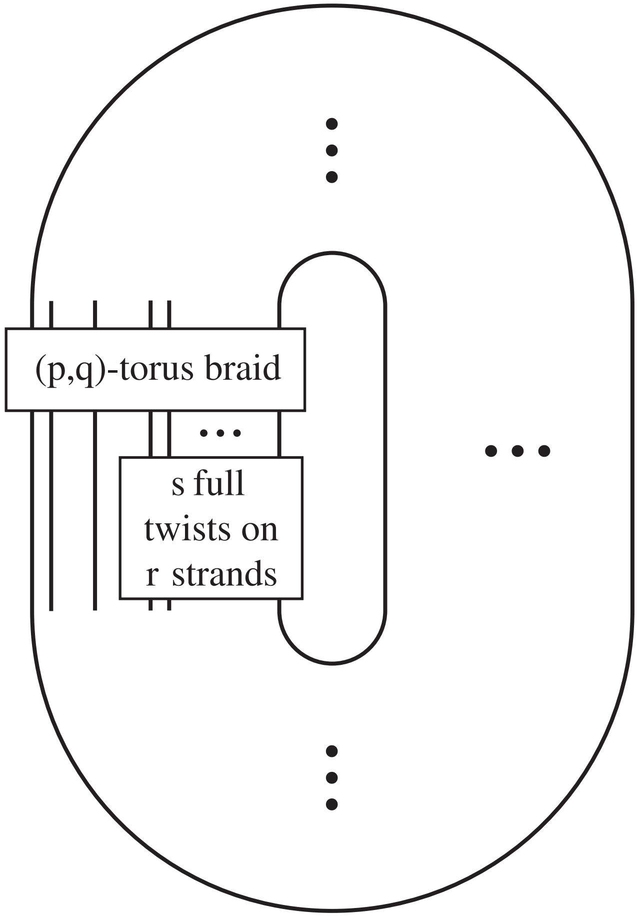
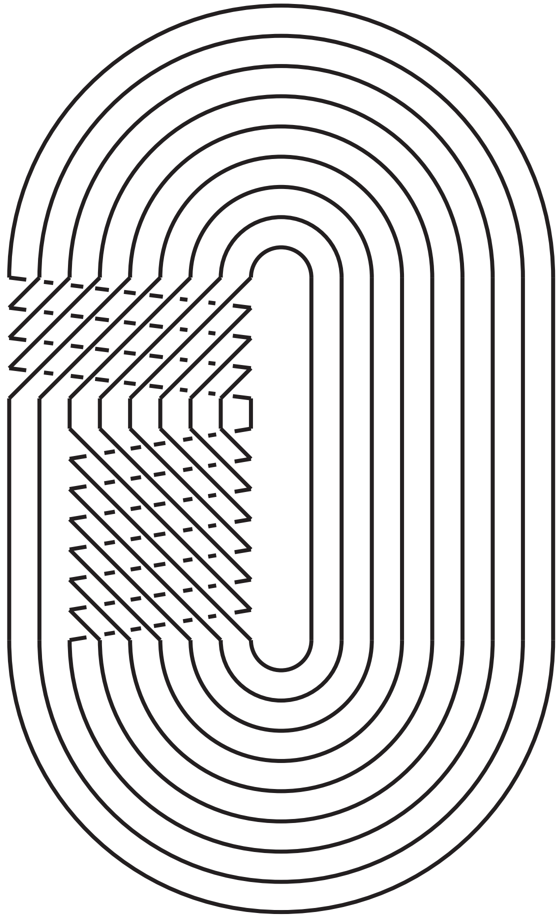
Twisted torus knots $T(p, q, r, s)$

p, q : coprime integers with $p > q \geq 1$
r : an integer with $2 \leq r < p$
s : a nonzero integer
$T(p, q)$: the torus knot of type (p, q)

$T(p, q, r, s)$ is a knot obtained from $T(p, q)$ by performing s full twists on adjacent r strands.

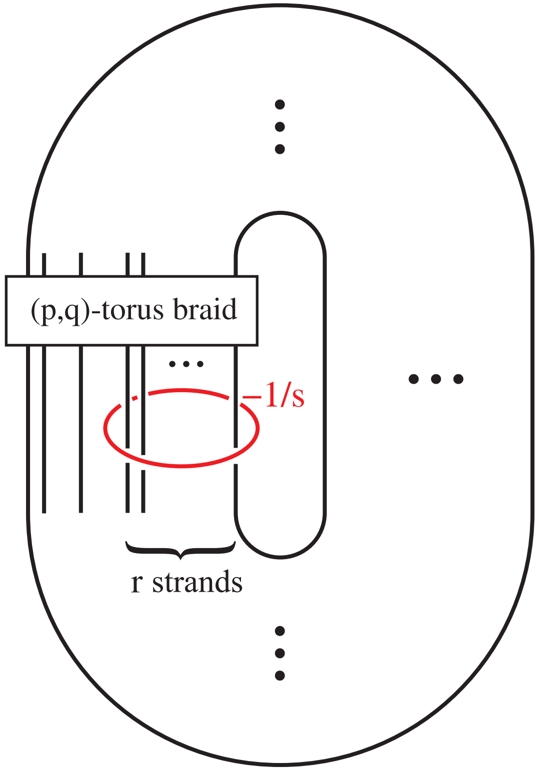


T(p,q,r,s)

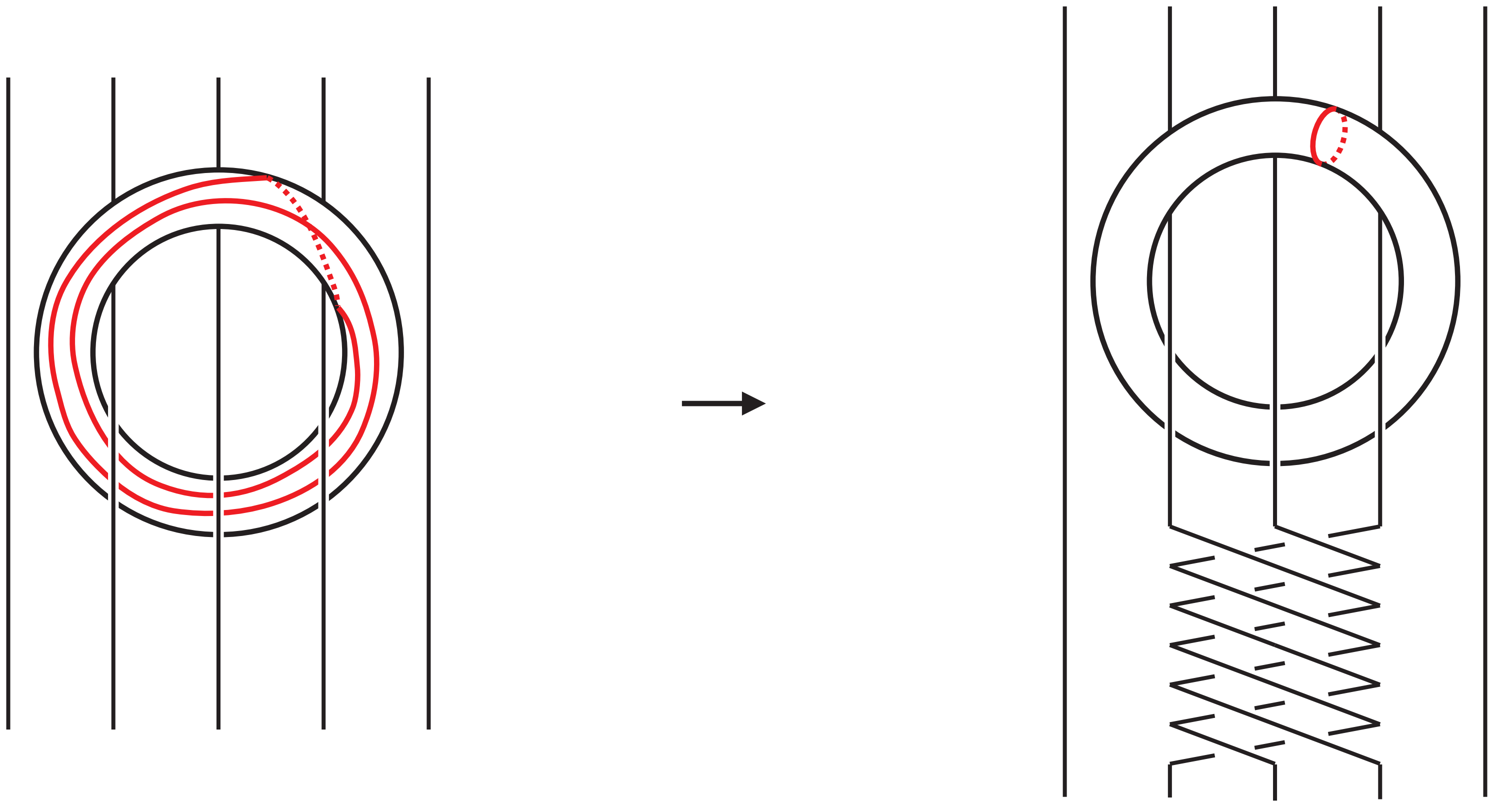


T(9,4,7,−1)

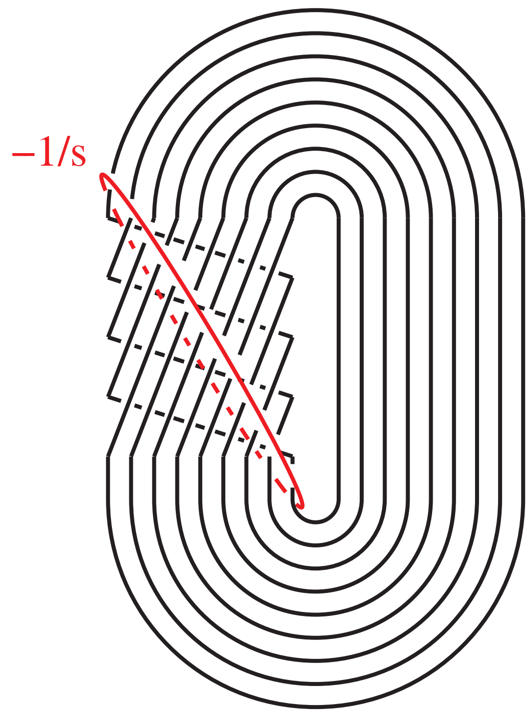
Twisted torus knots $T(p, q, r, s)$ and Dehn surgery



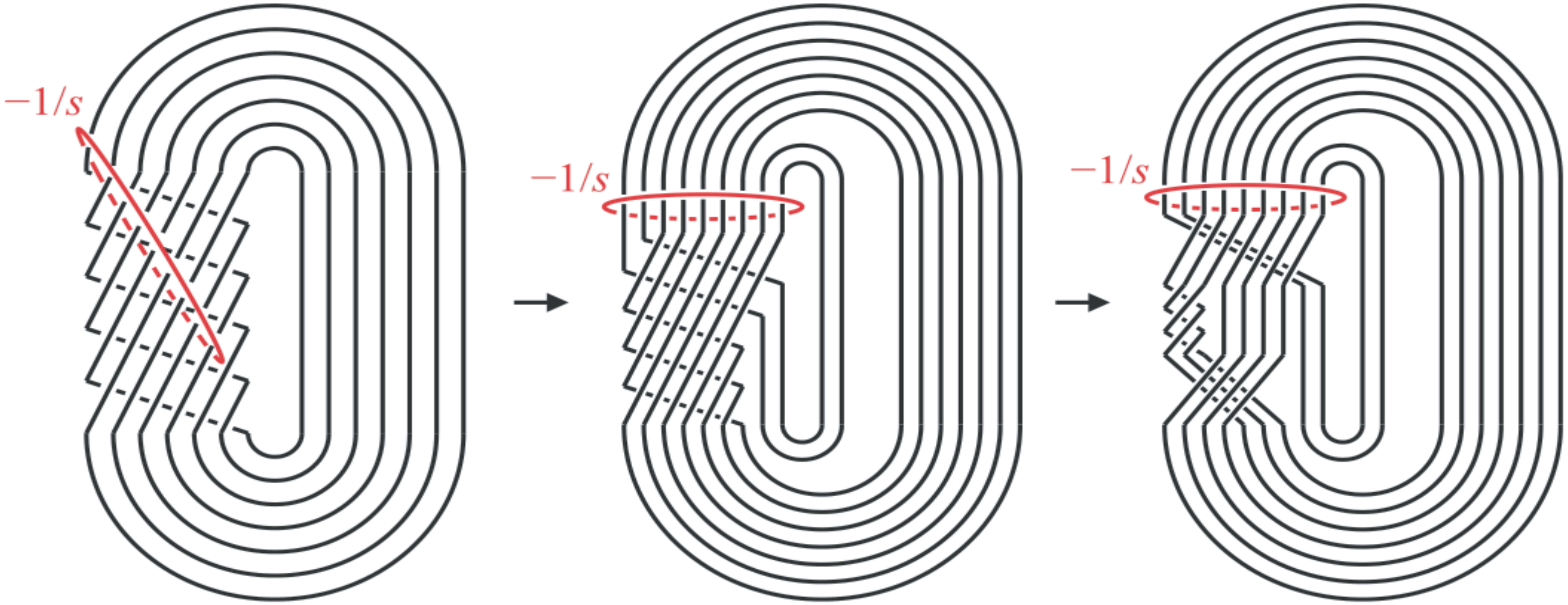
T(p,q,r,s)



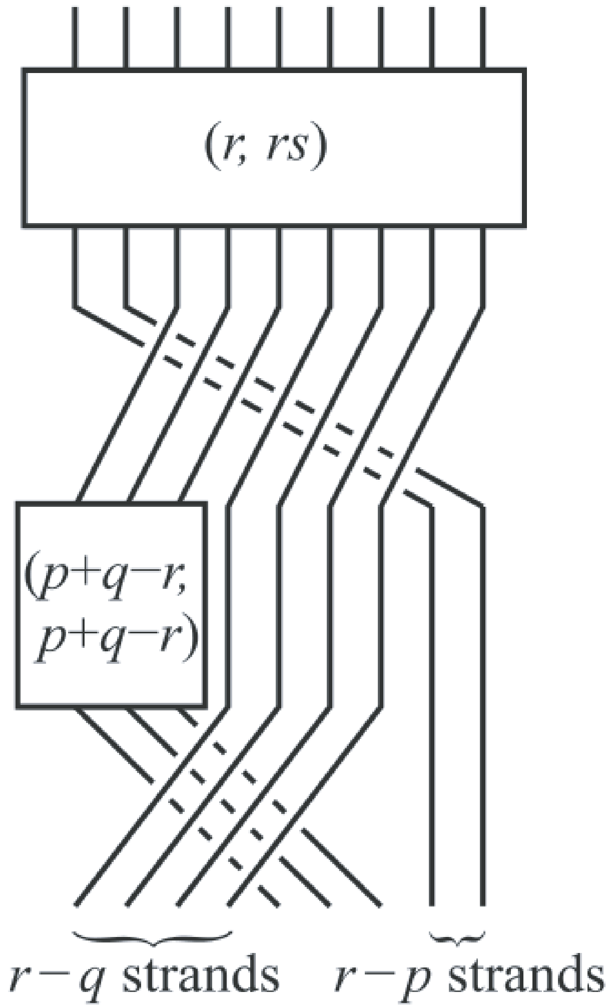
We may define $T(p, q, r, s)$ for $2 \leq r \leq p + q$.

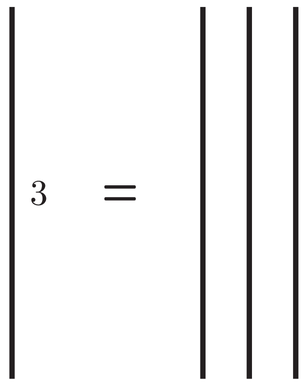


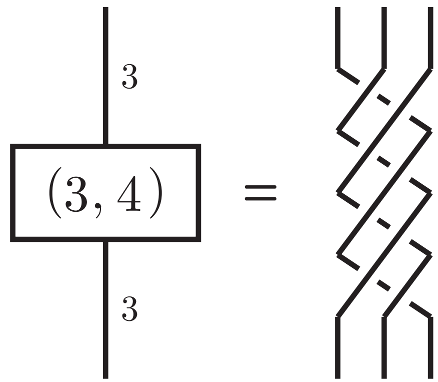
T(9,4,13, s)

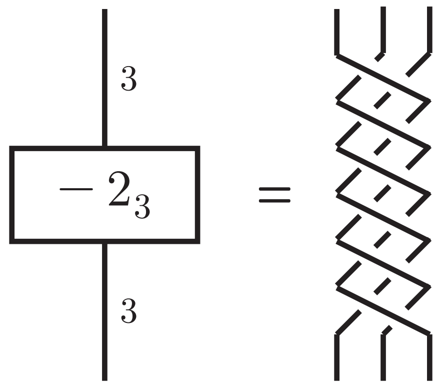


−1/s
−1/s
−1/s


(r, rs)
(p+q−r,
p+q−r)
r − q strands
r − p strands


3
=


3
(3, 4)
3
=


3
− 2₃
3
=

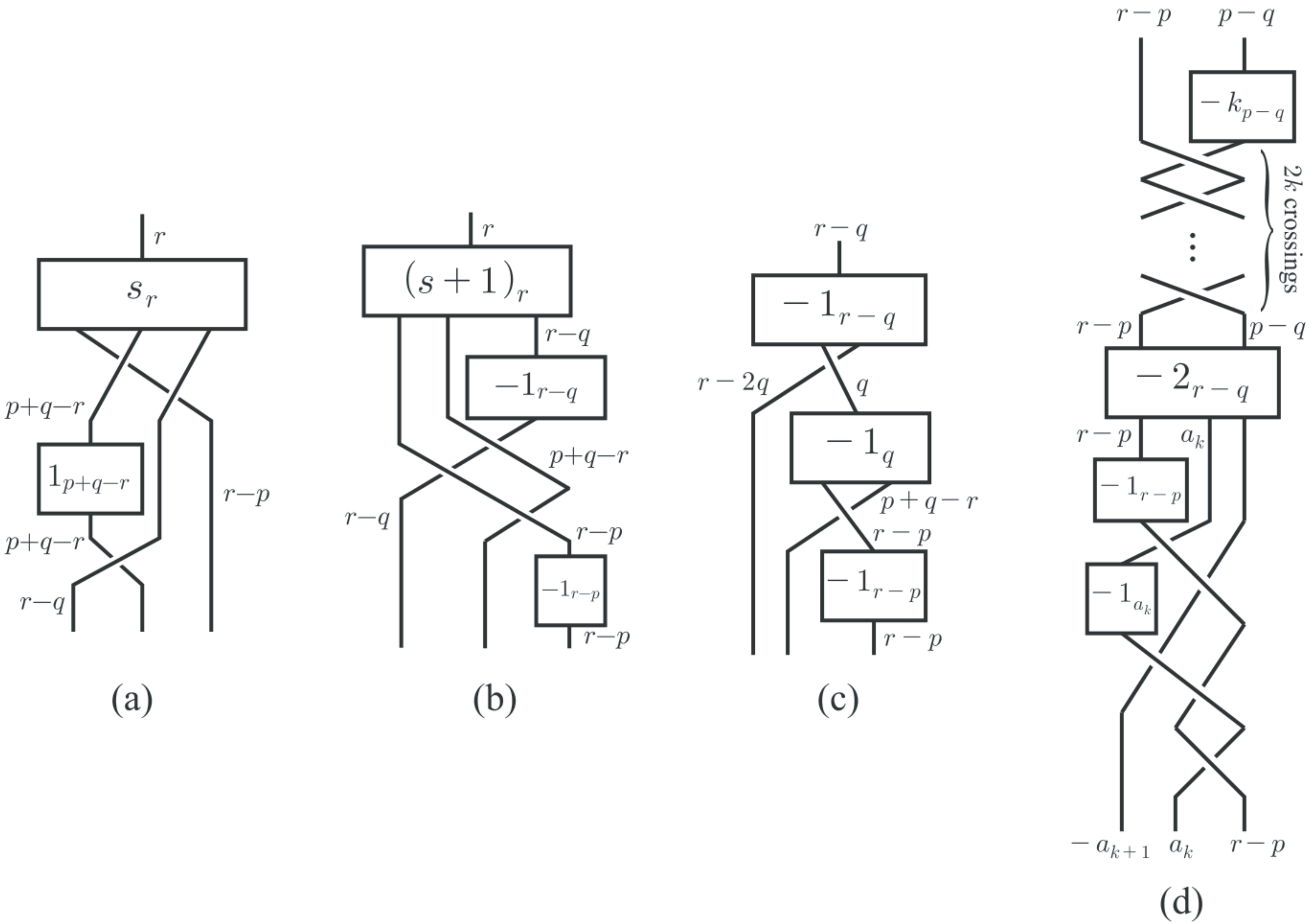
Lemma (Franks and Williams). *If a positive* b*-braid contains a full twist on* b *strands, then its closure has braid index* b.

Lemma (Murasugi). *Let* γ *be a homogeneous* n*-braid. Let* d_i *be the exponent sum of* σ_i *in* γ*. If* $b(\widehat{\gamma}) = n$ *or if* γ *is a reduced pure* n*-braid, then* $c(\widehat{\gamma}) = \sum_{i=1}^{n-1} |d_i|$.

Theorem. *Let p, q, r, s be integers such that p and q are coprime, r is not a multiple of q, $1 < q < p < r \leq p + q$, and $s \neq 0$. Then $T(p, q, r, s)$ is a fibered knot with braid index $b(p, q, r, s)$ given by*

$$b(p,q,r,s) = \begin{cases} r & \text{for } s \geq 1, \\ r - q & \text{for } s = -1, \\ r & \text{for } s \leq -2. \end{cases}$$

Moreover, a diagram of $T(p, q, r, s)$ with a minimal number of crossings is obtained by closing
(a) the braid in (a) if $s \geq 1$, or
(b) the braid in (b) if $s \leq -2$, or
(c) the braid in (c) if $s = -1$ and $r > 2q$, or
(d) the braid in (d) if $s = -1, r < 2q$ and $a_{k+1} \leq 0 < a_k$ for some $k \geq 0$, where $a_k = (k + 2)q - kp - r$.


r
s_r
p+q−r
1_{p+q−r}
p+q−r
r−q
r−p
(a)
r
(s+1)_r
r−q
−1_{r−q}
p+q−r
r−q
r−p
−1_{r−p}
r−p
(b)
r − q
− 1_{r − q}
r − 2q
q
− 1_q
p + q − r
r − p
− 1_{r − p}
r − p
(c)
r − p
p − q
− k_{p − q}
2k crossings
r − p
p − q
− 2_{r − q}
r − p
a_k
− 1_{r − p}
− 1_{a_k}
− a_{k + 1}
a_k
r − p
(d)

Lemma (Stallings). *Let* K *be a knot obtained by closing a positive braid with* b *strands and* c *crossings. Then* K *is a fibered knot with genus* g *given by*

$$g = \frac{1 - b + c}{2}.$$

Theorem. *Let p, q, r be integers such that p and q are coprime, r is not a multiple of q, and $1 < q < p < r \leq p + q$. Then $T(p, q, r, -1)$ is a torus knot if and only if either*

(a) $(p, q, r) = (mn + m + 1, mn + 1, mn + m + 2)$ *for some integers* $m \geq 1, n \geq 2$*; or*

(b) $(p, q, r) = (n + 1, n, 2n - 1)$ *for some integer* $n \geq 3$.

In the former, $T(p, q, r, -1) = T(mn + m + n + 2, -m - 1)$ *and in the latter,* $T(p, q, r, -1) = T(3n - 2, -n + 1)$.

Lemma. $r < 2q$ *if* $T(p, q, r, -1)$ *is a torus knot.*

We assume that $r < 2q$. Define a sequence $\{a_j\}_{j=0}^{\infty}$ as follows:

$$\begin{aligned} a_j &= (j+2)q - jp - r \\ &= (2q - r) - j(p - q). \end{aligned}$$

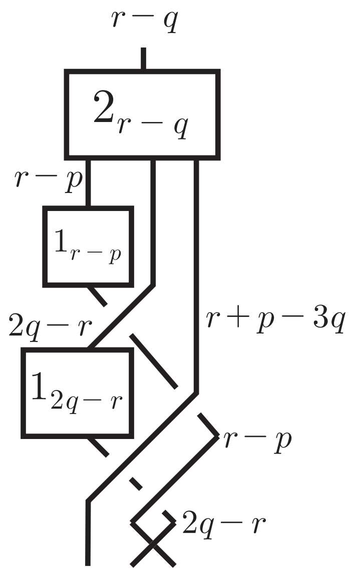
Then $\{a_j\}_{j=0}^{\infty}$ is a decreasing sequence and we can find a nonnegative integer k satisfying $a_{k+1} \leq 0 < a_k$.

Lemma. *Suppose* $k = 0$. *If* $T(p, q, r, -1)$ *is a torus knot, then*

$$(p, q, r) = (3m + 1, 2m + 1, 3m + 2) \text{ or}$$
$$(p, q, r) = (n + 1, n, 2n - 1)$$

for some integers $m \geq 1$ *and* $n \geq 3$.

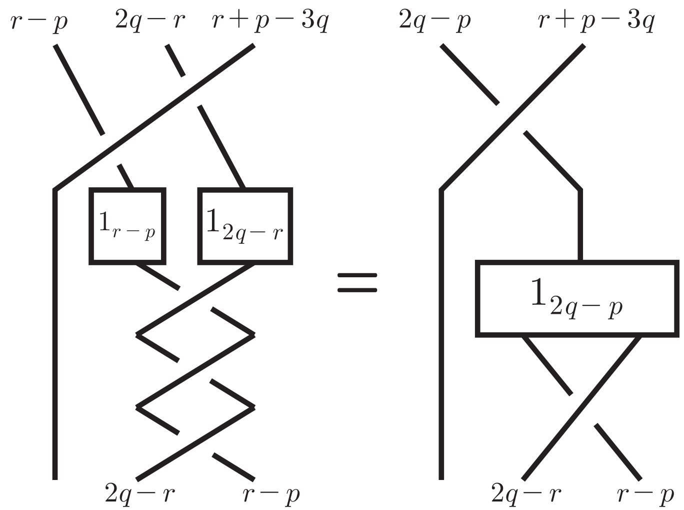
Proof. Suppose $T(p,q,r,-1)^*$ is a torus knot. Then by Theorem $T(p,q,r,-1)^*$ has braid index $r-q$ and is obtained by closing the braid below.

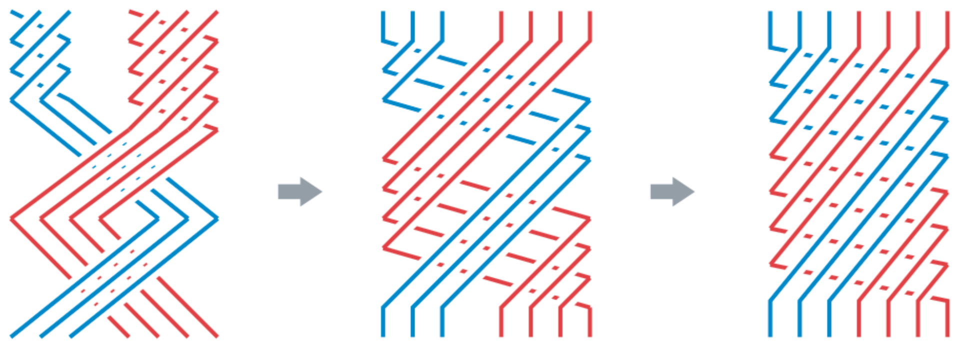


Thus $T(p,q,r,-1)^*$ may be assumed to be the torus knot $T(r-q,x)$ for some integer $x > r-q$.

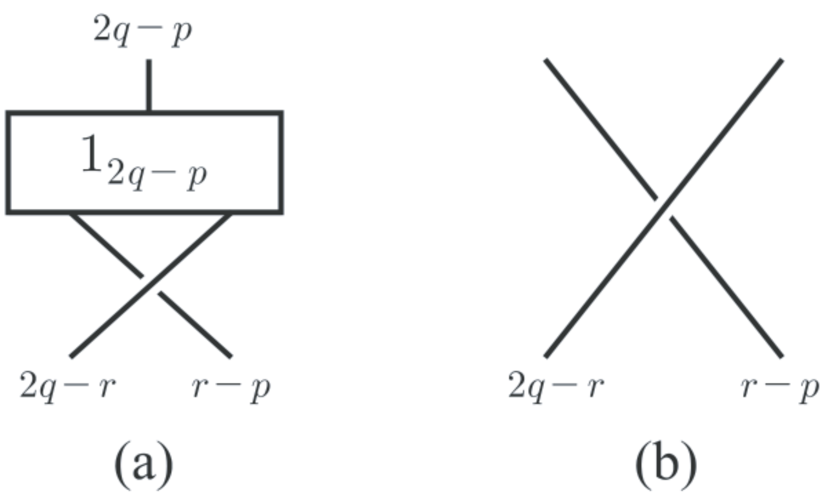
Thus $T(p, q, r, -1)^*$ may be assumed to be the torus knot $T(r - q, x)$ for some integer $x > r - q$.

By removing the two full twists on the top, we obtain the braid below. Hence the braid is closed to form the torus knot $T(r - q, x - 2r + 2q)$.





Claim. $r + p - 3q = 0$.



Since $r + p - 3q = 0$, the previous braid becomes the braid in (a).

The braid in (b), which is obtained from the braid in (a) by deleting a full twist on the top, is closed to form the torus knot $T(r - q, x - 3r + 3q)$.

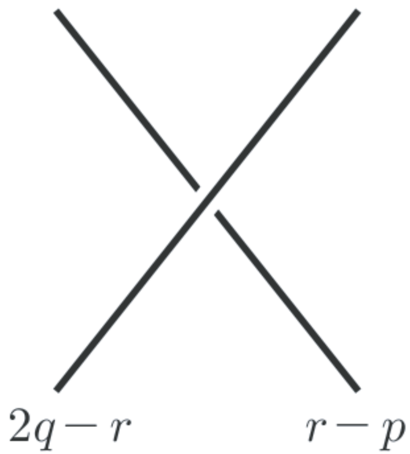
The braid in (b) has $(2q-r)+(r-p)$ strands and $(2q-r)(r-p)$ crossings, so by Stallings twice the genus of the torus knot $T(r-q, x-3r+3q)$ is

$$
\begin{aligned}
&(r-q-1)(x-3r+3q-1) \\
=&1-(2q-r)-(r-p)+(2q-r)(r-p) \\
=&(1-2q+r)(1-r+p) \qquad (*) \\
=&(1-2q+3q-p)(1-3q+p+p) \\
=&(1+q-p)(1-3q+2p).
\end{aligned}
$$

Note that both integers $2q - r$ and $r - p$ are positive.

If $2q - r = r - p$, then along with $r + p - 3q = 0$, we see that $p = 4q/3$ and $r = 5q/3$.

Since p and q are coprime, this implies that $(p, q, r) = (4, 3, 5)$, that is, $(p, q, r) = (n + 1, n, 2n - 1)$ with $n = 3$, as desired in the lemma.



Hence we may assume that $2q - r$ and $r - p$ are distinct positive integers.

Then the closure of the braid in (b), which is the torus knot $T(r - q, x - 3r + 3q)$, has braid index $2q - r$ if $2q - r < r - p$, or braid index $r - p$ if $2q - r > r - p$.

Suppose that the torus knot $T(r-q, x-3r+3q)$ has braid index $r-q$, that is, $r-q < x-3r+3q$.

Then either $r-q = 2q-r$ or $r-q = r-p$.

In the former, we get $2r = 3q = r+p$ and hence $r = p$.

In the latter, we get $q = p$.

Any of these is impossible. Thus the torus knot $T(r-q, x-3r+3q)$ has braid index $x-3r+3q$.

If $2q - r < r - p$, then $x - 3r + 3q = 2q - r$ and from (*) we get

$$\begin{aligned}
&(1 + q - p)(1 - 3q + 2p) \\
=&(r - q - 1)(x - 3r + 3q - 1) \\
=&(r - q - 1)(2q - r - 1) \\
=&(3q - p - q - 1)(2q - 3q + p - 1) \\
=&(2q - p - 1)(p - q - 1),
\end{aligned}$$

which gives

$$(p - q)(p - q - 1) = 0.$$

Since $p - q \neq 0$, we have $p = q + 1$ and hence $r = 3q - p = 2q - 1$. Since $r > p$, $q \geq 3$. Letting $q = n$, we have $(p, q, r) = (n + 1, n, 2n - 1)$.

If $2q - r > r - p$, then $x - 3r + 3q = r - p$ and from (*) we get

$$\begin{aligned}
&(1 + q - p)(1 - 3q + 2p) \\
=&(r - q - 1)(x - 3r + 3q - 1) \\
=&(r - q - 1)(r - p - 1) \\
=&(3q - p - q - 1)(3q - p - p - 1) \\
=&(2q - p - 1)(3q - 2p - 1),
\end{aligned}$$

which gives

$$(2p - 3q)(2p - 3q + 1) = 0.$$

If $2p - 3q = 0$, then $r = 3q - p = p$, a contradiction.

Thus $2p = 3q - 1$ (and hence q is an odd integer) and $r = 3q - p = p + 1$. It follows that $(p, q, r) = (3m + 1, 2m + 1, 3m + 2)$ for some positive integer m. □

Thank you