Arc presentations of Montesinos links

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February 2, 2021

Knots and Spatial Graphs 2021

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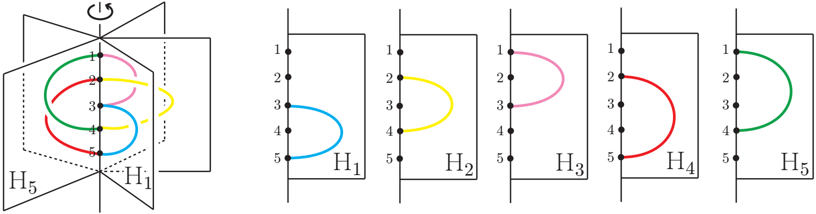
Arc presentation

An *arc presentation* of a knot or a link L is an embedding of L contained in the union of finitely many half planes with a common boundary line, called *binding axis*, in such a way that each half plane contains a properly embedded single simple arc.



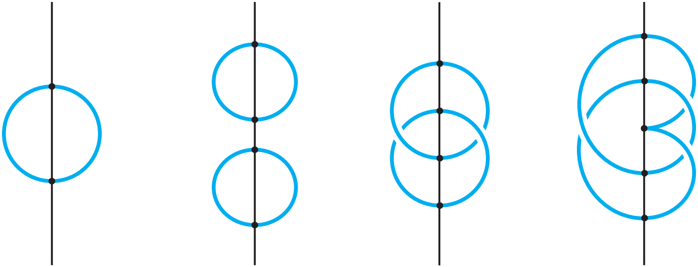
Cromwell, 1995

Every link admits an arc presentation.

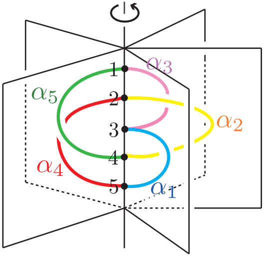
Arc index

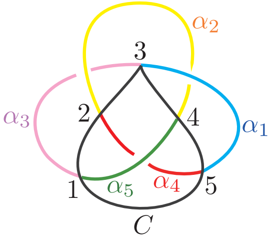
The minimum number of pages among all arc presentations of a link L is called the *arc index* of L and is denoted by $\alpha(L)$.

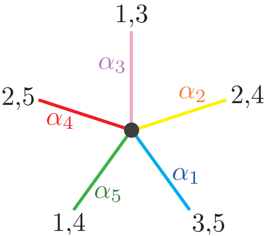
$\alpha(L)$	2	3	4	5
L	unknot	none	2-component unlink, Hopf link	trefoil

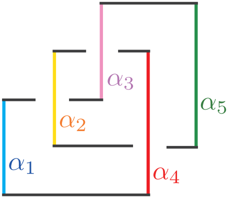


Representations of arc presentation









Known results I

- [Beltrami, 2002] Arc index for prime knots up to 10 crossings are determined.
- [Ng, 2006] Arc index for prime knots up to 11 crossings are determined.

★ [Nutt, 1999] All knots up to arc index 9 are identified.

★ [Jin et al., 2006] All prime knots up to arc index 10 are identified.

★ [Jin-Park, 2010] All prime knots up to arc index 11 are identified.

★ [Jin-Kim, 2021] All prime knots up to arc index 12 are identified.

Known results II

Let $F_L(a,z)$ denote the Kauffman polynomial of a link L.

- L : **non-split alternating link** $\Longrightarrow \alpha(L) = c(L) + 2$.
  - [Morton-Beltrami, 1998] For any link L, $\alpha(L) \geq \text{spread}_a(F_L(a,z)) + 2$.
  - [Thistlethwaite, 1988] If L is an alternating link, $\text{spread}_a(F_L(a,z)) \geq c(L)$.
  - [Bae-Park, 2000] If L is a non-split link, then

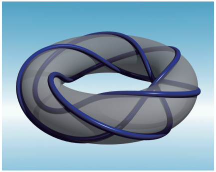
$$\alpha(L) \leq c(L) + 2.$$

- L : **nonalternating prime** $\Longrightarrow \text{spread}_a(F_L(a,z)) + 2 \leq \alpha(L) \leq c(L)$.
  - [M-B] For any link L, $\alpha(L) \geq \text{spread}_a(F_L(a,z)) + 2$.
  - [Jin-Park, 2010] A prime link L is nonalternating *if and only if*

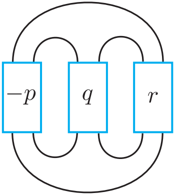
$$\alpha(L) \leq c(L).$$

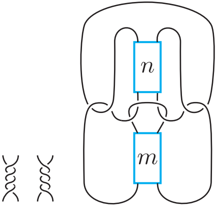
Known Results III

- ★ [Etnyre-Honda, 2001] $\alpha(T_{p,q}) = |p| + |q|$
- ★ [Beltrami, 2002] Arc index of n-semi alternating links
- ★ [L.-Jin, 2014] Arc index of pretzel knots of type $(-p, q, r)$
- ★ [L.-Takioka, 2017] Arc index of Kanenobu knots
- ★ [L.-Takioka, 2016] On the arc index of cable knots and Whitehead doubles



http://makerhome.blogspot.kr/2014/01/day−150−trefoil−torus−knots.html

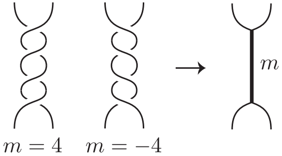


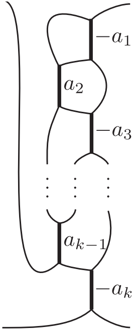


Rational tangle

The *rational tangle diagram* with Conway notation $[a_1, a_2, \ldots, a_k]$ can be represented by a slop $\frac{\beta}{\alpha}$ of two co-prime integers $\alpha > 1$ and β which is defined by the continued fraction expansion

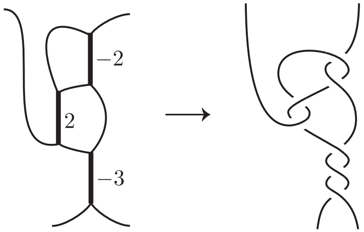
$$\frac{\beta}{\alpha} = \cfrac{1}{a_k + \cfrac{1}{a_{k-1} + \cdots + \cfrac{1}{a_2 + \frac{1}{a_1}}}} := [a_1, a_2, \ldots, a_k].$$





Rational tangle (Cont.)

$$\frac{5}{17} = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{2}}} := [2, 2, 3]$$



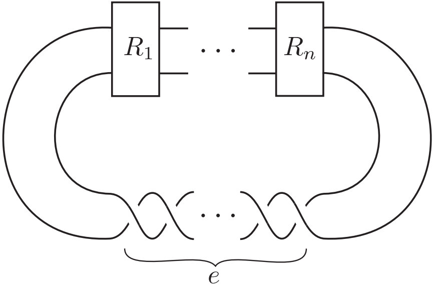
If all a_i's are positive (or negative), then the rational tangle diagram is said to be *positively* (or *negatively*) *alternating*. Hereinafter, for any rational tangle diagram, we assume the following:

(1) k is odd, and

(2) it is positively or negatively alternating.

Montesinos link

Let R_i be the rational tangle diagram with slope $\frac{\beta_i}{\alpha_i}$ for $1 \leq i \leq n$. A *Montesinos link* $L = M\left(e; \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n}\right)$ is a link that admits a diagram D composed of n rational tangle diagrams $R_1, R_2, \ldots, R_n$ and e half-twists. If $e = 0$, it will be omitted in the notation as $M\left(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n}\right)$.



Reduced Montesinos diagram

Owing to the study of Lickorish and Thistlethwaite, we may assume that Montesinos diagram D satisfies exactly one of the following conditions:

(M1) D is alternating;

(M2) $e = 0$, and each R_i is an alternating diagram, with at least two crossings, placed in D, such that the two lower ends of R_i belong to arcs incident to a common crossing in R_i.

Then, we call D a *reduced Montesinos diagram*.

Two conditions above can be rephrased as follows:

(M1*) $e \geq 0$, and all R_i's are positively alternating in D;

(M2*) $e = 0$, and $R_1, \ldots, R_k$ are negatively alternating in D, whereas $R_{k+1}, \ldots, R_n$ are positively alternating for some k, where $1 \leq k < n$.

Main Theorem A

Jin-Park, 2010

A prime link L is nonalternating *if and only if* $\alpha(L) \leq c(L)$.

Corollary

Let $L = M(-p, q, r)$ *be a Montesinos link. Then* $\alpha(L) \leq c(L)$.

Theorem A

(1) If $L = M(-p, q, r)$ is a link with $0 < p < 1$, $0 < q < \frac{1}{2}$ and $0 < r < \frac{1}{2}$, then

$$\alpha(L) \leq c(L) - 1.$$

(2) Let n be an integer greater than 2. If $L = M\left(-\frac{1}{n}, q, r\right)$ is a link with $0 < q < \frac{1}{3}$, $0 < r < \frac{1}{3}$, then

$$\alpha(L) \leq c(L) - 2.$$

Main Theorem B

Theorem B

(1) Let n be an integer greater than 1.
If $L = M\left(-\frac{1}{n}, \frac{2}{3}, \frac{2}{3}\right)$ or $L = M\left(-\frac{1}{n}, \frac{1}{2}, \frac{5}{17}\right)$, then

$$\alpha(L) = c(L).$$

(2) Let n be an integer greater than 2.
If $L = M\left(-\frac{1}{n}, \frac{2}{5}, \frac{2}{5}\right)$ or $L = M\left(-\frac{1}{n}, \frac{1}{3}, \frac{5}{17}\right)$, then

$$\alpha(L) = c(L) - 1.$$

(3) Let n be an integer greater than 3.
If $L = M\left(-\frac{1}{n}, \frac{2}{7}, \frac{2}{7}\right)$ or $L = M\left(-\frac{1}{n}, \frac{1}{4}, \frac{5}{17}\right)$, then

$$\alpha(L) = c(L) - 2.$$

Strategy

Our strategy is ...

- For the upper bound of $\alpha(L)$,
  we find arc presentations of L with the minimum number of arcs for various values of p, q and r.
- For the lower bound of $\alpha(L)$, we compute the

$$\mathrm{spread}_a(F_L(a, z)).$$

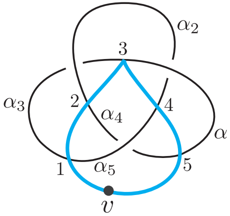
([Morton-Beltrami] For any link L, $\alpha(L) \geq \mathrm{spread}_a(F_L(a, z)) + 2$.)

Arc presentation

Let D be a diagram of a link L which lies on a plane P. Suppose that there is a simple closed curve C in P meeting D in $k(> 1)$ distinct points which divide D into k arcs $\alpha_1, \alpha_2, \ldots, \alpha_k$ with the following two properties:

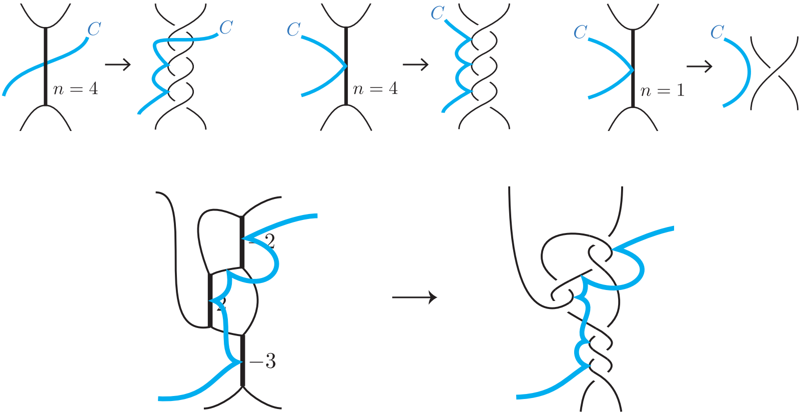
(C1) Each α_i has no self-crossings.

(C2) Let R_I be the inner region bounded by the curve C in the plane P and R_O the outer region. If α_i crosses over α_j at a crossing point in R_I(resp. R_O), then $i > j$(resp. $i < j$) and it crosses over α_j at any other crossing points with α_j.



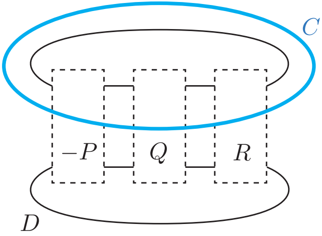
Then the pair (D, C) is called an *arc presentation* of L with k arcs, and C is called the *dividing circle* of the arc presentation.

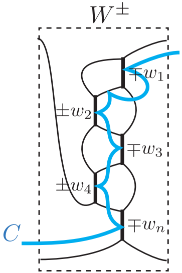
Constructing dividing circles



Arc presentations of $M(-p, q, r)$ with $c(L)$ arcs

Let L be a Montesinos link $M(-p, q, r)$. Let $-P$, Q, and R be the negatively, positively, and positively alternating rational tangle diagrams represented by $-p := [-p_1, -p_2, \ldots, -p_\ell]$, $q := [q_1, q_2, \ldots, q_m]$, and $r := [r_1, r_2, \ldots, r_n]$, respectively.





$$-P \leftrightarrow W^-, \;\; Q \leftrightarrow W^+, \;\; \text{and} \;\; R \leftrightarrow W^+.$$

Example 1

Let n be a positive integer greater than one. Let L_1^n and L_2^n be the Montesinos links of $M\left(-\frac{1}{n}, \frac{2}{3}, \frac{2}{3}\right)$ and $M\left(-\frac{1}{n}, \frac{1}{2}, \frac{5}{17}\right)$, respectively. Then the figure below shows arc presentations of the link L_1^n and L_2^n with $c(L_1^n) = n + 6$ and $c(L_2^n) = n + 9$ arcs, respectively.

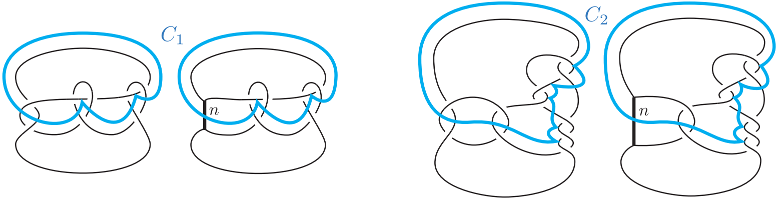
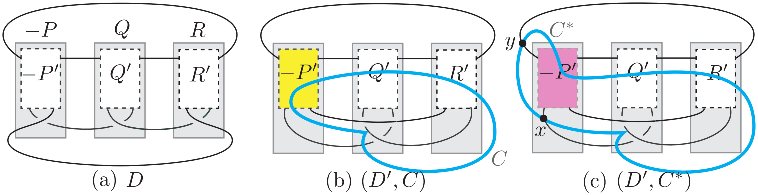
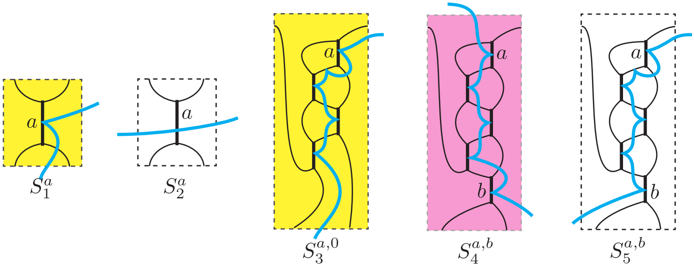


Figure: Arc presentations of L_1^2, L_1^n, L_2^2, and L_2^n

Arc presentations of $M(-p, q, r)$ with $c(L) - 1$ arcs

Suppose that both q and r are less than $\frac{1}{2}$.





Example 2

Let n be an integer greater than 1. Let L_3^n and L_4^n be the Montesinos links of $M\left(-\frac{1}{n}, \frac{2}{5}, \frac{2}{5}\right)$ and $M\left(-\frac{1}{n}, \frac{1}{3}, \frac{5}{17}\right)$, respectively. Then the figure below shows arc presentations of the link L_3^n and L_4^n with $c(L_3^n) - 1 = n + 7$ and $c(L_4^n) - 1 = n + 9$ arcs, respectively.

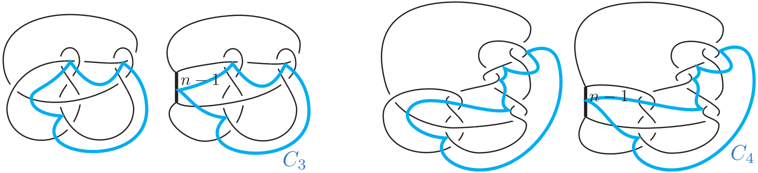
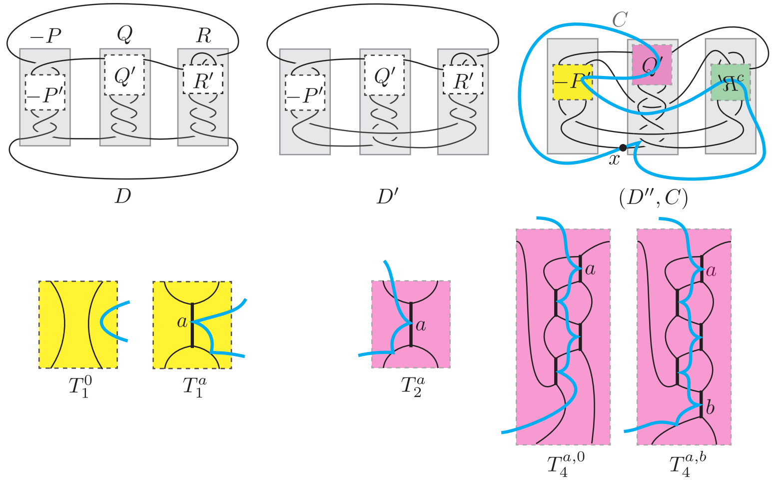


Figure: Arc presentations of L_3^n and L_4^n for $n \geq 2$

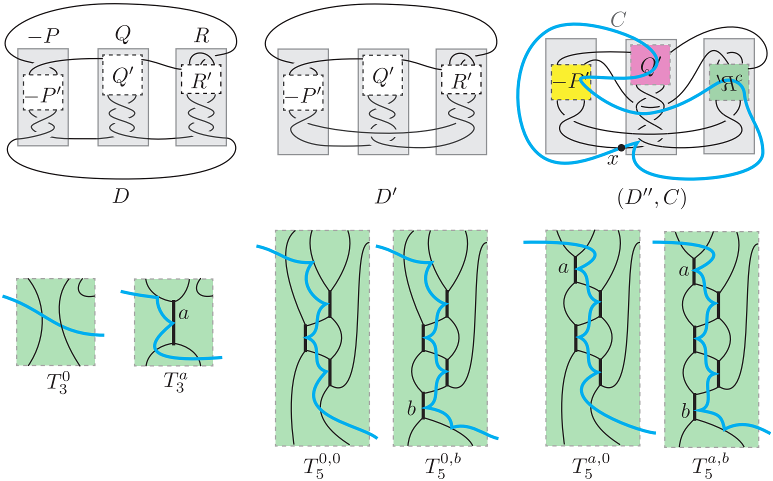
Arc presentations of $M(-p, q, r)$ with $c(L) - 2$ arcs

Suppose that $p = \frac{1}{n}$ $(n > 2)$ and both q and r are less than $\frac{1}{3}$.



Arc presentations of $M(-p, q, r)$ with $c(L) - 2$ arcs

Suppose that $p = \frac{1}{n}$ $(n > 2)$ and both q and r are less than $\frac{1}{3}$.



Example 3

Let n be an integer greater than 2. Let L_5^n and L_6^n be the Montesinos links of $M\left(-\frac{1}{n}, \frac{2}{7}, \frac{2}{7}\right)$ and $M\left(-\frac{1}{n}, \frac{1}{4}, \frac{5}{17}\right)$, respectively.

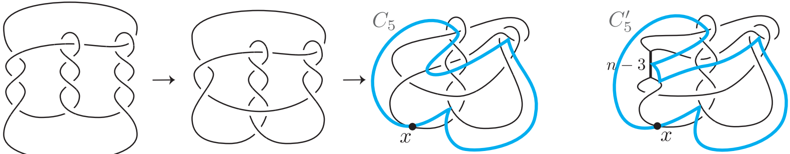


Figure: Arc presentations of L_5^3 and L_5^n for $n \geq 4$

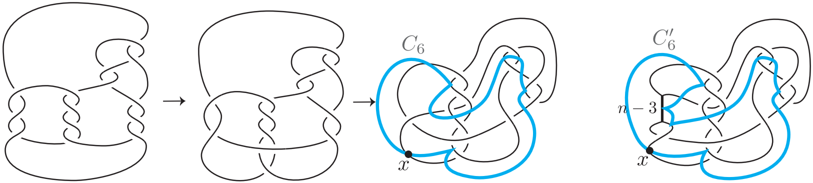


Figure: Arc presentations of L_6^3 and L_6^n for $n \geq 4$

Kauffman polynomial $F_L(a,z)$

The *Kauffman polynomial* of an oriented knot or link L is defined by

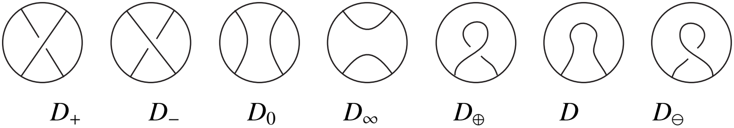
$$F_L(a,z) = a^{-w(D)}\Lambda_D(a,z)$$

where D is a diagram of L, $w(D)$ the writhe of D and $\Lambda_D(a,z)$ the polynomial determined by the rules (K1), (K2) and (K3).

(K1) $\Lambda_O(a,z) = 1$ where O is the trivial knot diagram.

(K2) $\Lambda_{D_+}(a,z) + \Lambda_{D_-}(a,z) = z(\Lambda_{D_0}(a,z) + \Lambda_{D_\infty}(a,z))$.

(K3) $a\,\Lambda_{D_\oplus}(a,z) = \Lambda_D(a,z) = a^{-1}\Lambda_{D_\ominus}(a,z)$.



Let the polynomial σ_n be a symmetric polynomial defined by

$$\sigma_n = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\dfrac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} & \text{if } n < 0, \end{cases}$$

for $\alpha + \beta = z$ and $\alpha\beta = 1$.

$$\sigma_1 = 1, \quad \sigma_2 = z, \quad \sigma_3 = z^2 - 1, \quad \sigma_4 = z^3 - 2z, \quad \cdots$$

Then the following properties holds immediately.

(1) $\sigma_{-n} = -\sigma_n$

(2) $z\sigma_n = \sigma_{n-1} + \sigma_{n+1}$

We use notation $\Lambda_{(-p,q,r)}$ for Λ_D when $D = M(-p, q, r)$. Specifically, Λ_n means the Λ-polynomial of the integer tangle with n crossings.

Let m and n be positive integers greater than 1. Using skein relation (K1), (K2) and (K3) repeatly, we have

$$\Lambda_n = \sigma_n \Lambda_+ - \sigma_{n-1}\Lambda_0 + z\sum_{i=1}^{n-1} \sigma_i a^{-(n-i)} \Lambda_\infty$$

For example, since the Λ-polynomial of the integer tangle with Conway notation of $[-7]$ can be expressed as

$$\Lambda_7 = \sigma_7 \Lambda_+ - \sigma_6 \Lambda_0 + z\sum_{i=1}^{6} \sigma_i a^{-(7-i)} \Lambda_\infty,$$

the Λ-polynomial for $\Lambda_D = M\left(-\frac{1}{7}, q, r\right)$ can be written as

$$\Lambda_{(-\frac{1}{7},q,r)} = \sigma_7 \Lambda_{(-\frac{1}{1},q,r)} - \sigma_6 \Lambda_{(-\frac{1}{0},q,r)} + z\sum_{i=1}^{6} \sigma_i a^{-(7-i)} \Lambda_{(-\frac{1}{\infty},q,r)}.$$

Proposition 1

Let n be an integer greater than 1 and L a Montesinos link $M\left(-\frac{1}{n}, \frac{2}{3}, \frac{2}{3}\right)$; then,

$$\operatorname{spread}_a(F_L) = n + 4.$$

Sketch of proof:

$$\Lambda_n = \sigma_n \Lambda_+ - \sigma_{n-1}\Lambda_0 + z\sum_{i=1}^{n-1}\sigma_i a^{-(n-i)}\Lambda_\infty$$

$$\Lambda_{(-\frac{1}{n},\frac{2}{3},\frac{2}{3})} = \sigma_n \Lambda_{(-\frac{1}{1},\frac{2}{3},\frac{2}{3})} - \sigma_{n-1}\Lambda_{(-\frac{1}{0},\frac{2}{3},\frac{2}{3})} + z\sum_{i=1}^{n-1}\sigma_i a^{-(n-i)}\Lambda_{(-\frac{1}{\infty},\frac{2}{3},\frac{2}{3})}.$$

Proposition 2

Let n be an integer greater than 2 and L a Montesinos link $M\left(-\frac{1}{n}, \frac{2}{5}, \frac{2}{5}\right)$; then,

$$\mathrm{spread}_a(F_L) = n + 5.$$

Proposition 3

Let n be an integer greater than 3 and L a Montesinos link $M\left(-\frac{1}{n}, \frac{2}{7}, \frac{2}{7}\right)$; then,

$$\mathrm{spread}_a(F_L) = n + 6.$$

Main Theorem B

Theorem B

(1) Let n be an integer greater than 1.
If $L = M\left(-\frac{1}{n}, \frac{2}{3}, \frac{2}{3}\right)$ or $L = M\left(-\frac{1}{n}, \frac{1}{2}, \frac{5}{17}\right)$, then

$$\alpha(L) = c(L).$$

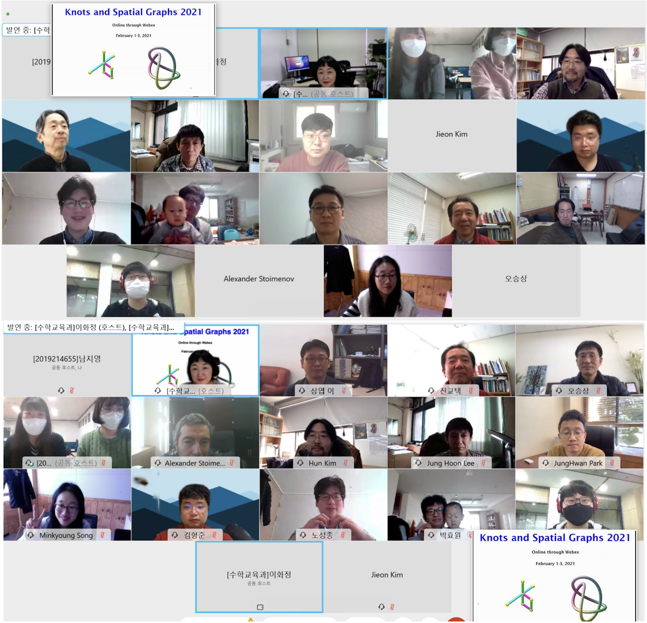
(2) Let n be an integer greater than 2.
If $L = M\left(-\frac{1}{n}, \frac{2}{5}, \frac{2}{5}\right)$ or $L = M\left(-\frac{1}{n}, \frac{1}{3}, \frac{5}{17}\right)$, then

$$\alpha(L) = c(L) - 1.$$

(3) Let n be an integer greater than 3.
If $L = M\left(-\frac{1}{n}, \frac{2}{7}, \frac{2}{7}\right)$ or $L = M\left(-\frac{1}{n}, \frac{1}{4}, \frac{5}{17}\right)$, then

$$\alpha(L) = c(L) - 2.$$

Thank you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