# Symmetric unions and essential tori

Toshifumi Tanaka
Gifu University

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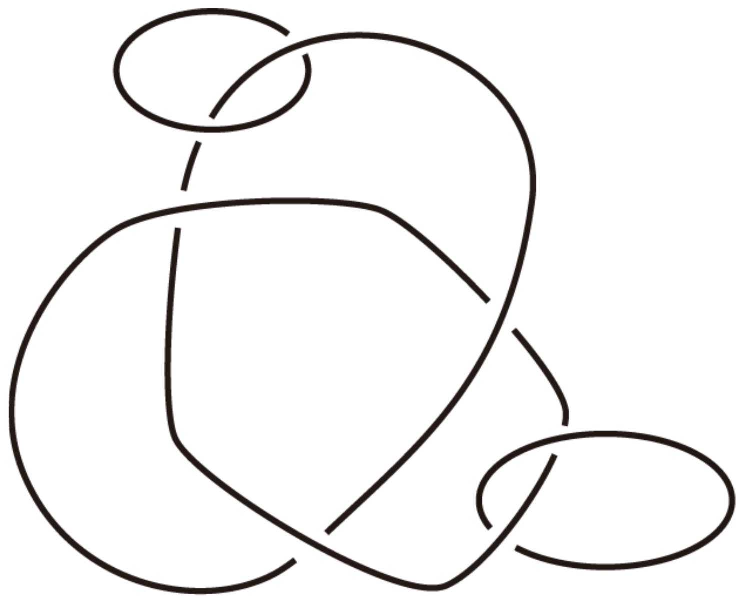
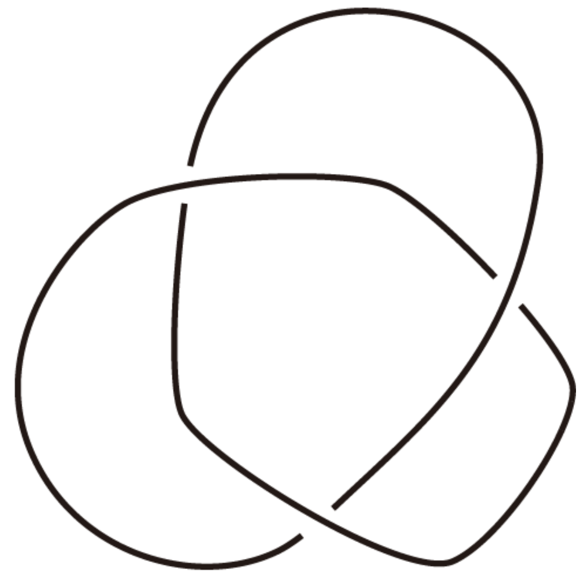
Def. A link is a disjoint union of embedded circles in $S^3$.

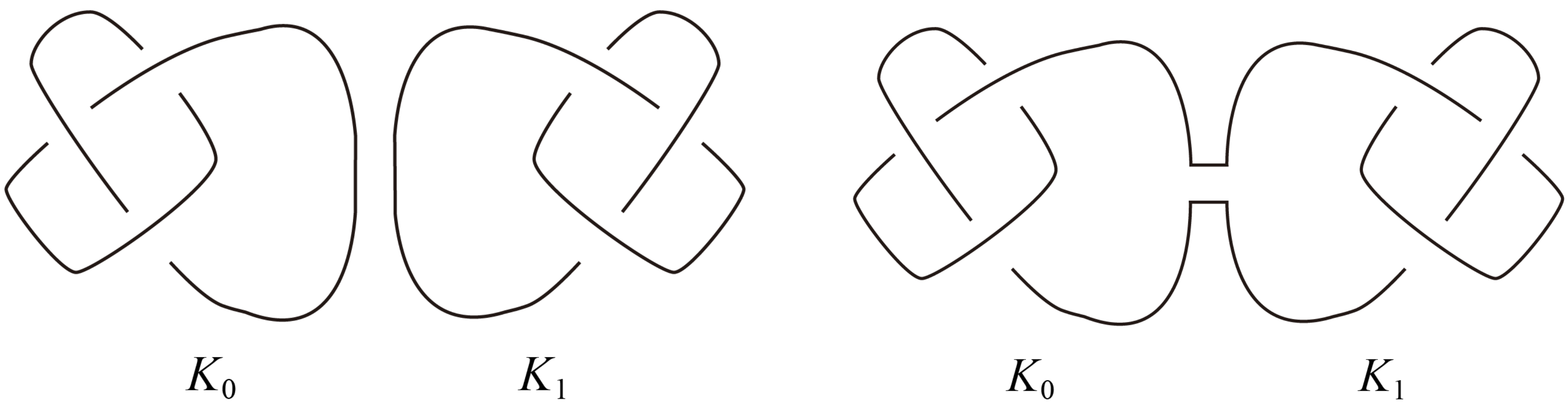


Link

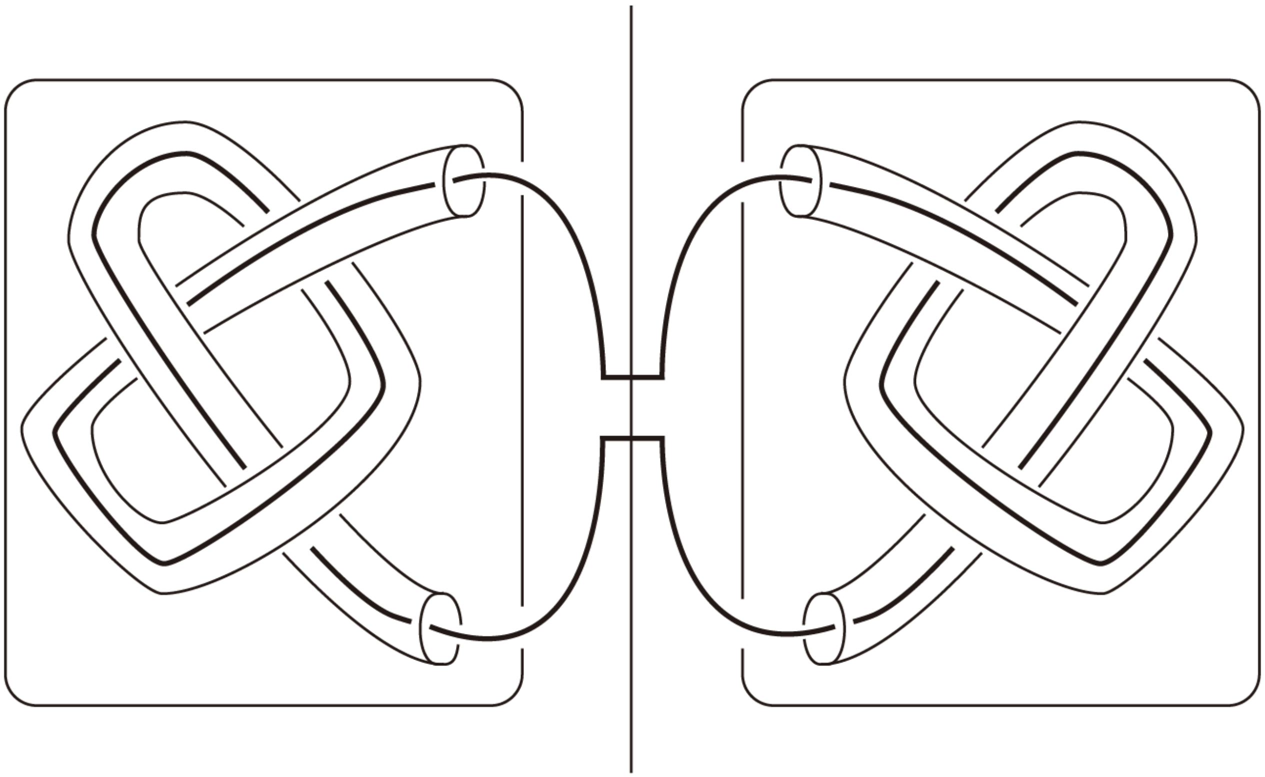


Knot

Def. If $K_1$ and $K_2$ are knots, the connected sum $K_1 \# K_2$ is defined as follows:



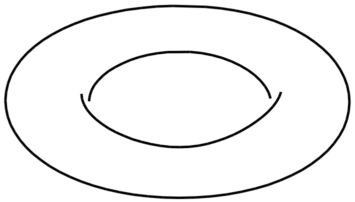
The connected sum

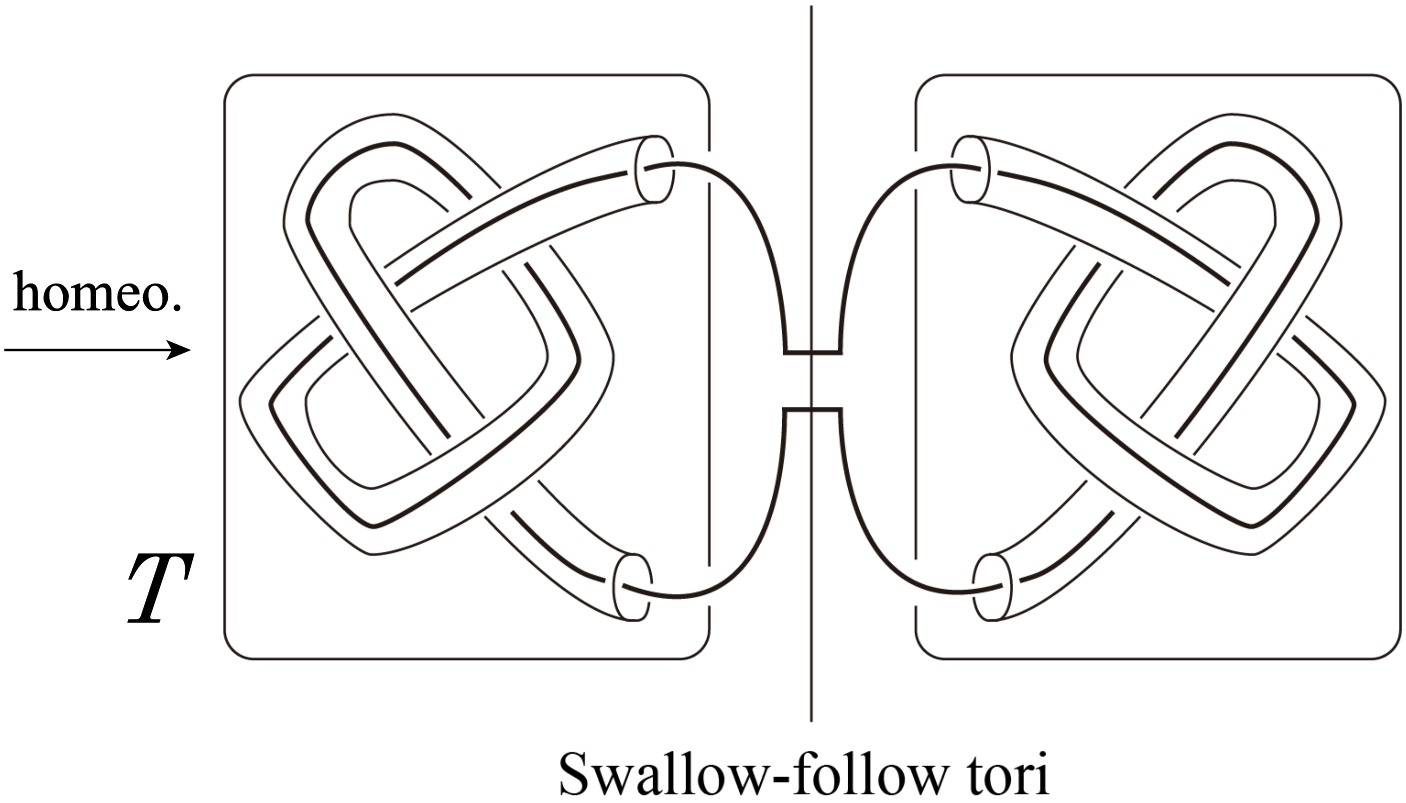


Swallow-follow tori

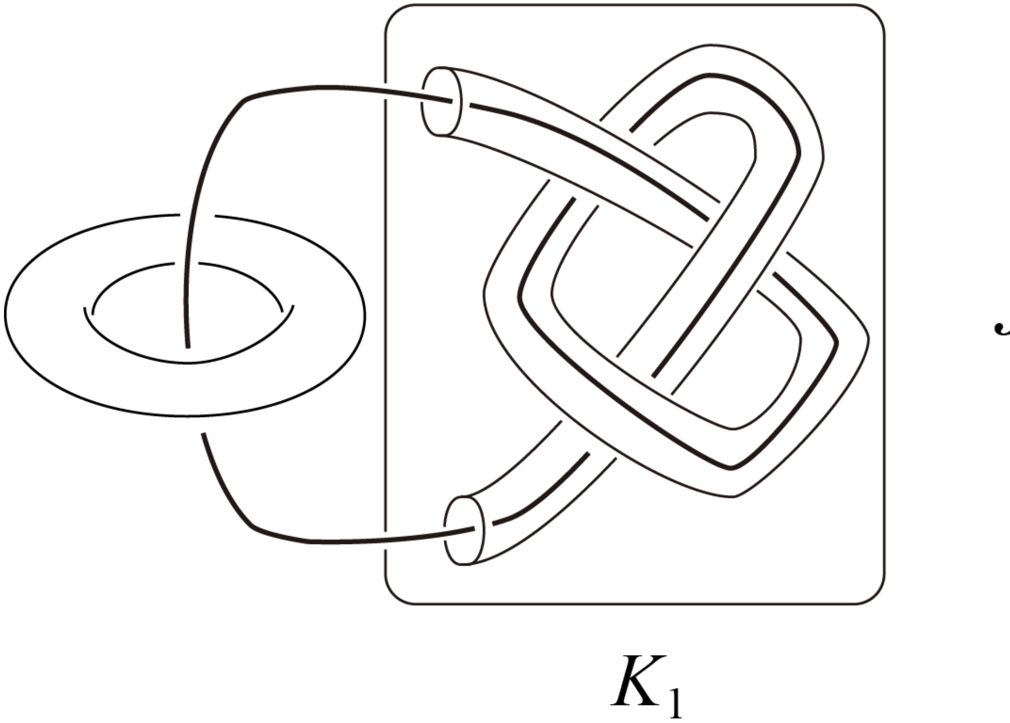
We have two non-parallel essential tori in the complement.

# The connected sum






We have two non-parallel essential tori in the complement.




We have an essential torus in the complement of the pattern link $J \cup K_1$.

# *Satellite knot*

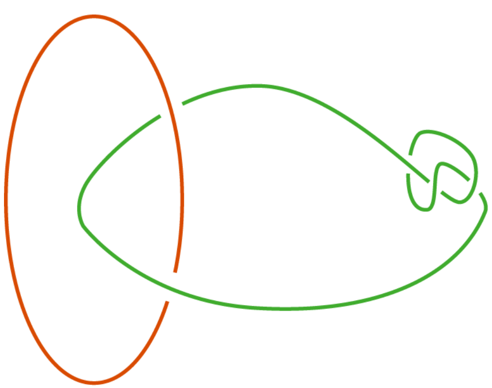
Def. Let $V_h$ be a solid torus which is the complement of the unknot $J$ in $S^3$.

Let $K_h$ be a knot in $V_h$ such that $K_h$ is a geometrically essential.

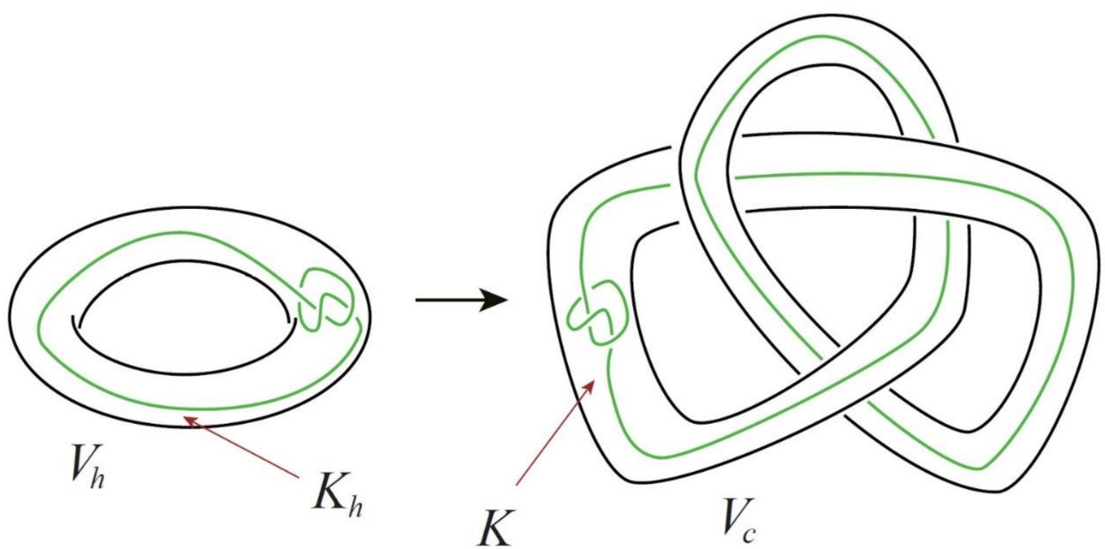
We define the order of the pair $(V_h, K_h)$ as the geometric intersection number of $K_h$ to any meridian disk of $V_h$.

Let $V_c$ be a regular neighborhood of a non-trivial knot $K_c$ in $S^3$.

We call a knot $K$ is a satellite knot if $K$ is the image $\Phi(K_h)$ for a homeo. $\Phi : V_h \longrightarrow V_c$, the order of $(V_h, K_h)$ is not zero and $K_h$ is not the core of $V_h$.

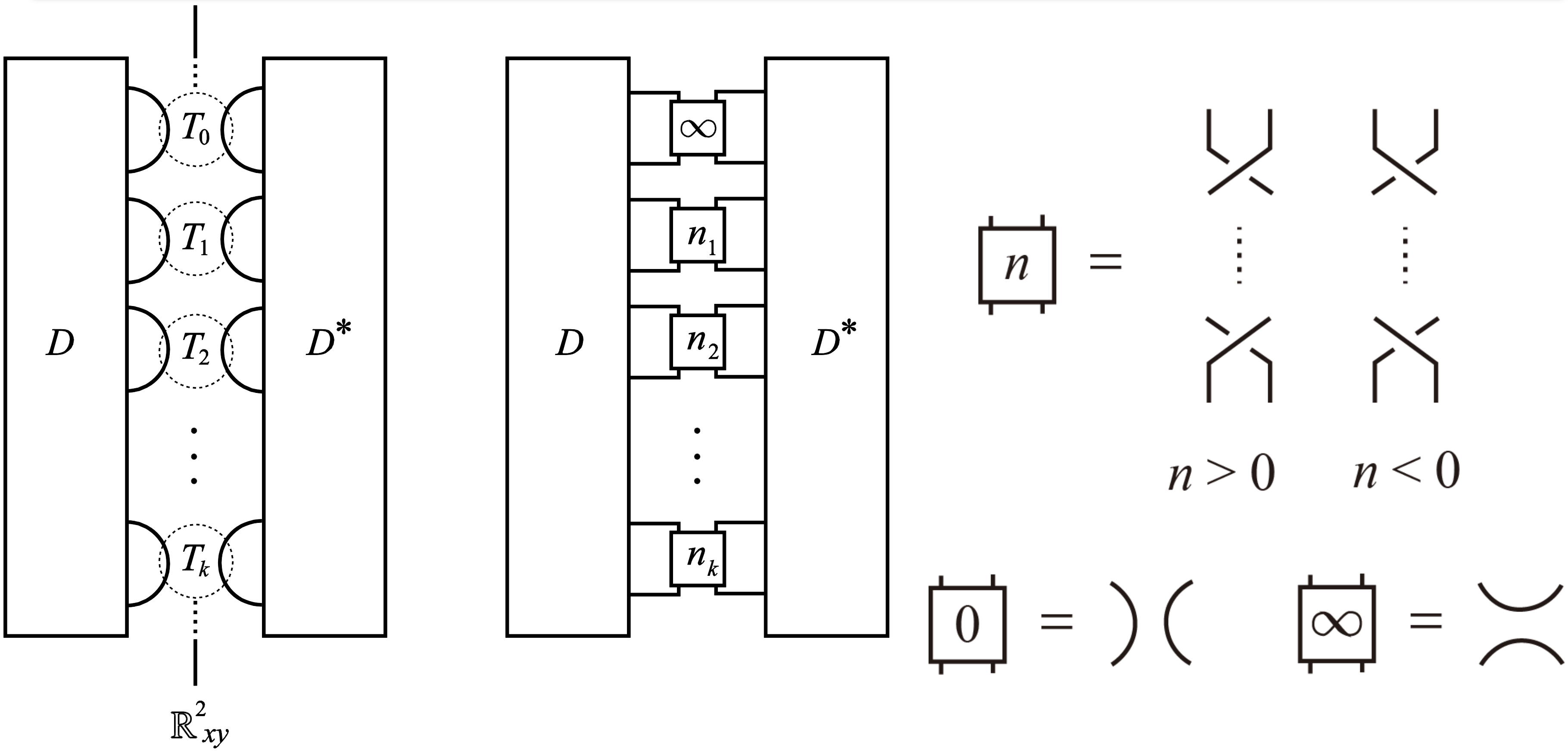


$J \cup K_h$ : *Pattern link*

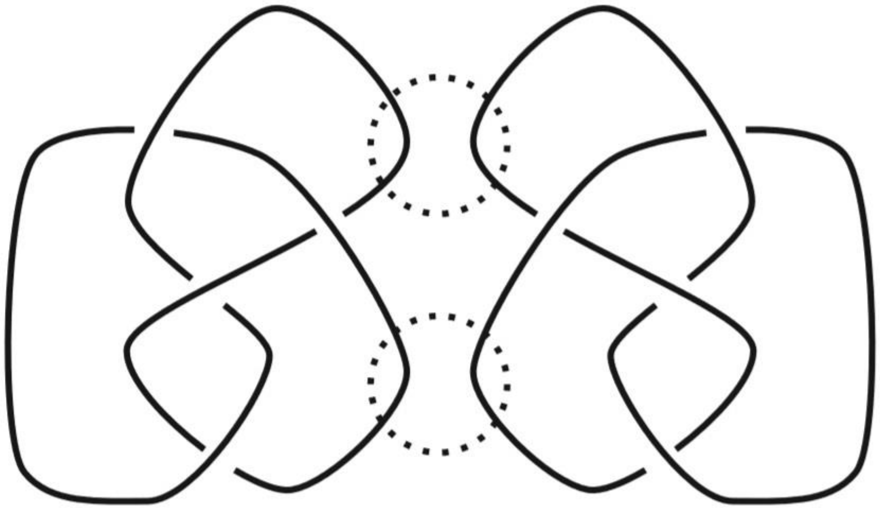


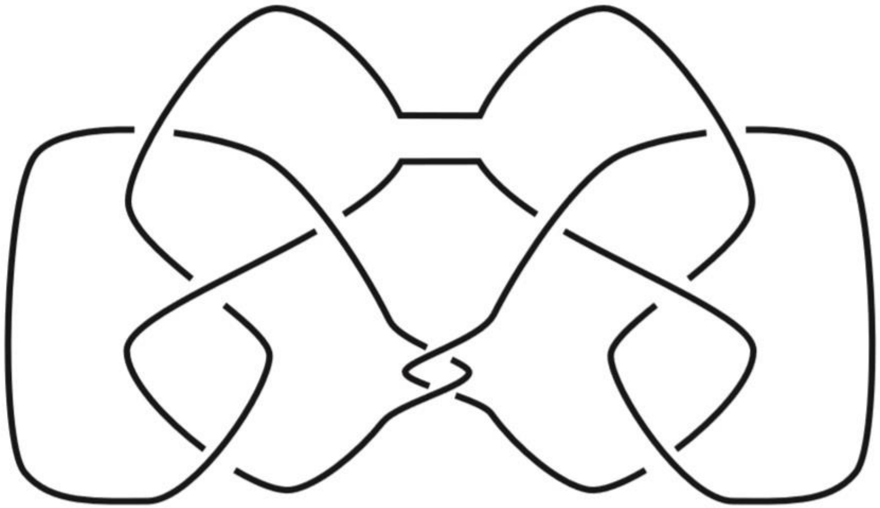
# *Symmetric Union*

*A* ***symmetric union*** $D \cup D^*(n_1, \cdots, n_k)$ $(\mathbb{Z} \ni n_i \neq \infty)$ is defined by the following diagram:



# *Symmetric Union*





Tangles

## *Property*

Fact.
(1) $D \cup D^*(n)$ ( S. Kinoshita-H. Terasaka (OMJ 1957))
(2) $D \cup D^*(n_1, \cdots, n_k)$ (C. Lamm (OJM 2000))

Fact (Lamm).
(1) Every symmetric union is a ribbon knot.

(2) $\triangle(D \cup D^*(n_1, \cdots, n_k)) = \triangle(D \cup D^*(n'_1, \cdots, n'_k))$
if $n_i \equiv n_i'$ (mod 2) for all $i$.
($\triangle$ : *Alexander polynomial*. )

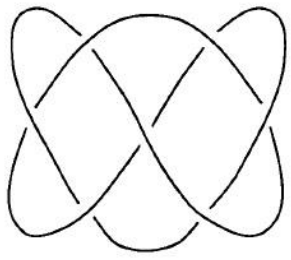
(3) $\det(D \cup D^*(n_1, \cdots, n_k)) = \det(D)^2$ .

# *Minimal twisting number*

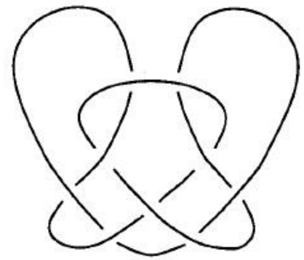
Def.

We call the number of *non-zero* elements in $\{n_1, \cdots, n_k\}$ the twisting number for $D \cup D^*(n_1, \cdots, n_k)$.

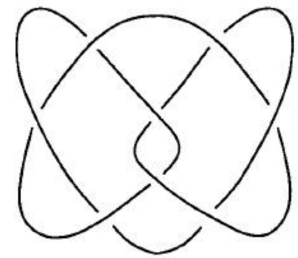
The minimal twisting number of a symmetric union $K$ is the smallest number of the twisting numbers of all symmetric union presentations to $K$ denoted by $\mathrm{tw}(K)$.



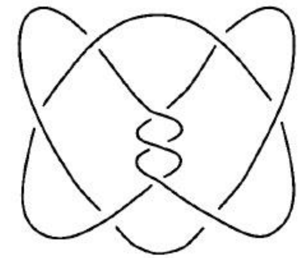
$6_1$



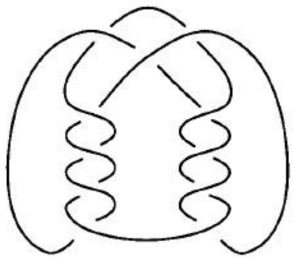
$8_8$



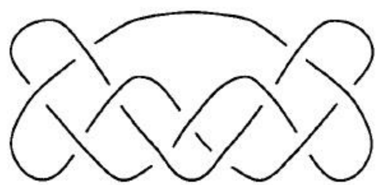
$8_{20}$



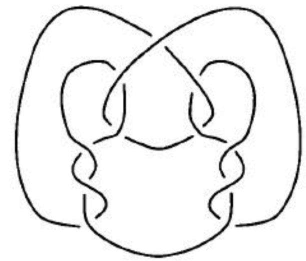
$9_{46}$



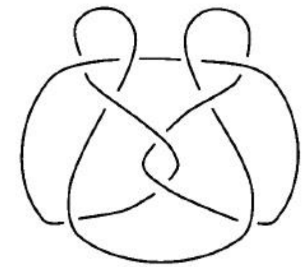
$10_3$



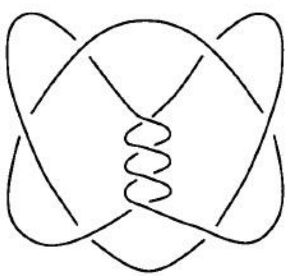
$10_{22}$



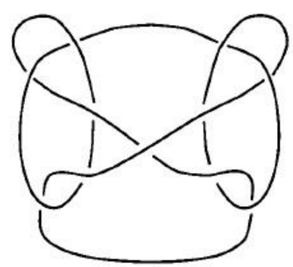
$10_{35}$



$10_{137}$



$10_{140}$



$10_{153}$

$\text{tw}\,(K) = 1.$

# *Fact*

Prop. (T. JKTR (2019)).

*Let* $K_1$ and $K_2$ be prime symmetric union with

$\mathrm{tw}(K_1) = \mathrm{tw}(K_2) = 1$.

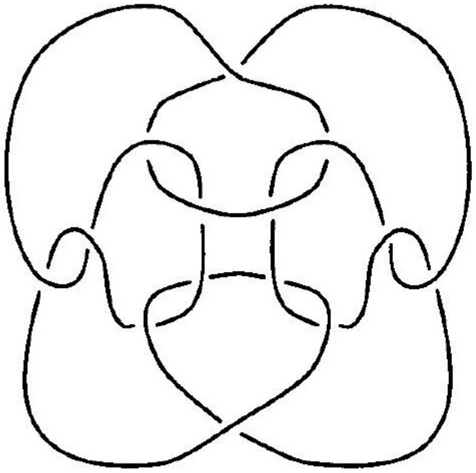
Then

$\mathrm{tw}(K_1 \# K_2) = 2$ iff $K_1$ is not the mirror image of $K_2$.

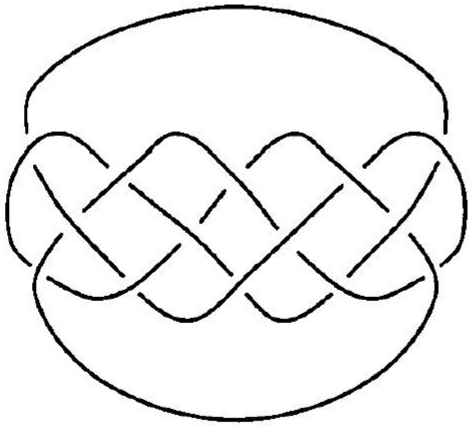
$K_1,\ K_2 \in \{6_1, 8_8, 8_{20}, 9_{46}, 10_3, 10_{22}, 10_{35}, 10_{137}, 10_{140}, 10_{153}\}$

$\Rightarrow\ \mathrm{tw}\,(K_1 \# K_2) = 2.$

Prop. (T. JKMS (2015)).
tw ($10_{99}$) = tw ($10_{123}$)= 2.



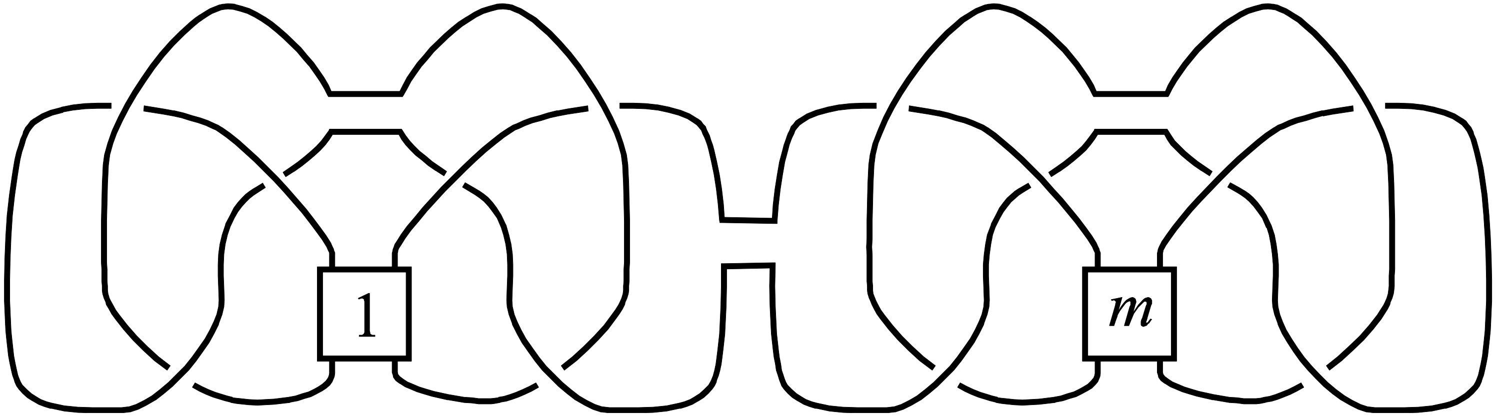
$10_{99}$



$10_{123}$

Th. (T. JKTR (2019)).

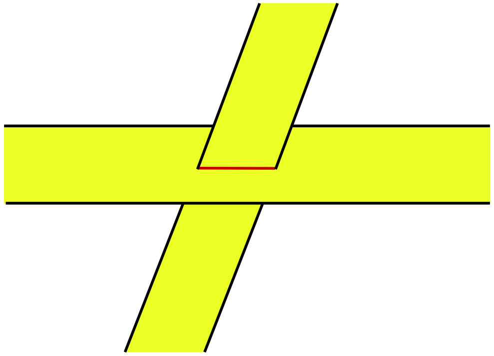
There are infinitely many symmetric unions with minimal twisting number two.

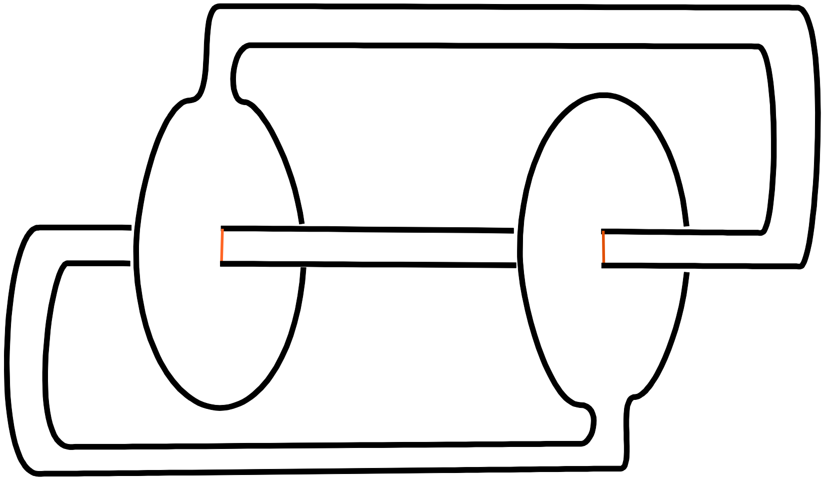



$m > 1$

# *Ribbon knot*

Def. A ribbon knot is a knot that bounds a self-intersecting disk with only ribbon singularities.





Ribbon singularity

# *Open problem*

Every symmetric union is a ribbon knot.

Problem (Lamm OJM (2000)).
Is every ribbon knot, symmetric union?

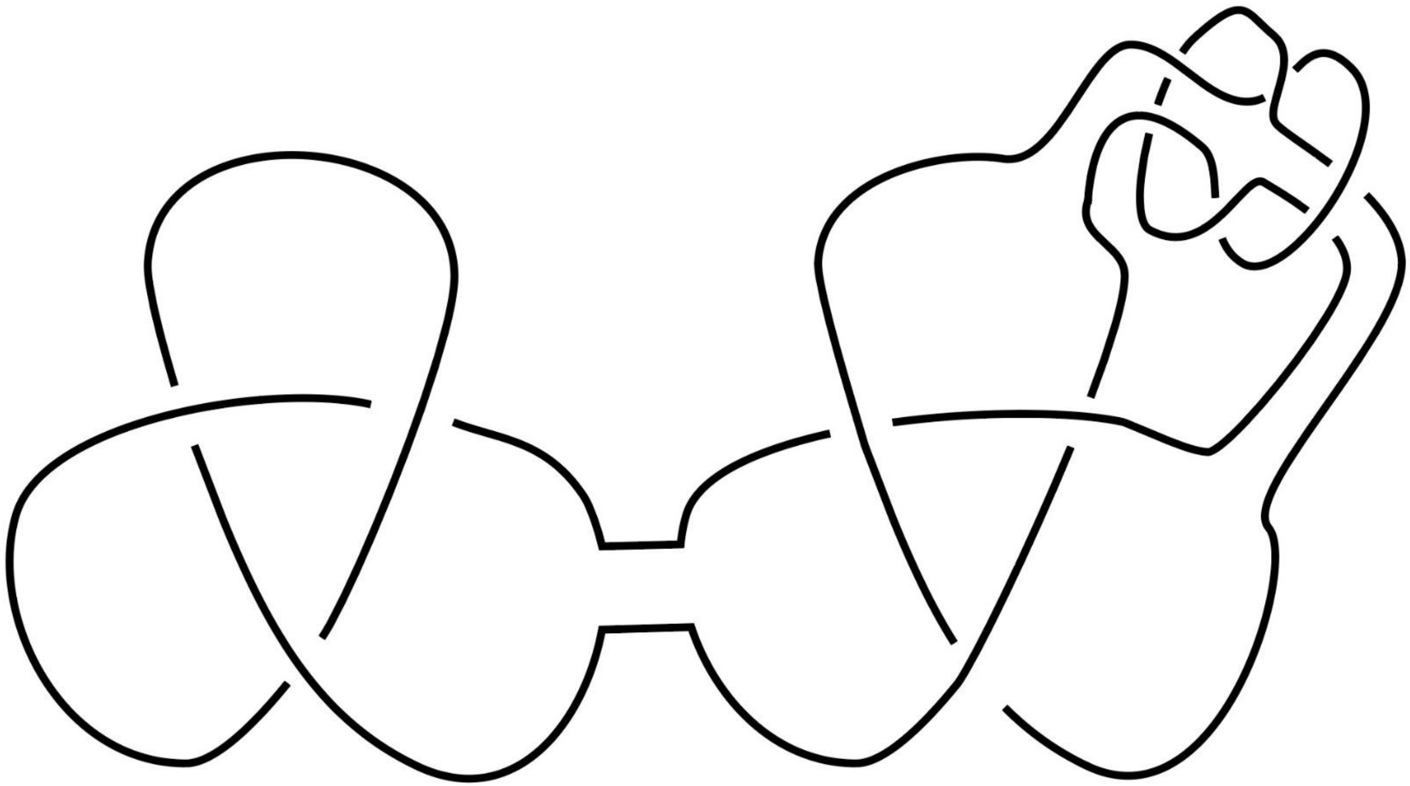
(1) Every ribbon knot with crossing number $\leqq 10$ is a symmetric union.

(2) Every two-bridge ribbon knot is a symmetric union.

C. Lamm, The search for non-symmetric ribbon knots, arXiv:1710.06909, 2017.

# *Potential counterexample*

C. Lamm, *The search for non-symmetric ribbon knots*, arXiv:1710.06909, 2017.



$3_1$ $8_{10}$

Th. (T).

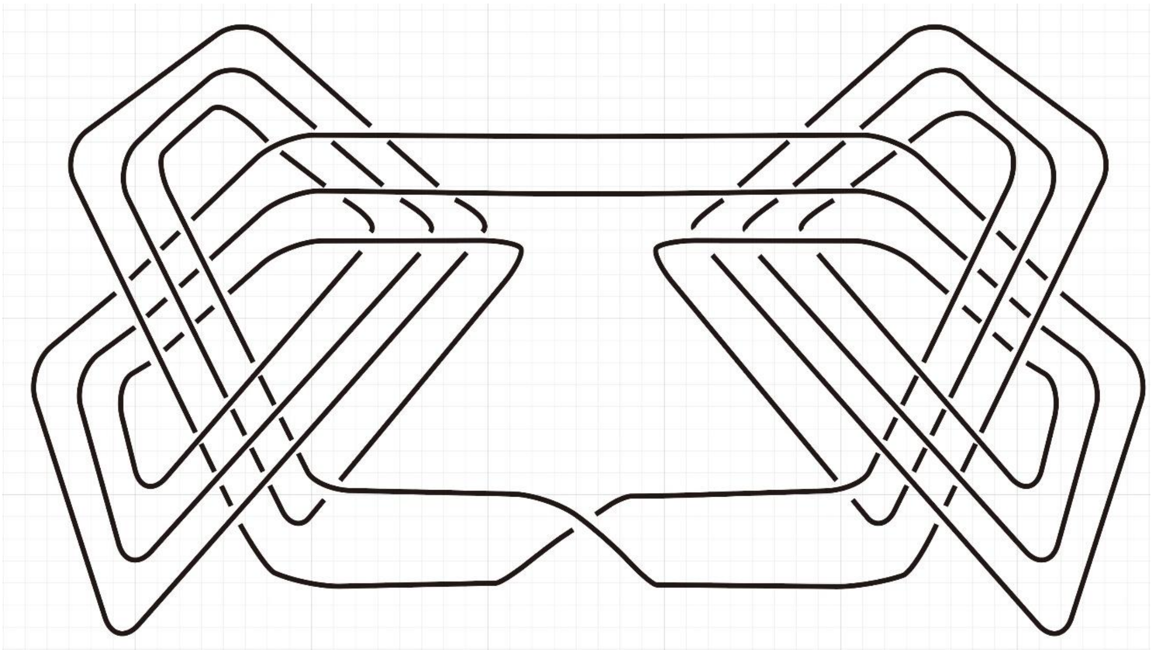
Let $K$ be a satellite symmetric union with *minimal twisting number one*.

If the *order* of the pattern of $K$ is an odd number $\geqq 3$, then the complement of $K$ has two disjoint non-parallel *essential tori* which are symmetric with respect to a plane.

In particular, the pattern link complement contains an *essential torus*.

Conj. The condition for minimum twisting number is unnecessary.

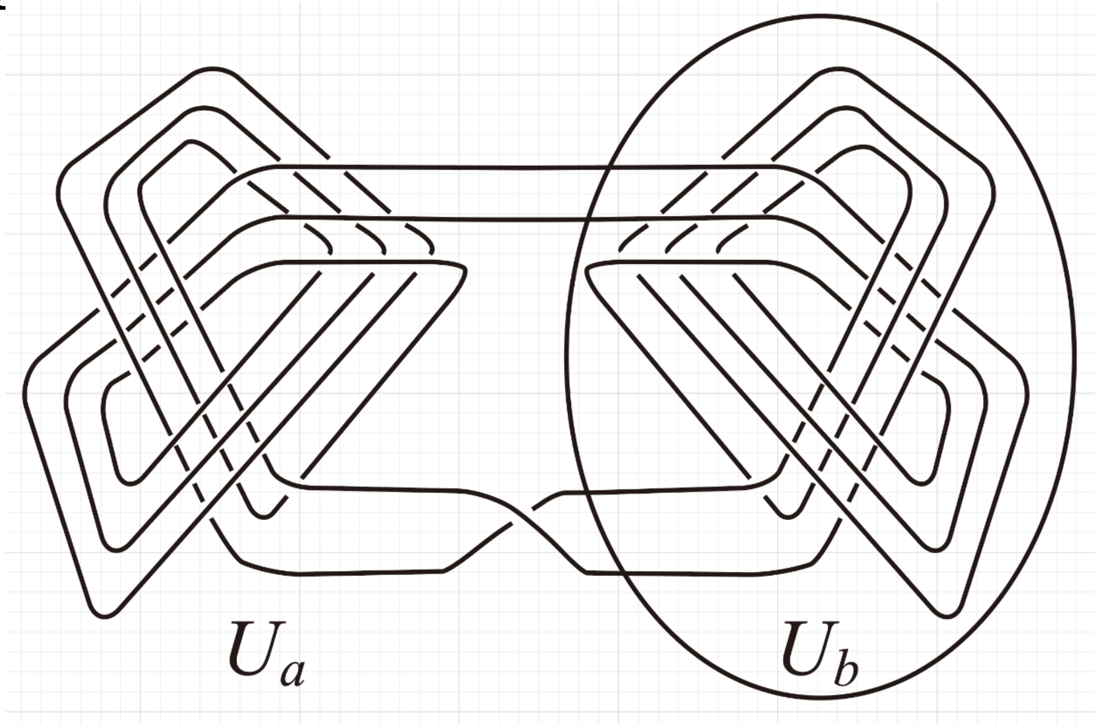
# *Example*



$K$

$K$ is a symmetric union with minimal twisting number $\leq 1$.

# *Example*

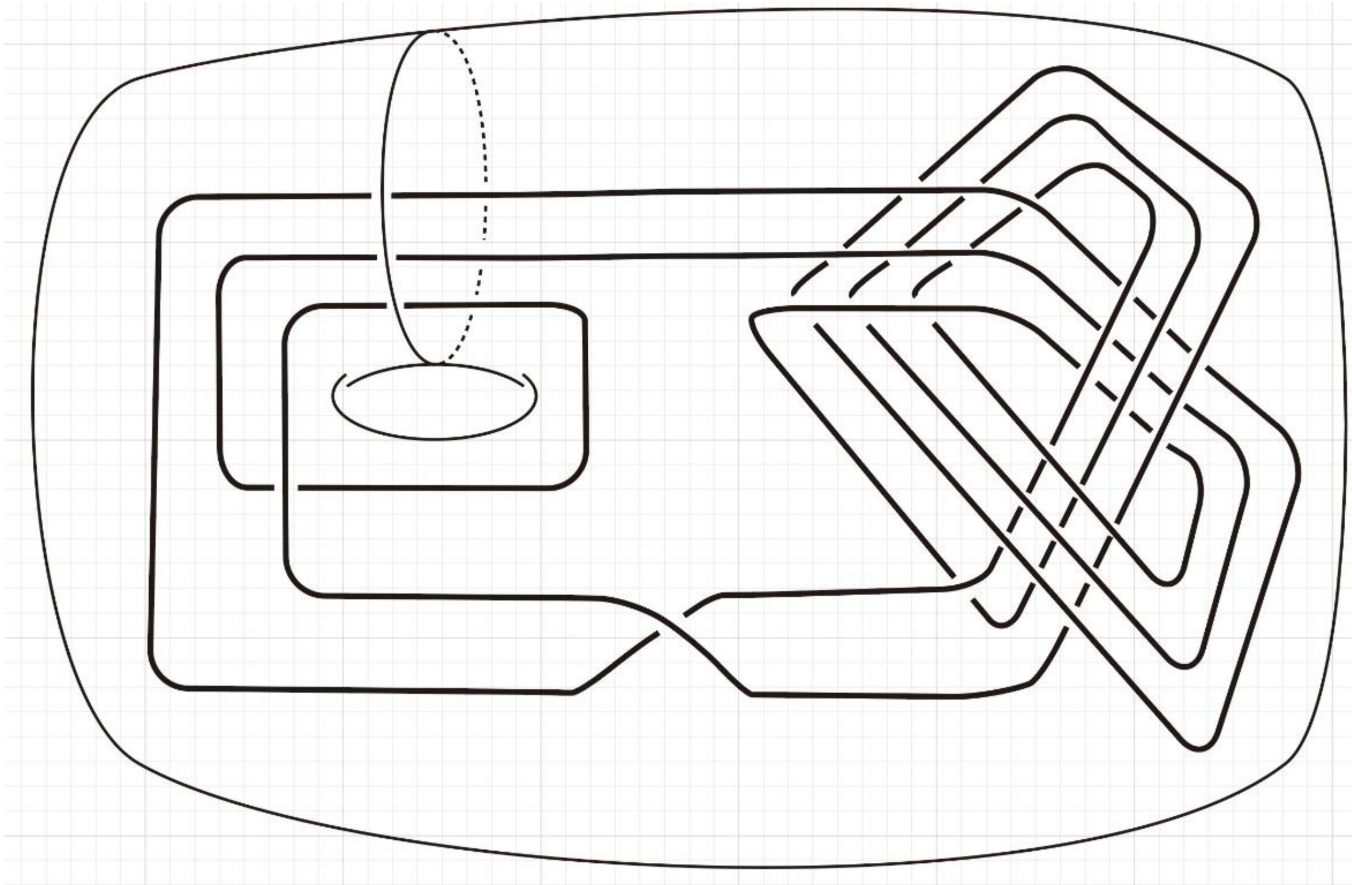



$K$

W. B. R. Lickorish, Prime knots and tangles,
Trans. Amer. Math. Soc. 267 (1981)

$K$ is a prime knot $\Rightarrow$ $K$ has minimal twisting number one.

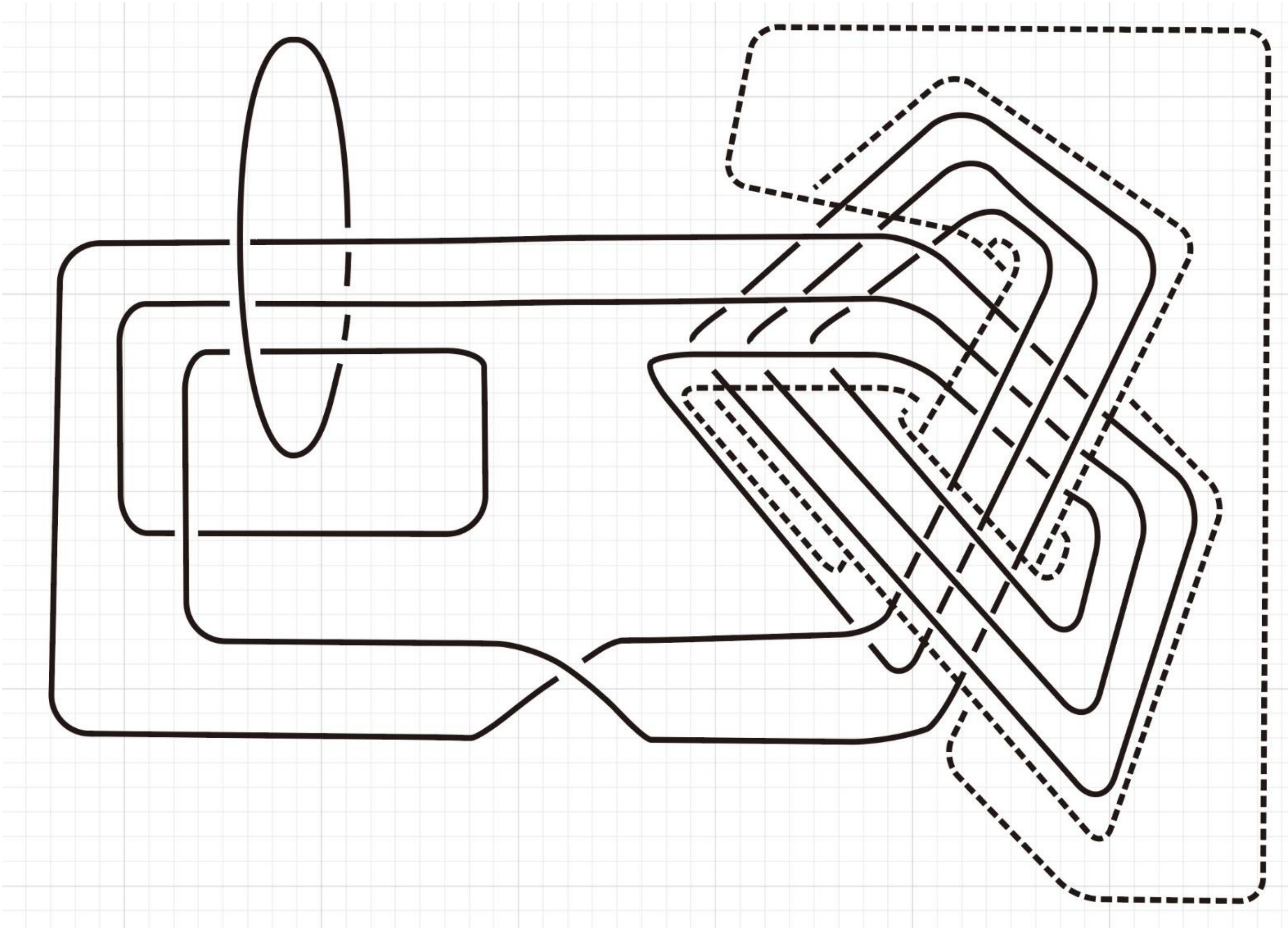
## *Example*



Pattern

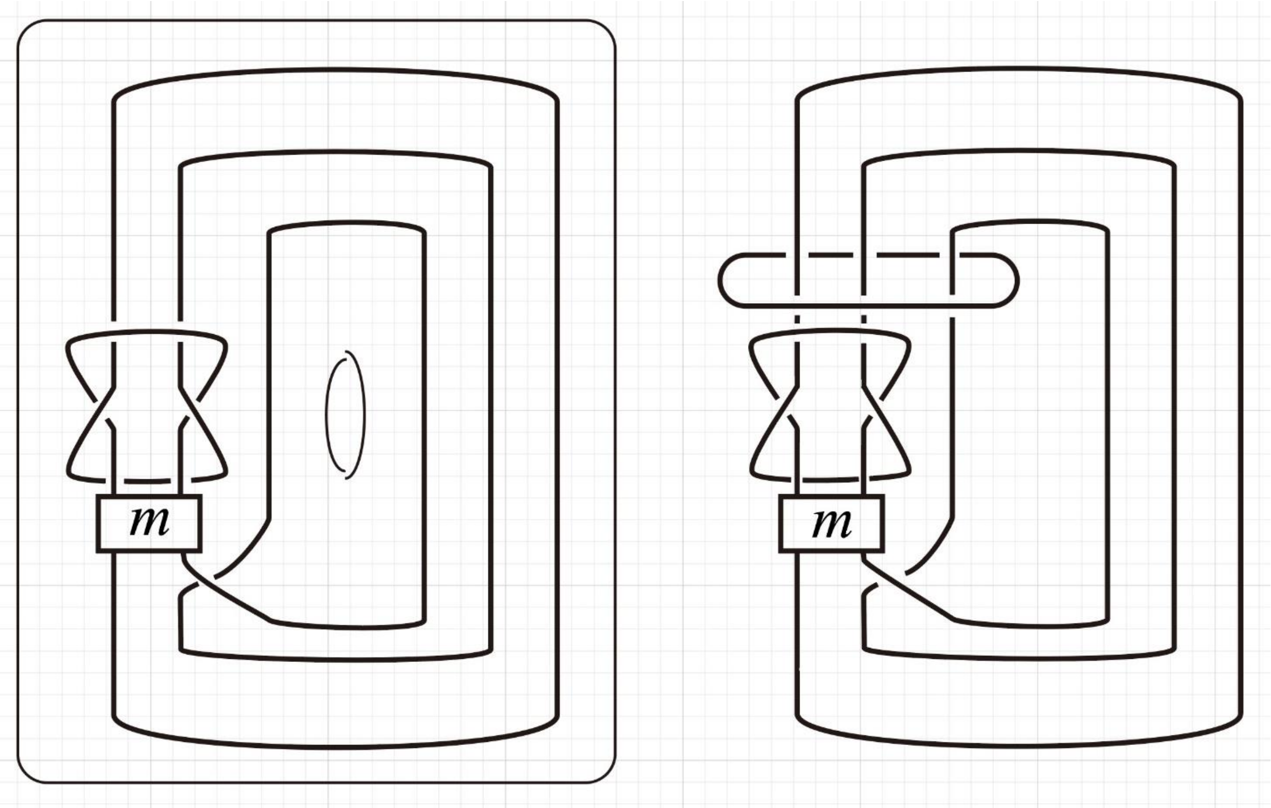
The order is 3.

*Example*



Pattern link

# *Example*

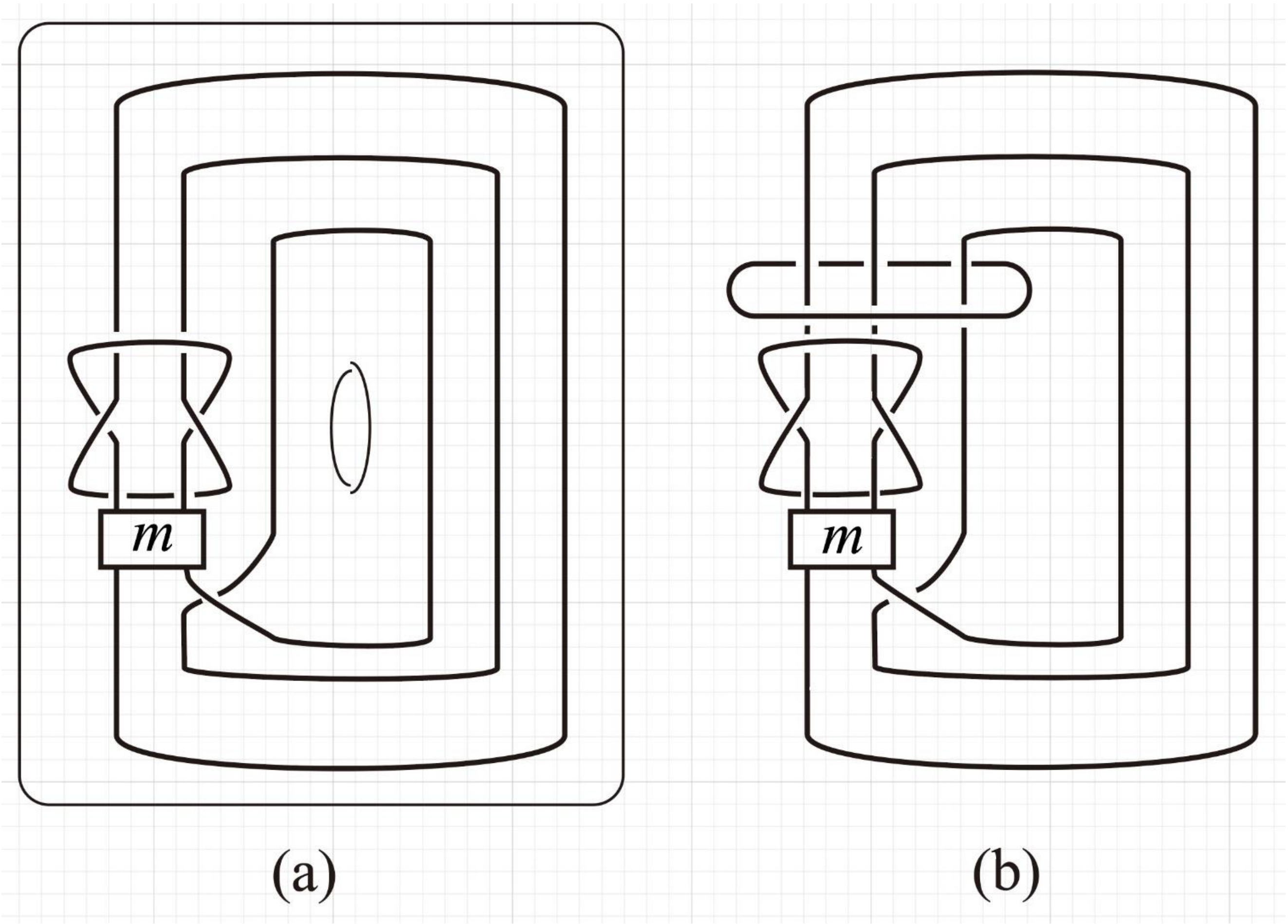


(a) (b)

*The order of this pattern* is 3.

*Let* $K_c$ be a non-trivial ribbon knot as a companion knot
$\Rightarrow$ the satellite knot $K$ is ribbon knot.

# *Example*




The computer program *HIKMOT*, which is integrated into *SnapPy* by M. Culler, N. Dunfield and J. Weeks, shows that the pattern link $J \cup K_h$ is hyperbolic and in particular it does not contain an essential torus.

Theorem $\Rightarrow$ The satellite knot $K$ is not a symmetric union with minimal twisting number one.

# *Reference*


1. S. Kinoshita and H.Terasaka, On unions of knots, Osaka J. Math. Vol. 9 (1957), 131-153.

2. C. Lamm, *Symmetric unions and ribbon knots*, Osaka J. Math., Vol. 37 (2000), 537-550.

3. T. Tanaka, *The Jones polynomial of knots with symmetric union presentations*, J. Korean Math. Soc. 52 (2015), no. 2, 389-402.

4. C. Lamm, The search for non-symmetric ribbon knots, arXiv:1710.06909, 2017.

5. T. Tanaka, On composite knots with symmetric union presentations, J. Knot Theory Ramifications 28 (2019), no. 10.


Thank you for your attention