Symmetric unions and essential tori

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Def. A **link** is a disjoint union of embedded circles in $S^3$. 

Link

Knot
Def. If $K_1$ and $K_2$ are knots, the connected sum $K_1 \# K_2$ is defined as follows:
The connected sum

Swallow-follow tori

We have two non-parallel essential tori in the complement.
The connected sum

We have two non-parallel essential tori in the complement.

Swallow-follow tori

We have two non-parallel essential tori in the complement.
We have an essential torus in the complement of the pattern link $J \cup K_1$. 
**Satellite knot**

Def. Let $V_h$ be a solid torus which is the complement of the unknot $J$ in $S^3$. Let $K_h$ be a knot in $V_h$ such that $K_h$ is a *geometrically essential*. We define the order of the pair $(V_h, K_h)$ as the geometric intersection number of $K_h$ to any meridian disk of $V_h$.

Let $V_c$ be a regular neighborhood of a non-trivial knot $K_c$ in $S^3$. We call a knot $K$ is a *satellite knot* if $K$ is the image $\Phi(K_h)$ for a homeo. $\Phi : V_h \longrightarrow V_c$, the order of $(V_h, K_h)$ is not zero and $K_h$ is not the core of $V_h$.

$J \cup K_h : \textit{Pattern link}$
A symmetric union $D \cup D^*(n_1, \ldots, n_k)$ ($\mathbb{Z} \ni n_i \neq \infty$) is defined by the following diagram:
Symmetric Union

Tangles
Property

Fact.
(1) \( D \cup D * (n) \) ( S. Kinoshita-H. Terasaka (OMJ 1957))
(2) \( D \cup D *(n_1, \cdots, n_k) \) (C. Lamm (OJM 2000))

Fact (Lamm).
(1) Every symmetric union is a ribbon knot.

(2) \( \triangle(D \cup D *(n_1, \cdots, n_k)) = \triangle(D \cup D *(n_1', \cdots, n_k')) \)
    if \( n_i \equiv n_i' \) (mod 2) for all \( i \).
    (\( \triangle \) : Alexander polynomial.)

(3) \( \det(D \cup D *(n_1, \cdots, n_k)) = \det(D)^2 \).
Minimal twisting number

Def.
We call the number of non-zero elements in \( \{n_1, \cdots, n_k\} \) the twisting number for \( D \cup D^*(n_1, \cdots, n_k) \).
The minimal twisting number of a symmetric union \( K \) is the smallest number of the twisting numbers of all symmetric union presentations to \( K \) denoted by \( \text{tw}(K) \).
$\text{tw} \ (K) = 1.$
Fact

Prop. (T. JKTR (2019)).

Let $K_1$ and $K_2$ be prime symmetric union with $\text{tw}(K_1) = \text{tw}(K_2) = 1$.

Then

$$\text{tw}(K_1 \# K_2) = 2 \iff K_1 \text{ is not the mirror image of } K_2.$$
Prop. (T. JKMS (2015)).
\[
tw (10_{99}) = tw (10_{123}) = 2.
\]
Th. (T. JKTR (2019)).

There are infinitely many symmetric unions with minimal twisting number two.
**Ribbon knot**

Def. A ribbon knot is a knot that bounds a self-intersecting disk with only ribbon singularities.

Ribbon singularity
Open problem

Every symmetric union is a ribbon knot.

Problem (Lamm OJM (2000)). Is every ribbon knot, symmetric union?

(1) Every ribbon knot with crossing number $\leq 10$ is a symmetric union.

(2) Every two-bridge ribbon knot is a symmetric union.

Potential counterexample

Th. (T).
Let $K$ be a satellite symmetric union with *minimal twisting number one*.
If the *order* of the pattern of $K$ is an odd number $\geq 3$, then the complement of $K$ has two disjoint non-parallel *essential tori* which are symmetric with respect to a plane.
In particular, the pattern link complement contains an *essential torus*.

**Conj.** The condition for minimum twisting number is unnecessary.
Example

$K$ is a symmetric union with minimal twisting number $\leq 1$. 
Example

$K$ is a prime knot $\Rightarrow$ $K$ has minimal twisting number one.


$K$ is a prime knot $\Rightarrow$ $K$ has minimal twisting number one.
Example

Pattern

The order is 3.
Example

Pattern link
Let $K_c$ be a non-trivial ribbon knot as a companion knot
⇒ the satellite knot $K$ is ribbon knot.

The order of this pattern is 3.
Example

The computer program HIKMOT, which is integrated into SnapPy by M. Culler, N. Dunfield and J. Weeks, shows that the pattern link $J \cup K_h$ is hyperbolic and in particular it does not contain an essential torus.

Theorem $\Rightarrow$ The satellite knot $K$ is not a symmetric union with minimal twisting number one.
Reference


Thank you for your attention