Non-minimal bridge position of 2-cable links

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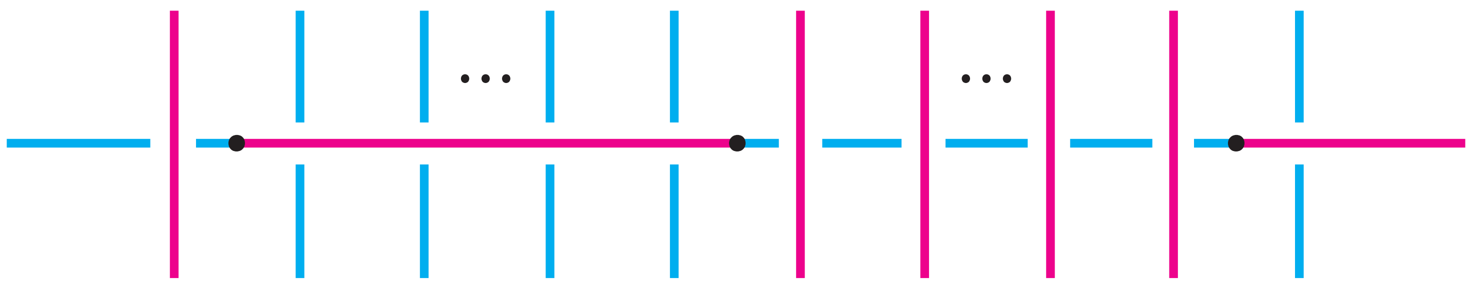
Outline

- bridge number, bridge position
- perturbed bridge position
- main result
- three ingredients in the proof

• bridge number

[Schubert, 1954]

D : a regular diagram of a knot (or link) K
$b(D)$: the number of over-bridges(or under-bridges) of D



$b(K) = \min\{b(D) \mid D$ is a regular diagram of $K\}$ is called the *bridge number* of K.

[Schubert]

$b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$

a satellite knot K

companion J, wrapping number n

$b(K) \geq n \cdot b(J)$

In particular, $K =$ a (p, q)-cable of J (longitudinally p times)

$b(K) = p \cdot b(J)$

• bridge position

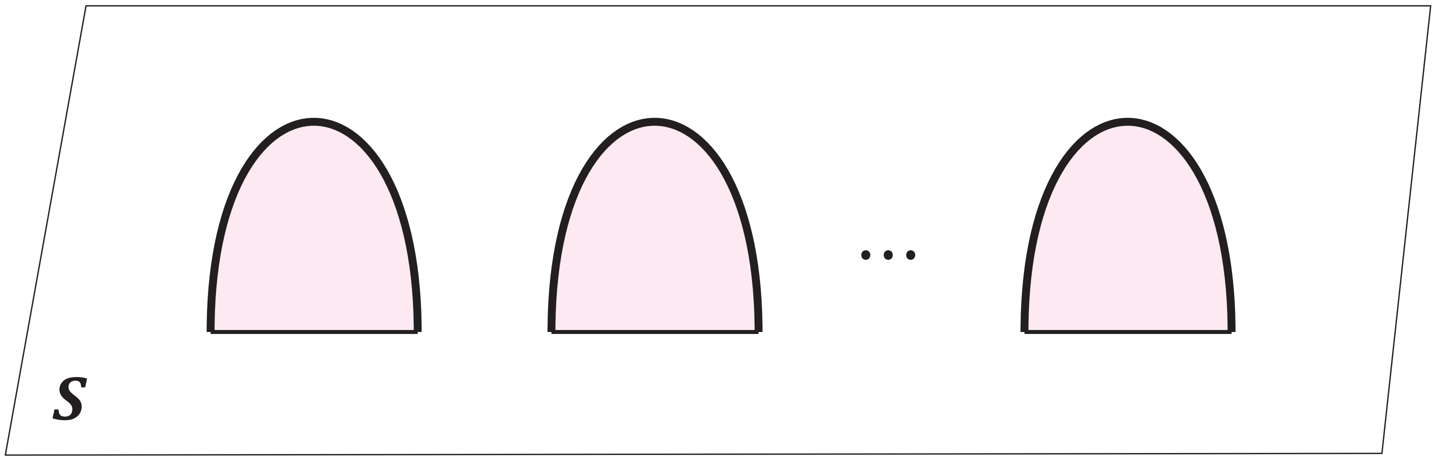
generalized to bridge splittings for (M^3, K).

$S^3 = B_1 \cup_S B_2$, B_1, B_2 : 3-balls
$K \cap B_i =$ a collection of ∂-parallel arcs (= a trivial tangle),
say $K \cap B_1 = t_1 \cup \cdots \cup t_n$.

each t_i : a *bridge*
We say that K is in *n-bridge position* with respect to S.

A bridge cobounds a *bridge disk* D with an arc in S.
$((\text{int}\, D) \cap K = \emptyset)$
not properly embedded in B_i

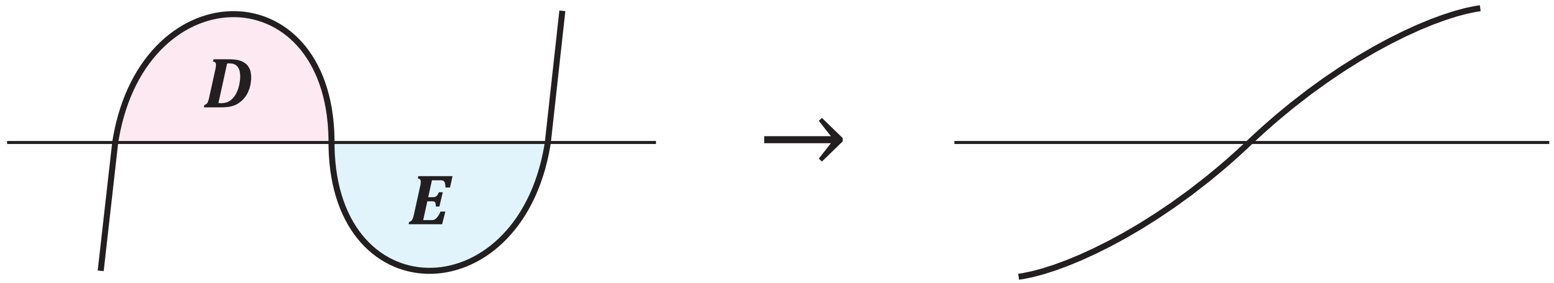
$\exists$ a collection of n pairwise disjoint bridge disks,
called a *complete bridge disk system*.



• perturbed bridge position

D : a bridge disk in B_1, E : a bridge disk in B_2
If $D \cap E = \{$a single point of $K\}$, then each of D and E is called a *cancelling disk*.
(D, E) : a *cancelling pair*

A bridge position is *perturbed* if it admits a cancelling pair.
It gives rise to a lower index bridge position.



Every n-bridge ($n > 1$) position of the unknot is perturbed [Otal].

Non-minimal bridge positions of the following classes of knots are perturbed:
2-bridge knots [Otal]
torus knots [Ozawa]
iterated torus knots, iterated cable of 2-bridges knots [Zupan]

$*$ $\exists$ examples of knots with non-minimal bridge positions that are not perturbed [Jang-Kobayashi-Ozawa-Takao].

If K is an mp-small knot and
every non-minimal bridge position of K is perturbed,
then every non-minimal bridge position of a (p, q)-cable of K is also perturbed [Zupan].

(A knot is *mp-small* if its exterior does not contain essential meridional planar surface.)

Theorem. [L.]
K : a knot such that every non-minimal bridge position of K is perturbed
L : a $(2, 2q)$-cable link of K

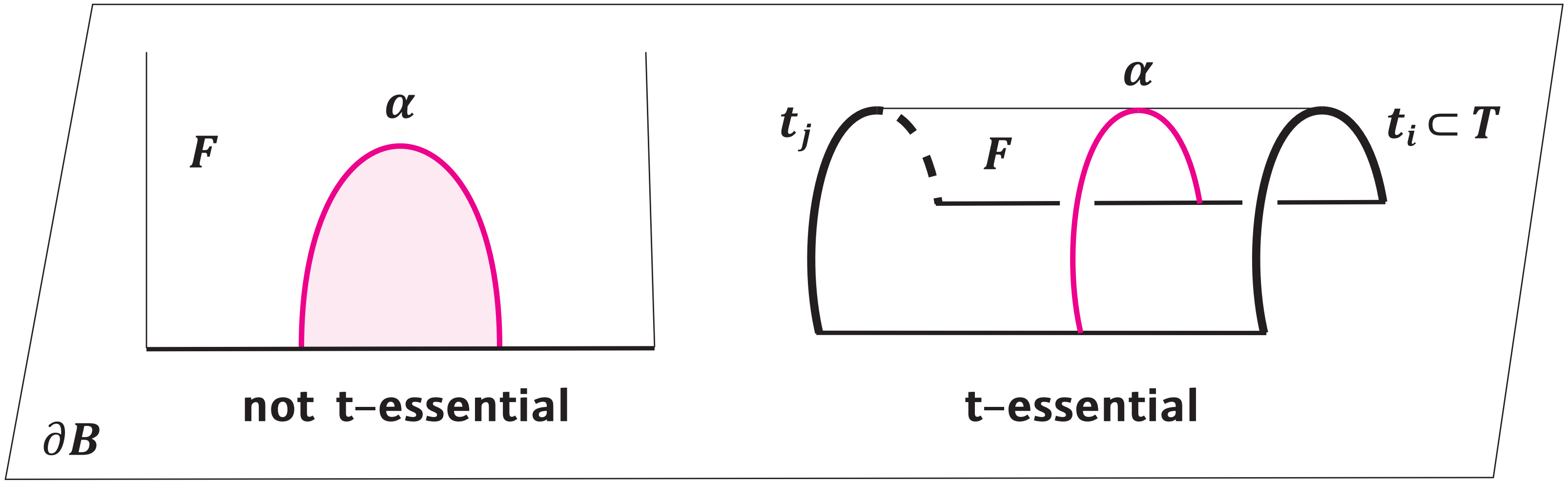
Then every non-minimal bridge position of L is also perturbed.

- t-incompressibility, t-∂-incompressibility [Hayashi-Shimokawa]
- changing the order of t-∂-compressions [Doll]
- a sufficient condition for a perturbed bridge position [L.]

• t-essential arc

T : a trivial tangle in a 3-ball B
F : a surface in B satisfying $F \cap (\partial B \cup T) = \partial F$

α : an arc properly embedded in F, with $\partial\alpha \subset F \cap \partial B$
α is *t-essential* if
α does not cobound a disk in F with a subarc of $F \cap \partial B$.



t-compressing disk D for F

$D \subset B - T$
$D \cap F = \partial D$ does not bound a disk in F.

F is *t-compressible* if $\exists$ a t-compressing disk for F.
Otherwise, F is *t-incompressible*.

t-∂-compressing disk Δ for F

$\Delta \subset B - T$
$\partial\Delta = \alpha \cup \beta$, $\alpha = \Delta \cap F$ is t-essential, $\beta = \Delta \cap \partial B$

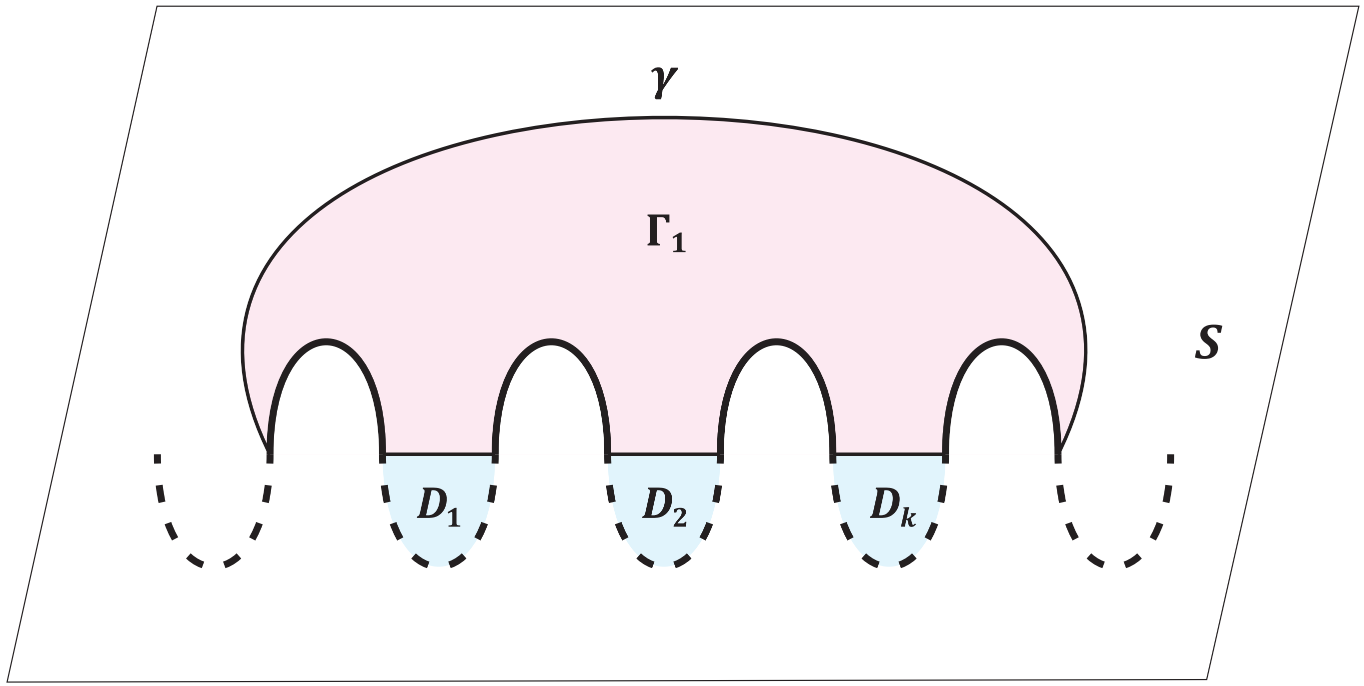
F is *t-∂-compressible* if $\exists$ a t-∂-compressing disk for F.
Otherwise, F is *t-∂-incompressible*.

- **a sufficient condition for a perturbed bridge position**

K : a knot (or link) in n-bridge position w.r.t. $S^3 = B_1 \cup_S B_2$

F : a surface bounded by K

Lemma. Suppose a separating arc γ of $F \cap S$ cuts off a disk Γ from F such that
(1) $\Gamma \cap B_1$ is a single disk Γ_1, and
(2) $\Gamma \cap B_2 (\neq \emptyset)$ consists of bridge disks $D_1, \ldots, D_k$.
Then the bridge position of K is perturbed.



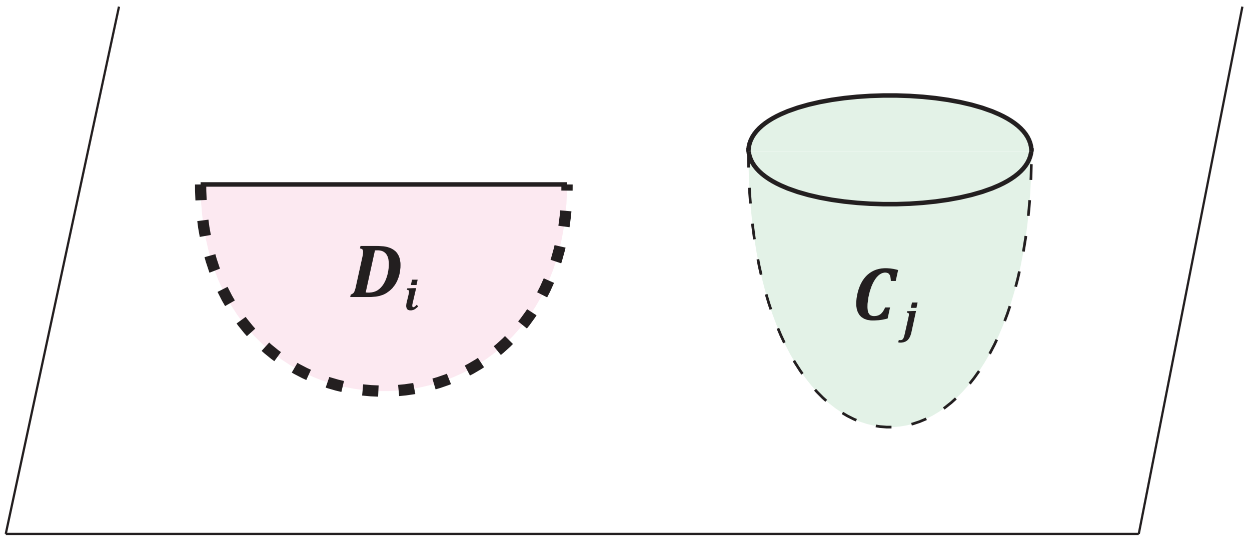
- **setting**

K : a knot s.t. every non-minimal bridge position is perturbed
L : a $(2,2q)$-cable link of K

$S^3 = B_1 \cup_S B_2$, B_1, B_2 : 3-balls
Suppose L is in non-minimal bridge position with respect to S.
A : an annulus bounded by L

Maximally t-compress and t-∂-compress A in B_2.

By [Hayashi-Shimokawa],
$A \cap B_2$ consists of
bridge disks D_i's and properly embedded disks C_j's ($C_j \cap L = \emptyset$).

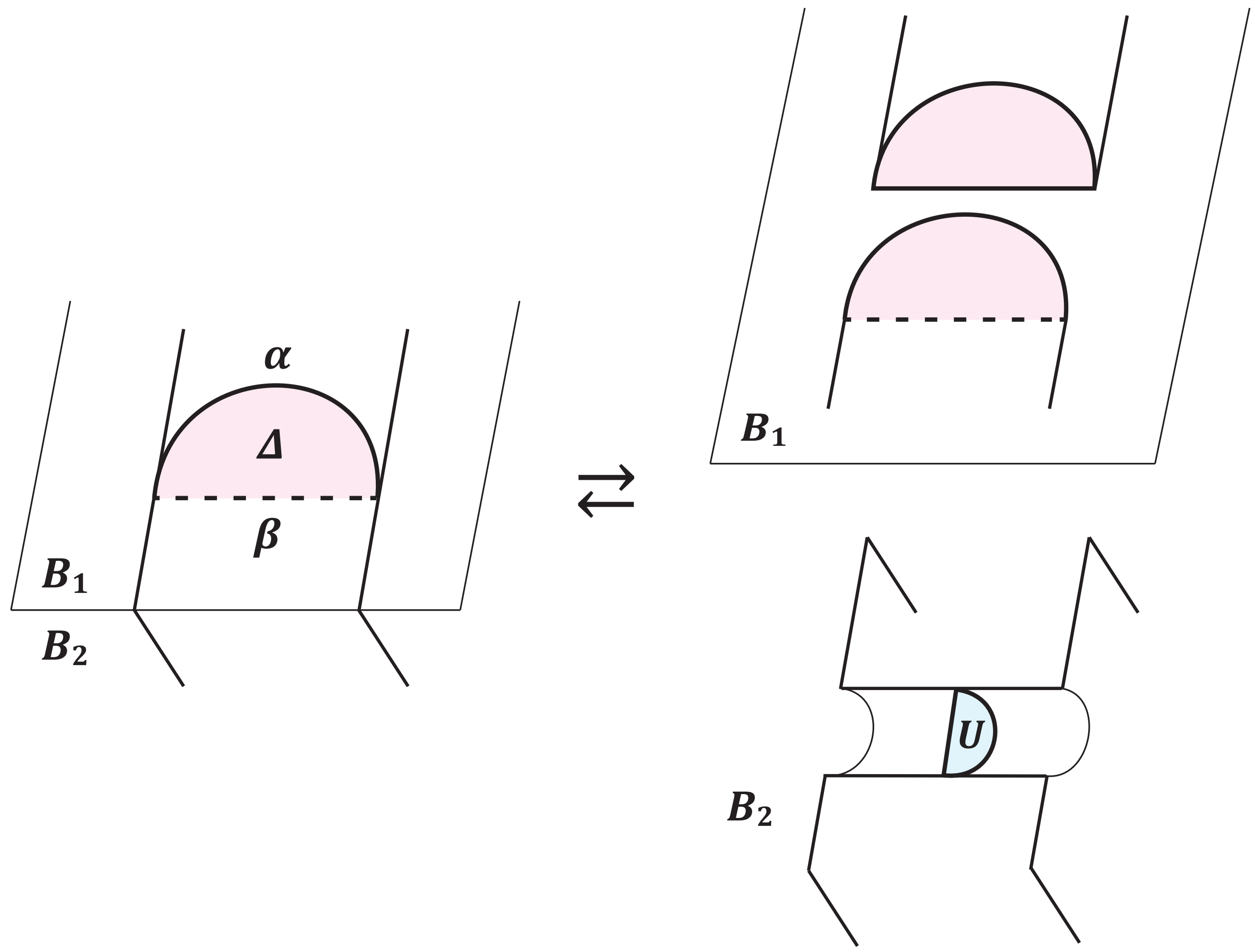


bridge disk D_i and properly embedded disk C_j(= cap disk)

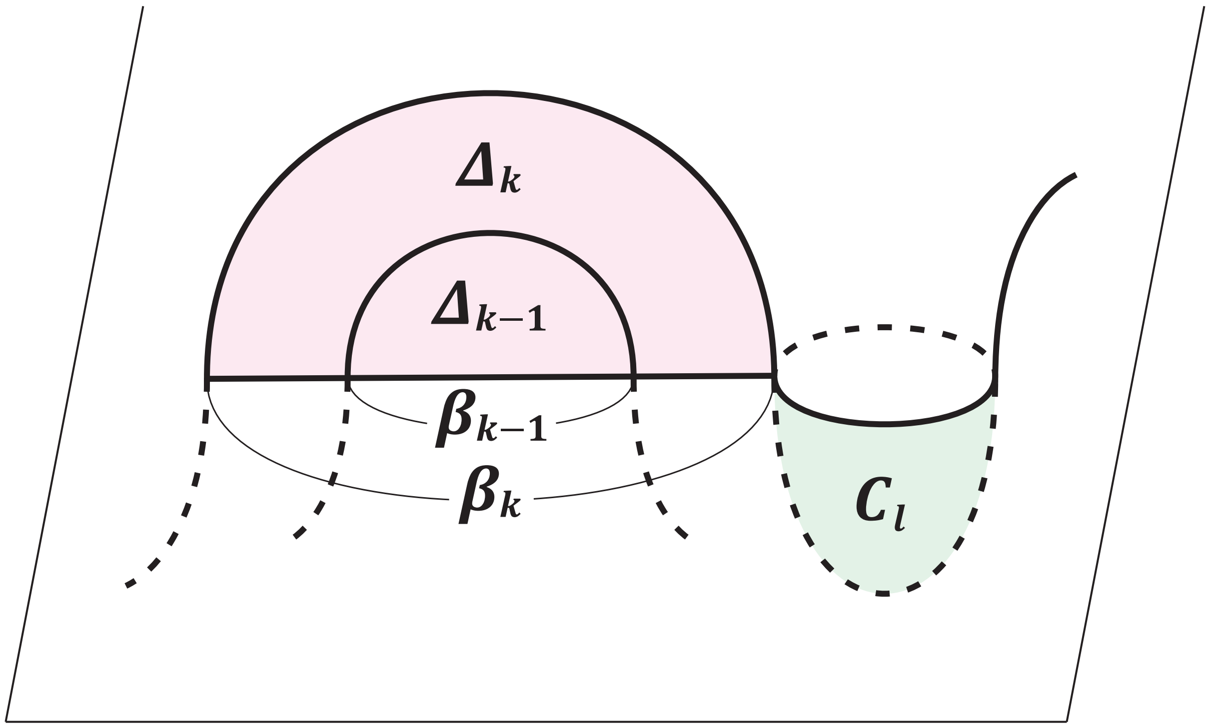
Take an annulus A bounded by L so that
the number m of properly embedded disks C_j is minimal.
$\Longrightarrow m = 0$.

For the proof, suppose that $m > 0$.

• t-∂-compression and its dual operation

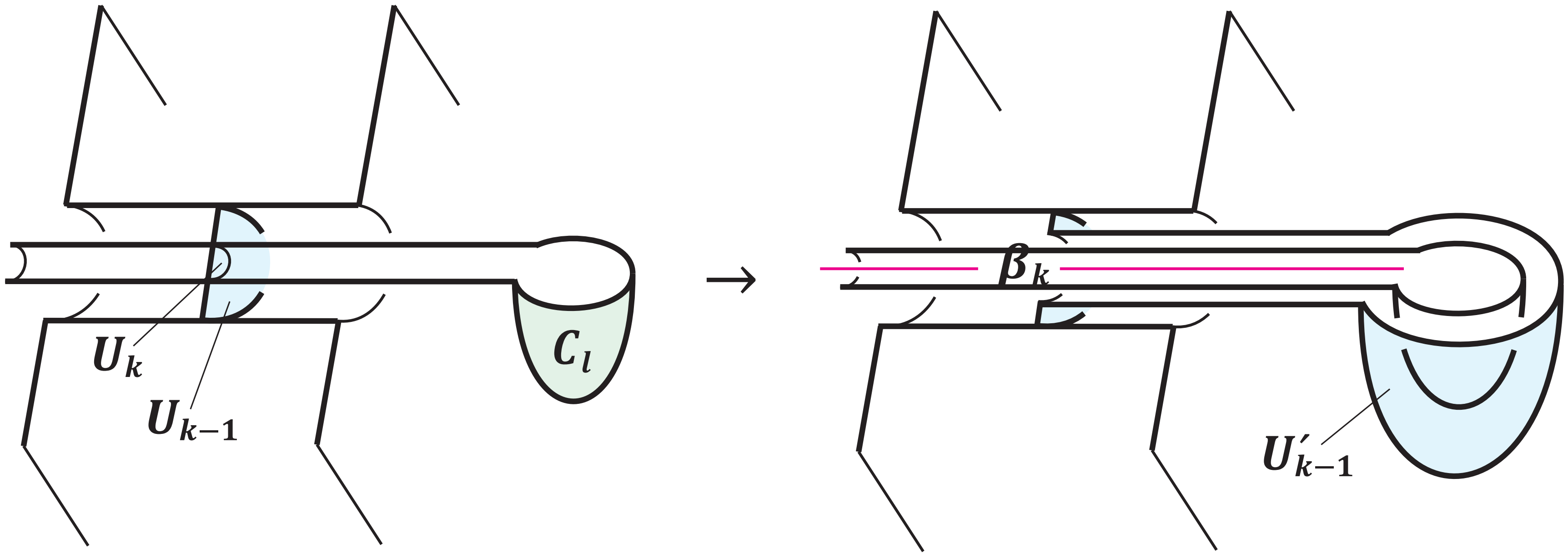


- **nested t-∂-compressing disks**

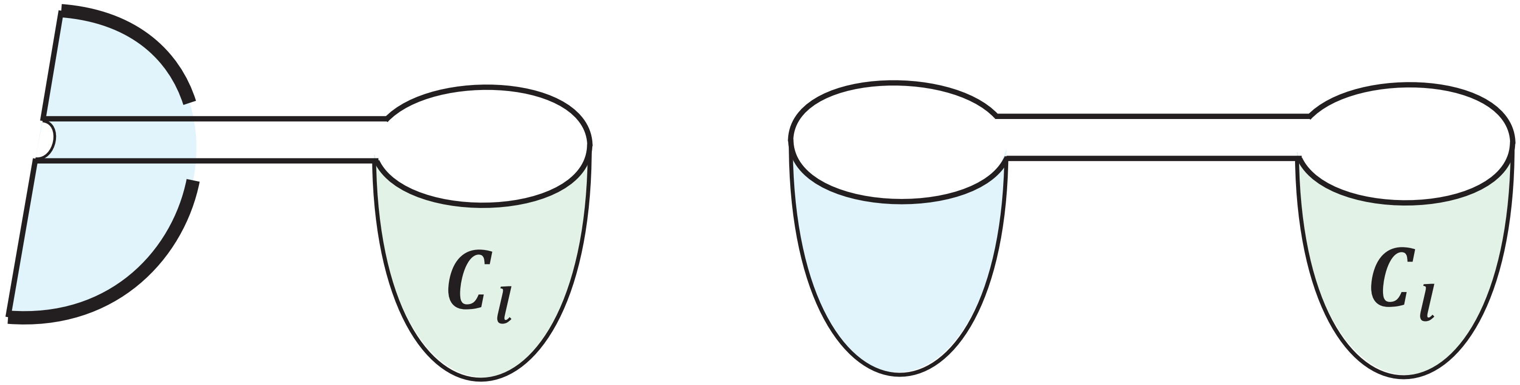


Δ_k after Δ_{k-1}
C_l : cap disk

Replace U_{k-1} by U'_{k-1} so that $\beta_k \cap U'_{k-1} = \emptyset$

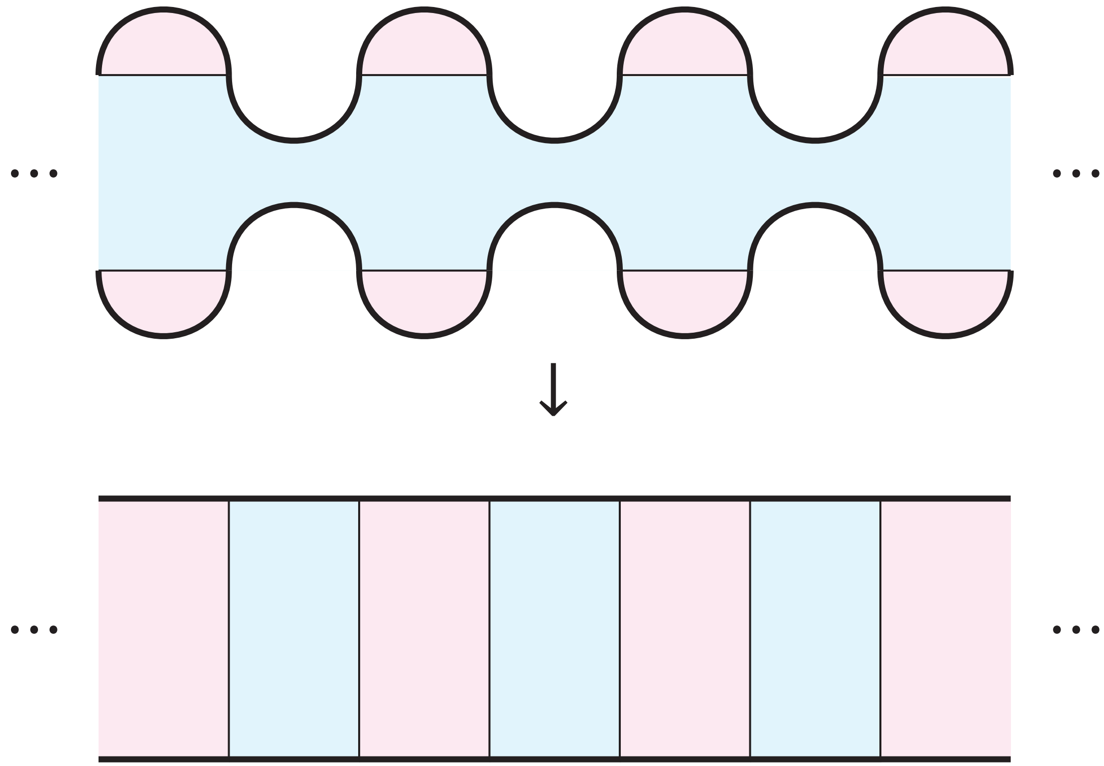


We can do the t-∂-compression along Δ_k first.



The number m of properly embedded disks C_j is reduced.

$A \cap B_2 =$ bridge disks.



By a further argument, we find a cancelling pair of bridge disks.

Thank you for your attention.