Link polynomial computation

and application to braid index

will specifically focus on

skein HOMFLY-PT

$$\begin{tikzcd} & & \text{Young diagrams} \\ P \arrow[r] \arrow[rd] & \text{Hecke algebra (braid)} \arrow[ru] \arrow[r] & \text{permutation braids} \\ & \text{skein theory} & \end{tikzcd}$$

$$\begin{tikzcd} & P \arrow[ld] \arrow[rd] & \\ \Delta & & V \end{tikzcd}$$

works for them as well, but there are other approaches

Seifert mtx. Kauffman bracket etc.

(1.1.) Hecke algebra

symm group algebra $\mathbb{Z}[S_n] \xleftarrow{q=1} H(n,q) = \langle g_1 \dots g_{n-1} \mid g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},\ g_i g_j = g_j g_i \ |i-j|>1,\ g_i^2 = (q-1)g_i + q \rangle$

$\uparrow\dots\uparrow \underset{i\ i+1}{\times} \uparrow\dots\uparrow \leftarrow \uparrow\dots\uparrow \underset{\sigma_i}{\times} \uparrow\dots\uparrow$

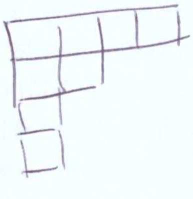
$\parallel$

$\mathbb{Z}[S_n] \ni \tau_i \longleftarrow \underset{\underset{B_n}{\cap}}{\sigma_i} \longmapsto g_i$ gives a repres of B_n by right mult.

$Y \vdash n$ partition $\simeq$ Young diagram

$d_Y = \dim \pi_Y$ known (hook length formula)

$Y = 4+2+1+1$



$\pi_Y \quad Y \vdash n$ irreps of $S_n \overset{\text{deform}}{\rightsquigarrow} \pi_Y$ irreps of B_n

$\beta \in B_n$

$P(\hat{\beta}) = \dots \sum_{Y \vdash n} W_Y \operatorname{tr} \pi_Y(\beta)$

(reminder: $K = \hat{\beta}$ for some β)

can be defined (Ocneanu) Hoste

complexity linear in length of β

Fact: mult. of π_Y in $\mathbb{Z}[S_n] = d_Y$ $\rightsquigarrow$ same for $H(n,q)$

so

$$\dim \mathbb{Z}[S_n] = n! = \sum_{Y \vdash n} d_Y^{\,2} \gg \sum_{Y \vdash n / \text{mirroring}} d_Y = \dim \bigoplus \pi_Y$$

all we need to compute traces

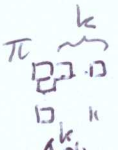
but

π_Y is symm. in rows, asymm. in cols. $\xrightarrow{\text{how to deform}}$ $\pi_Y(\sigma_i) = ?$

$\pi_Y(\sigma_i)$ writable

for special Y ok

= Burau Ψ_n



$\Lambda^k \Psi_n$

Ex. $n=4$ $\dim(\overset{3}{\Psi_4} \oplus (\overset{2}{\Psi_3} \circ B_4 \to B_3)) = 5$ vs. $24 = \dim H(4,q)$

$n=5$ $\Psi_5 \oplus \pi$

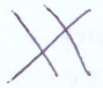
ⓘ (1.2.) 9 vs. 120

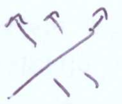
$\mathbb{Z}[S_n]$ basis $\{g \in S_n\}$

permutation braids $\{\beta_g : g \in S_n\}$
"
$H(n,q)$ has a basis $\mathcal{B}$

β_g make all crsgs in g pos.

$n=3$ $1 \to 3 \to 2 \to 1$

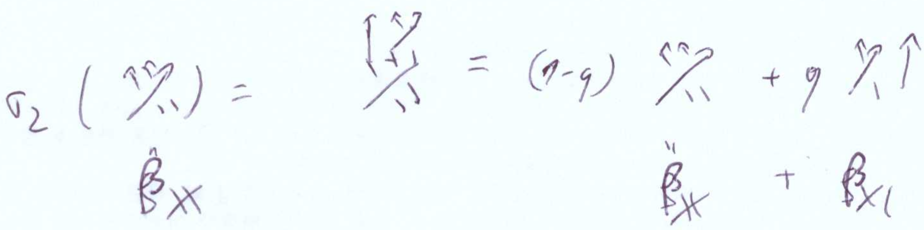




$= \sigma_1 \sigma_2$

in basis $\mathcal{B}$, rep. π can be written down directly

e.g.



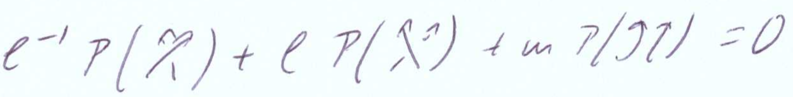
$P(\hat\beta) = \sum_{\beta_g \in \mathcal{B}} \cdots P(\hat{\beta}_g)$

linear in length of β

but $\dim \pi = n!$

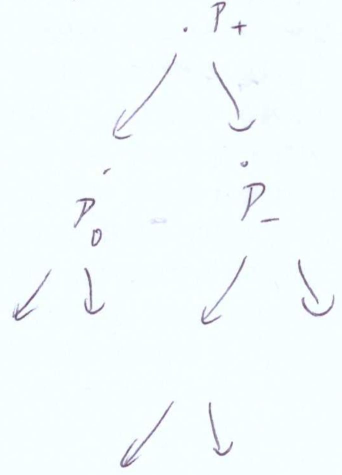
Morton-Short program $n \leq 10$

2. skein relation



$$\ell^{-1} P(\times) + \ell P(\times) + m P(\asymp) = 0$$

Millett - Ewing program (ME)



skein tree
terminal nodes unlinks (O O..O)

. uses a separate notation (ME notation)
valid also for link diagrams

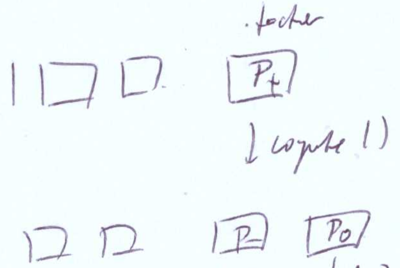
. skein tree recursion

compute(L) = ℓ^2. compute(L_-)
+ ℓm compute(L_0)

recursive call
will make exec. stack with all
procedure's data

much better

(data)
stack (ME)



factor i = monomial in l,m

(data) stack is more flexible

ME program can make stack dumps
and be restarted

has also arc analysis (is very simplified)

I tried to recreate the above functionality
+ provision for truncation calculations
(P mod m^k)

caution: no int overflow test

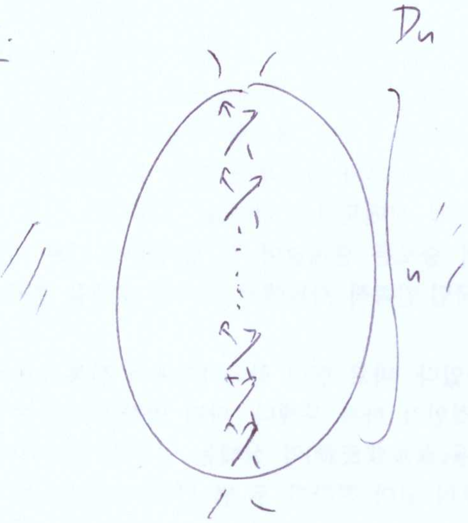
advantage: does not restrict to closed braid

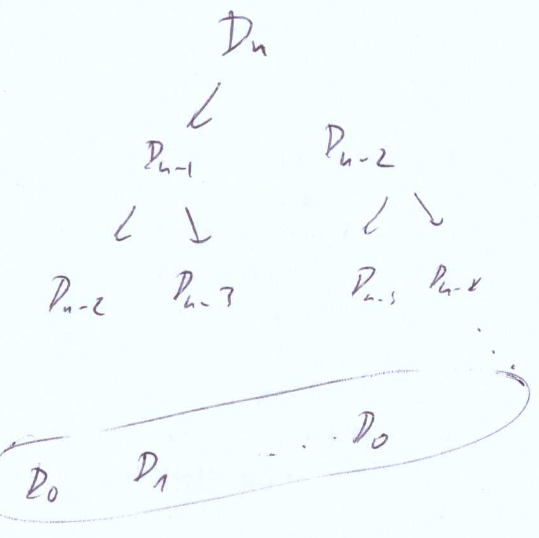
↑ can always be made, but many strings and crossings

disadvantage: for diagrams with long unknotting / unlinking sequences is very slow

↑ "braid like" and many pos or neg crossings

Ex.





of "terminal" nodes ~ Fibo

$\sim \left(\frac{1+\sqrt{5}}{2}\right)^n$ exp in n

but only two diagrams D_0, D_1 occur

$H(2,q)$ has basis { ↑↑ , ⤫ }

⇒ add their weights (will be polynomials) and compute $P(D_0)$, $P(D_1)$ only once

Project: "hybrid" braid/skein approach
recognize braid patterns and decompose them into permutation braids

other tasks: · parallelize (threads)
· ℓ truncations

application: braid index

goal: make table of braid indices of table knots

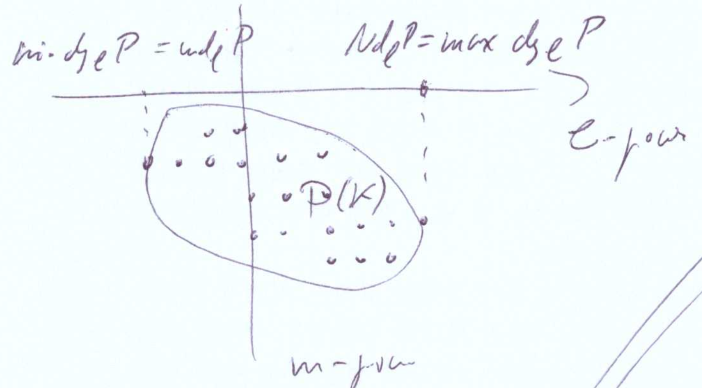
reminder

$$b(K) = \min\{ n : \exists \beta \in B_n : K = \hat\beta \}$$

= K

there are ∞ly many braid in B_n so testing any braid does not work

main tools: Morton-Williams-Franks (MWF)



$$b(L) \ge \overbrace{\frac{1}{2}\left(\underbrace{Md_\ell P - md_\ell P}_{span_\ell P}\right) + 1}^{MWF(L)}$$

obstruction

$$b(L) \le s(D) - ind(D)$$

Seifert circles

MP-index

realization

more precisely if $\exists \beta \in B_n$ $\hat\beta = K$ e = exp sum

$B_n \to \mathbb{Z}$

$\beta \mapsto e$

$$\left.\begin{array}{l} Md_\ell P(K) \le e+n-1 \\ md_\ell P(K) \ge e-n+1 \end{array}\right\} span_\ell P \le 2n-2$$

e = admissible exp sum / writhe

Rem: $e+n \equiv$ # components of $\hat\beta$ mod 2

so admissible writhe has a parity for fixed n, K

b(K) = MWF(K) when
- K is alternating & c(K) ≤ 17
- —"— & MWF(K) ≤ 4

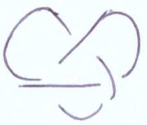
so non-alternating knot more complicated

$MWF(9_{42}) = 3 < 4 = b(9_{42})$ same 9_{49}

how to prove?

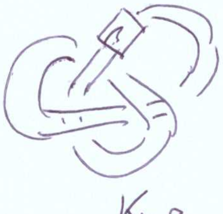
(n-)cabled MWF

"ncMWF"



K

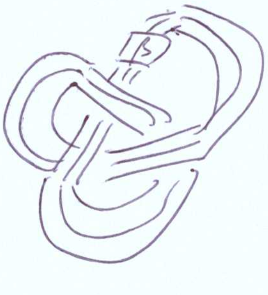
n=2



$K_{n,\beta}$

$\beta \in B_n$ pattern

n=3



$b(K_{n,\beta}) \le n\, b(K)$ ("=" Birman-Menasco) $K \neq O$

$$b(K) \ge \left\lceil \frac{MWF(K_{n,\beta})}{n} \right\rceil = \left\lceil \frac{\mathrm{span}\ P(K_{n,\beta})+2}{2n} \right\rceil \quad (*)$$

complexity grows with n $\#\mathrm{cr}(\mathrm{gs}) \sim c(D)\cdot n^2$

+ auxiliary tools $b=3$ (many) …… 村上 - 金信

$b=4$ Rurvan unitarity test → Ph: to exact real vers.

Rem: $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$ (Birman-Menasco)

Rem: for fixed n, $\max\limits_{\beta}(*)$ is determined by $\frac{\text{perm. braids } \beta}{\text{conjugacy}}$ e.g. $n=3$ $\beta = 1d, \sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1$

have determined $b(K)$ $c(K) \le 13$ prime, all $b(K)=3$ knots $c(K) \le 16$

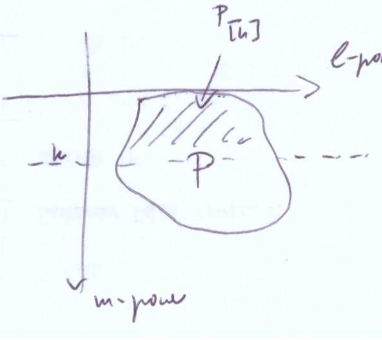
project: determine $b(K)=4$ knots

more difficult part: prohibit $b=4$

- MWF
- (1st truncated) 2cMWF ← Rem: do not use (*) (for $n=2$) in that naive way

- try to realize (find 4-braid representative) and relate by Kirbsape moves

there is a mod 4 condition on the admissible writhe of $K_{2,\beta}$ (e mod 2 fixed)



$2b \ge \mathrm{span}\, P \ge \mathrm{span}\, P_{[k]}$

$P_{[k]} = P \bmod m^k$ complexity $C^{\log C}$

$\Updownarrow$

P complexity B^C

2cMWF h=4 left 1215 15+16 crsg knts
h=6 801

15 × 9 = 135
16 × 9 = 144 crsgs

- 4-braid Burau unitarity test
- truncated 3cMWF

caveat: ∃ admissible e divisible by 3
for β = 111

$\mathrm{md}_m P = -2$

m^{-2} term gives nothing (known by theory) 801

↓

m^0 ~30s-1' — h — 801

↓

m^2 + unitarity test ~~few minutes~~ 1/2-1h - 121 18 15crsg 103 16crsg

↓

m^4 2h-4½d 9 all 16crsgs

laptop 2GHz

desktop 4.2GHz ×8

of these 9: 16_{641054} out by 2cMWF wlt truncation (h=8 ok)

$16_{1228986}$

$16_{1218750}$

$16_{1322968}$ realized as 4-braid (needed my extra pass-move tool to identify)

$16_{1377117}$

$16_{1360238}$

$16_{1239411}$ out by 3cMWF h=6

$16_{1058154}$? not found and not ruled out by 2cMWF, 3cMWF k≤6

$16_{1153788}$

101,812 non-al 15,16 crsg knts b>3 MWF≤4

67,980 realized → b=4

38,830 discardd → b>4

2 undecided

Can you take higher (n-)cable?

<u>Conjecture</u> $b(K) = \max_{n,\beta} \left\lceil \frac{MWF(K_{n,\beta})}{n} \right\rceil$

- (⟹ strong Jones conjecture proved by Menasco-L. Dynnikov-P.)
- alternating ... n=1
  ...
  fibred } Murasugi
  2-bridge
  genus ≤4 knot
  3-braid links (S.) n≤2
- but (for general n) practically useless