On the Γ-polynomial and its cabling for knots

Hideo Takioka

JSPS Research Fellow, Kyoto University

Knots and Spatial Graphs 2020,
A workshop in memory of Choon Bae Jeon

KAIST, Daejeon, Korea

February 18, 2020

Outline of my presentation

1. **Cabling for knot invariants**
2. **Polynomial invariants $\nabla, V, \Gamma, Q, P, F$**
3. **Γ-polynomial**
4. **$\Gamma_{2/q}$-polynomial**
5. **$\Gamma_{3/q}$-polynomial**
6. **Vassiliev knot invariants derived from $\Gamma_{p/q}$-polynomials**

1. Cabling for knot invariants

Cable knot

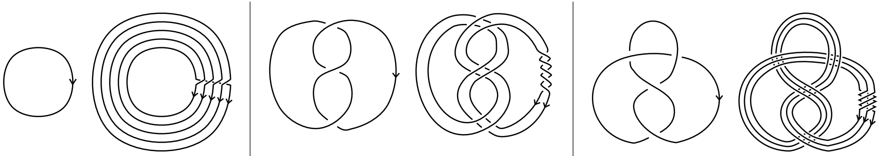
$p(>0)$, q: coprime integers.

K: a knot. $N(K)$: a tubular neighborhood of K.

$K^{(p,q)}$: the (p,q)-cable knot of K, that is, an essential loop in $\partial N(K)$ with $[K^{(p,q)}] = p[l] + q[m]$ in $H_1(\partial N(K); \mathbb{Z})$, where (m,l) is a meridian-longitude pair of K with $\mathrm{lk}(K \cup l) = 0$ and $\mathrm{lk}(K \cup m) = +1$.

D: a diagram of K. $w(D)$: the writhe of D.

$K^{(p,q)}$ has a diagram $D^{(p,q)}$ which consists of the p-parallel of D and the p-braid $(\sigma_1 \cdots \sigma_{p-1})^{q-pw(D)}$.



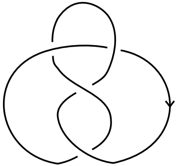
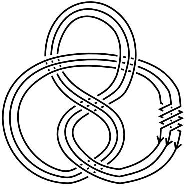
D $\quad D^{(5,1)}$ $\quad D$ $\quad D^{(2,1)}$ $\quad D$ $\quad D^{(3,4)}$

Cabling for knot invariants

$p(>0)$, q: coprime integers.

I: a knot invariant.

The map sending a knot K to the value $I(K^{(p,q)})$ is also a knot invariant, which is called the (p, q)-cabling of I denoted by $I_{p/q}$.

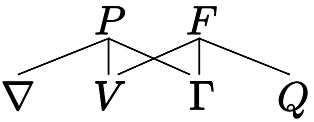
$I_{3/4}($  $) = I($  $).$

K $\qquad$ $K^{(3,4)}$

In this talk, we focus on cabling for polynomial invariants.

2. Polynomial invariants

$\nabla, V, \Gamma, Q, P, F$



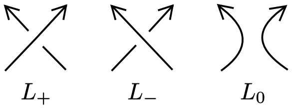
HOMFLYPT polynomial (P)

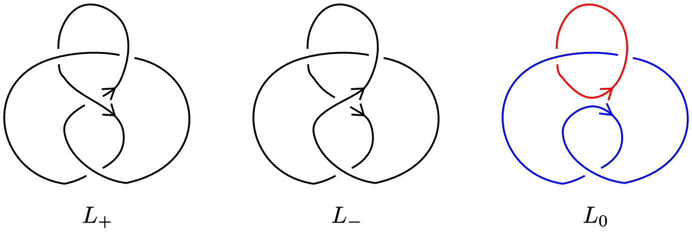
$P(L;t,z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$ **is an invariant for oriented links.**

For the trivial knot $\bigcirc$, we have $P(\bigcirc) = 1$.

For a skein triple (L_+, L_-, L_0), we have

$$t^{-1}P(L_+) - tP(L_-) = zP(L_0).$$





Kauffman polynomial (F)

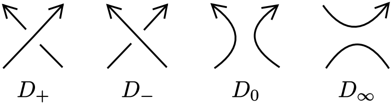
$F(L; a, z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ **is an invariant for oriented links.**

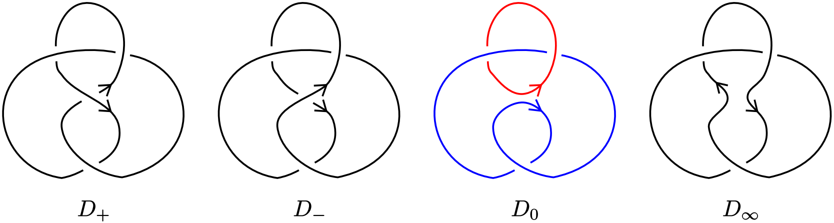
For the trivial knot $\bigcirc$, we have $F(\bigcirc) = 1$.

For a skein quadruple $(D_+, D_-, D_0, D_\infty)$, we have

$$aF(D_+) + a^{-1}F(D_-) = z(F(D_0) + a^{-2\nu}F(D_\infty)),$$

where $2\nu = w(D_+) - w(D_\infty) - 1$ and $w(D_+)$, $w(D_\infty)$ are the writhes of D_+, D_∞, respectively.





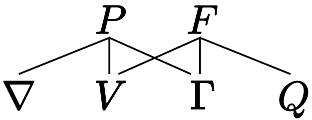
Alexander-Conway (∇), Jones (V), Q-polynomials

$$\nabla(L; z) = P(L; 1, z) \in \mathbb{Z}[z^{\pm 1}].$$

$$V(L; t) = P(L; t, t^{1/2} - t^{-1/2})$$

$$= F(L; -t^{-3/4}, t^{1/4} + t^{-1/4}) \in \mathbb{Z}[t^{\pm 1}].$$

$$Q(L; x) = F(L; 1, x) \in \mathbb{Z}[x^{\pm 1}].$$



Γ-polynomial

L: an oriented r-component link.

$$P(L) = (-t^{-1}z)^{-r+1} \sum_{i \geq 0} P_{2i}(L;t) z^{2i},$$

$$F(L) = (az)^{-r+1} \sum_{i \geq 0} F_i(L;a) z^i,$$

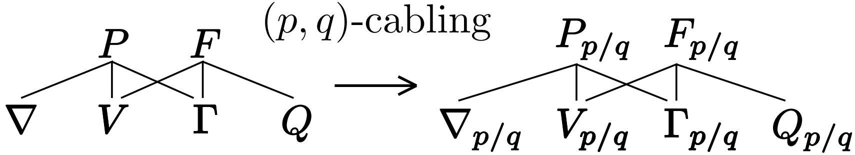
where $P_{2i}(L;t) \in \mathbb{Z}[t^{\pm 1}]$ and $F_i(L;a) \in \mathbb{Z}[a^{\pm 1}]$ are called the $2i$th coefficient polynomial of $P(L)$ and the ith coefficient polynomial of $F(L)$, respectively.

Fact [Lickorish 1988] $P_0(L;t) = F_0(L;\sqrt{-1}t^{-1})$.

$P_0(L;t)$ is a Laurent polynomial in t^{-2}. Putting $t^{-2} = x$, we call it the Γ-polynomial $\Gamma(L;x) \in \mathbb{Z}[x^{\pm 1}]$, that is,

$$\Gamma(L;t^{-2}) = P_0(L;t) = F_0(L;\sqrt{-1}t^{-1}).$$

Cabling for the polynomial invariants



3. Γ-polynomial

Skein relation of the Γ-polynomial

For the trivial knot $\bigcirc$, we have $\Gamma(\bigcirc) = 1$.

For a skein triple (L_+, L_-, L_0), we have

$$-x\Gamma(L_+) + \Gamma(L_-) = \begin{cases} \Gamma(L_0) & \text{if } \mu = 0, \\ 0 & \text{if } \mu = 1, \end{cases}$$

where $\mu = (r_+ - r_0 + 1)/2 \ (= 0, 1)$ for the numbers r_+, r_0 of components of L_+, L_0, respectively.

Proposition Let $L = K_1 \cup \cdots \cup K_r$ be an r-component link and $\mathrm{lk}(L)$ the total linking number of L. Then we have

$$\Gamma(L) = (1 - x)^{r-1} x^{-\mathrm{lk}(L)} \Gamma(K_1) \cdots \Gamma(K_r).$$

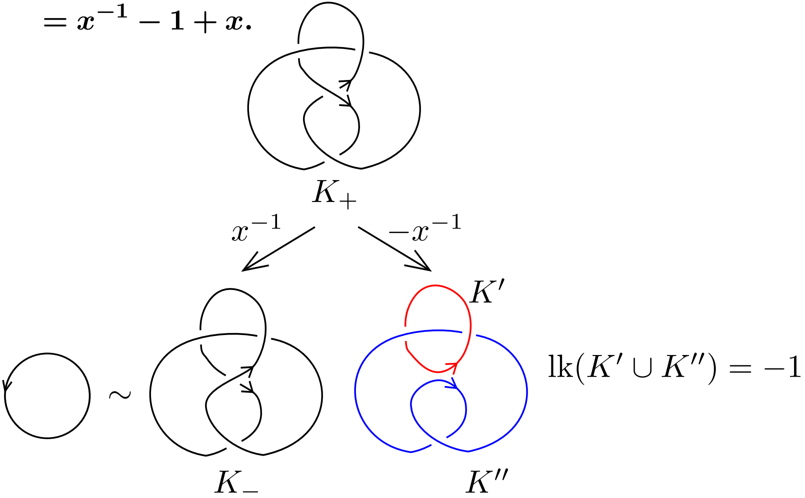
We obtain a special skein relation for a knot as follows:

$$-x\Gamma(K_+) + \Gamma(K_-) = (1 - x) x^{-\mathrm{lk}(K' \cup K'')} \Gamma(K') \Gamma(K''),$$

where $(K_+, K_-, K_0 = K' \cup K'')$ is a skein triple such that K_+, K_-, K', and K'' are knots.

Example

$$\Gamma(K_+) = x^{-1}\Gamma(K_-) - x^{-1}(1-x)x^{-\operatorname{lk}(K'\cup K'')}\Gamma(K')\Gamma(K'')$$
$$= x^{-1}\Gamma(\bigcirc) - x^{-1}(1-x)x^{-(-1)}\Gamma(\bigcirc)\Gamma(\bigcirc)$$
$$= x^{-1} - 1 + x.$$



Facts on the Γ-polynomial

Fact [Kawauchi 1994] Let $\mathcal{K}$ be the set of oriented knots. The image of $\mathcal{K}$ under Γ is the following:

$$\Gamma(\mathcal{K}) = \{1 + (1-x)^2 f(x) \mid f(x) \in \mathbb{Z}[x^{\pm 1}]\}.$$

Fact [Fujii 1999] Let $\mathcal{K}'$ be the set of 2-bridge knots with unknotting number one. Then we have

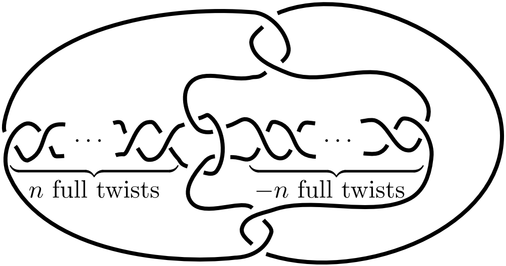
$$\Gamma(\mathcal{K}') = \Gamma(\mathcal{K}).$$

Fact [Morton-Ryder 2009] There exists a pair of genus two mutant knots such that their colored Jones polynomials coincide for all colors and their Γ-polynomials differ.

Fact [Przytycki 2017] Let n be the number of crossings of a diagram of a knot K. Then $\Gamma(K)$ can be computed in quadratic time (i.e. in $\mathcal{O}(n^2)$ time).

4. $\Gamma_{2/q}$-polynomial

Kanenobu knots $k(n)$ $(n \geq 0)$



$k(n)$

<u>Fact</u> [Kanenobu 1986] For any n, we have

$$P(k(n)) = P(k(0)) = (t^{-2} - 1 + t^2 - z^2)^2.$$

<u>Fact</u> [T. 2013] The $\Gamma_{2/1}$-polynomial classifies Kanenobu knots $k(n)$ $(n \geq 0)$ completely as follows:

$$\Gamma_{2/1}\big(k(0)\big) = 5x^4 - 22x^3 + 48x^2 - 60x + 39 \\ + 4x^{-1} - 34x^{-2} + 34x^{-3} - 17x^{-4} + 4x^{-5},$$

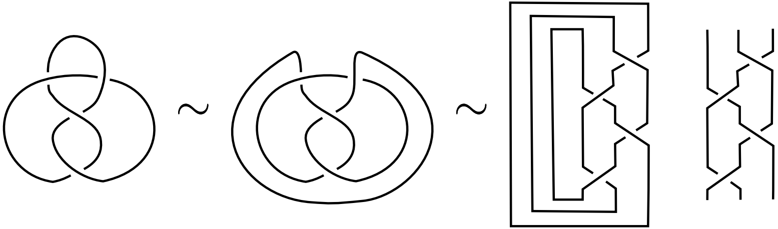
$$\Gamma_{2/1}\big(k(n)\big) - \Gamma_{2/1}\big(k(n-1)\big) \\ = -2x^{n+2} + 8x^{n+1} - 10x^n + 10x^{n-2} - 8x^{n-3} + 2x^{n-4} \\ + 2x^{-n+3} - 8x^{-n+2} + 10x^{-n+1} - 10x^{-n-1} + 8x^{-n-2} - 2x^{-n-3}.$$

We see that the difference between the maximum and minimum degrees of $\Gamma_{2/1}(k(n))$ is the following:

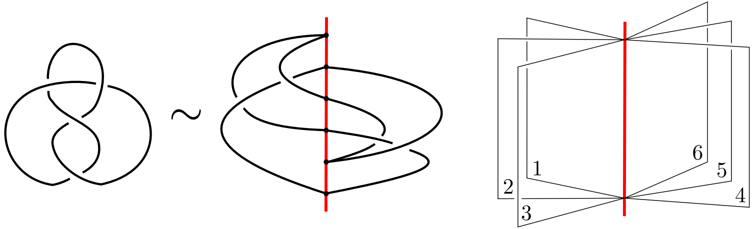
$$\operatorname{span}\Gamma_{2/1}(k(n)) = \begin{cases} 9 & (n = 0, 1, 2), \\ 2n + 5 & (n \geq 3). \end{cases}$$

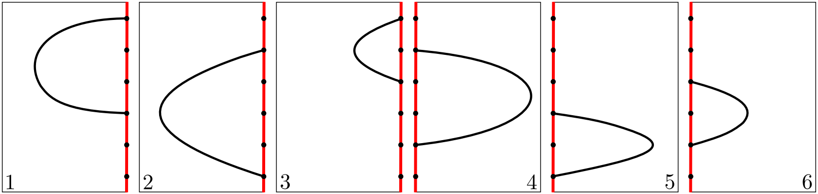
This information gives sharper lower bounds of the braid and arc indices of Kanenobu knots.

Braid index Every link has a closed braid presentation. The braid index, $\mathrm{braid}(L)$, of a link L is the minimum number of strings needed for L to be presented as a closed braid.



Arc index Every link has an arc presentation.
An arc presentation of a link L is an embedding of L in finitely many pages of an open-book such that L meets each page in a properly embedded single simple arc.
The arc index, $\mathrm{arc}(L)$, of a link L is the minimum number of pages needed for L to be presented as an arc presentation.





Fact Using the MFW inequality on t-span $P(k(n);t,z)$,

$$5 \leq \operatorname{braid}(k(n)) \quad (n \geq 0).$$

Fact [T. 2013] Using $\operatorname{span}\Gamma_{2/1}(k(n))$,

$$\begin{cases} \operatorname{braid}(k(n)) = 5 & (n = 0,1,2), \\ n+3 \leq \operatorname{braid}(k(n)) \leq 2n+1 & (n \geq 3). \end{cases}$$

Fact Using the MB inequality on a-span $F(k(n);a,z)$,

$$10 \leq \operatorname{arc}(k(n)) \quad (n \geq 0).$$

Fact [Lee-T. 2017] Using $\operatorname{span}\Gamma_{2/1}(k(n))$,

$$\begin{cases} \operatorname{arc}(k(n)) = 10 & (n = 0,1,2), \\ 2n+6 \leq \operatorname{arc}(k(n)) \leq 4n+4 & (n \geq 3). \end{cases}$$

Infinitely many knots with the trivial $\Gamma_{2/1}$-polynomial

For the trivial knot $\bigcirc$, we have

$$\nabla(\bigcirc) = V(\bigcirc) = \Gamma(\bigcirc) = Q(\bigcirc) = P(\bigcirc) = F(\bigcirc) = 1.$$

<u>Problems</u> Does there exist a non-trivial knot K s.t.

$\nabla(K) = 1?$ $\Longrightarrow$ Yes. Kinoshita-Terasaka knot, Conway knot, $(-3, 5, 7)$-pretzel knot, etc.

$V(K) = 1?$ $\Longrightarrow$ Open problem.

$\Gamma(K) = 1?$ $\Longrightarrow$ Yes. 8_{14} in Rolfsen's table, etc.

$Q(K) = 1?$ $\Longrightarrow$ Yes. $16n389841$, $16n491778$ ([Miyazawa 2019]).

$P(K) = 1?$ $\Longrightarrow$ Open problem.

$F(K) = 1?$ $\Longrightarrow$ Open problem.

Since $\bigcirc^{(p,1)} \sim \bigcirc$, we see that

$$\nabla_{p/1}(\bigcirc) = V_{p/1}(\bigcirc) = \Gamma_{p/1}(\bigcirc) = Q_{p/1}(\bigcirc)$$

$$= P_{p/1}(\bigcirc) = F_{p/1}(\bigcirc) = 1.$$

Problems $p \geq 2$. Does there exist a non-trivial knot K s.t.

$\nabla_{p/1}(K) = 1?$ $\Longrightarrow$ Yes. (Using satellite formula)

$V_{p/1}(K) = 1?$ $\Longrightarrow$ Open problem.

$\Gamma_{p/1}(K) = 1?$ $\Longrightarrow$ Yes. $(p = 2)$.

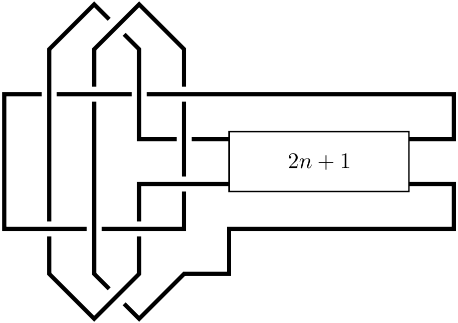
$Q_{p/1}(K) = 1?$ $\Longrightarrow$ Open problem.

$P_{p/1}(K) = 1?$ $\Longrightarrow$ Open problem.

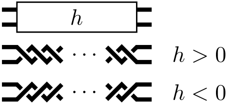
$F_{p/1}(K) = 1?$ $\Longrightarrow$ Open problem.

<u>Fact</u> [T. 2018] For any $n \in \mathbb{Z}$, we have

$$\boldsymbol{\Gamma_{2/1}(K_n) = \Gamma_{2/1}(\bigcirc) = 1.}$$

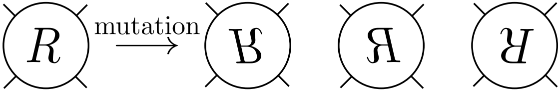


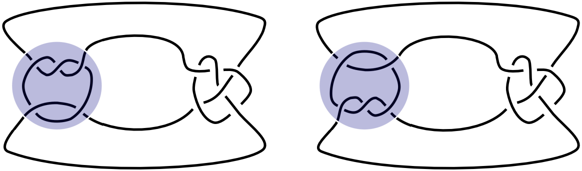
K_n



5. $\Gamma_{3/q}$-polynomial

Mutant knots





K-T C

Fact $\nabla_{p/q}$, $V_{p/q}$, P, F, $P_{2/q}$, $F_{2/q}$**-polynomials cannot distinguish mutant knots.**

Fact [Morton-Traczyk 1988, Murakami 1989]
The $P_{3/0}$**-polynomial (**3**-parallel HOMFLYPT polynomial) can distinguish Kinoshita-Terasaka and Conway knots.**

Fact [T. 2015] The $\Gamma_{3/q}$**-polynomial cannot distinguish mutant knots. (Using skein theoretic argument)**

Fact [Ito 2019] The $\Gamma_{p/q}$**-polynomial cannot distinguish mutant knots. (Using a theory of Kontsevich invariants)**

6. Vassiliev knot invariants derived from $\Gamma_{p/q}$-polynomials

In this section, we use P_0 not Γ to apply Kanenobu's results.

$$\Gamma(L; t^{-2}) = P_0(L; t) = F_0(L; \sqrt{-1}t^{-1}).$$

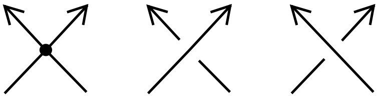
Singular knot invariant from a knot invariant

A singular knot is an oriented immersed circle in S^3 whose singularities are only transverse double points. We assume that each double point on a singular knot is a rigid vertex.

Let v be an invariant of an oriented knot in S^3, which takes values in $\mathbb{Q}$. Then v can be uniquely extended to a singular knot invariant by the Vassiliev skein relation:

$$v(K_\times) = v(K_+) - v(K_-),$$

where $K_\times$ is a singular knot with a double point $\times$ and K_+, K_- are singular knots obtained from $K_\times$ by replacing $\times$ by a positive crossing and a negative crossing, respectively.



$K_\times$ K_+ K_-

Vassiliev knot invariants

We call v a Vassiliev knot invariant of order d
if there exists an integer d such that $v(K^{>d}) = 0$ for any singular knot $K^{>d}$ with more than d double points and $v(K^d) \neq 0$ for a singular knot K^d with d double points.

The set of all Vassiliev knot invariants of order $\leq d$ forms a vector space over $\mathbb{Q}$, which is denoted by $\mathcal{V}_d$.

There is a filtration

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_d \subset \cdots$$

in the entire space of Vassiliev knot invariants. Each $\mathcal{V}_d$ is finite-dimensional. In particular, we have $\mathcal{V}_0 = \mathcal{V}_1 =< 1 >$.

Vassiliev knot invariants up to order six

Fact [Kanenobu 2001]

a_{2i}: the $2i$th coefficient of the Alexander-Conway polynomial.

$P_{2i}^{(d)}(1)$: the dth derivative of P_{2i} at $t = 1$.

$F_i^{(d)}(\sqrt{-1})$: the dth derivative of F_i at $a = \sqrt{-1}$.

$\mathcal{V}_2 = \langle 1, a_2 \rangle .$

$\mathcal{V}_3 = \langle 1, a_2, P_0^{(3)}(1) \rangle .$

$\mathcal{V}_4 = \langle 1, a_2, P_0^{(3)}(1), a_2^2, a_4, P_0^{(4)}(1) \rangle .$

$$\mathcal{V}_5 = \langle 1, a_2, P_0^{(3)}(1), a_2^2, a_4, P_0^{(4)}(1), a_2 P_0^{(3)}(1), P_0^{(5)}(1), P_4^{(1)}(1), F_4^{(1)}(\sqrt{-1})/\sqrt{-1} \rangle .$$

$$\mathcal{V}_6 = \langle 1, a_2, P_0^{(3)}(1), a_2^2, a_4, P_0^{(4)}(1), a_2 P_0^{(3)}(1), P_0^{(5)}(1), P_4^{(1)}(1), F_4^{(1)}(\sqrt{-1})/\sqrt{-1}, a_2^3, a_2 a_4, a_2 P_0^{(4)}(1), P_0^{(3)}(1)^2, P_0^{(6)}(1), P_4^{(2)}(1), a_6, F_4^{(2)}(\sqrt{-1}), F_5^{(1)}(\sqrt{-1}) \rangle .$$

Cabling for Vassiliev knot invariants

Fact [Bar-Natan 1995, Stanford 1994]

If v is a Vassiliev knot invariant of order d,
then the (p,q)-cabling $v_{p/q}$ is also a Vassiliev knot invariant of order $\leq d$.

Since $P_0^{(d)}(1)$ is a Vassiliev knot invariant of order d,
$(P_0)_{p/q}^{(d)}(1)$ is a Vassiliev knot invariant of order $\leq d$.

In this talk, we consider $(P_0)_{n/1}^{(d)}(1)$ for $1 \leq d \leq 6$ and $1 \leq n \leq 7$.

Results

By using Kodama's program "KNOT", we can calculate $(P_0)_{n/1}(K)$ with $1 \leq n \leq 7$ for a knot K with small crossings. Therefore, we obtain the following results.

Order ≤ 2 $\mathcal{V}_2 = < 1, a_2 >$.

$P_0^{(2)}(1) = -8a_2$	
$(P_0)_{2/1}^{(2)}(1) = -32a_2$	$(P_0)_{2/1}^{(2)}(1) = 4P_0^{(2)}(1)$
$(P_0)_{3/1}^{(2)}(1) = -72a_2$	$(P_0)_{3/1}^{(2)}(1) = 9P_0^{(2)}(1)$
$(P_0)_{4/1}^{(2)}(1) = -128a_2$	$(P_0)_{4/1}^{(2)}(1) = 16P_0^{(2)}(1)$
$(P_0)_{5/1}^{(2)}(1) = -200a_2$	$(P_0)_{5/1}^{(2)}(1) = 25P_0^{(2)}(1)$
$(P_0)_{6/1}^{(2)}(1) = -288a_2$	$(P_0)_{6/1}^{(2)}(1) = 36P_0^{(2)}(1)$
$(P_0)_{7/1}^{(2)}(1) = -392a_2$	$(P_0)_{7/1}^{(2)}(1) = 49P_0^{(2)}(1)$

Proposition $(P_0)^{(2)}_{n/1}(1) = n^2 P_0^{(2)}(1)$.

Proof. Let $\Delta_K(t)$ be the normalized Alexander polynomial of a knot K, which satisfies $\Delta_K(t) = \nabla_K(t^{1/2} - t^{-1/2})$.

Then we have $\Delta_K^{(1)}(1) = 0$ and $\Delta_K^{(2)}(1) = 2a_2(K)$.

By using satellite formula, we have $\Delta_{K^{(n,1)}}(t) = \Delta_K(t^n)$.

Therefore, we have

$$a_2(K^{(n,1)}) = (1/2)\Delta^{(2)}_{K^{(n,1)}}(1) = (1/2)\Delta_K^{(2)}(t^n)\Big|_{t=1}$$

$$= (1/2)(n^2\Delta_K^{(2)}(t^n)t^{2n-2} + n(n-1)\Delta_K^{(1)}(t^n)t^{n-2})\Big|_{t=1}$$

$$= n^2 a_2(K).$$

By $P_0^{(2)}(K;1) = -8a_2(K)$, we have

$$(P_0)^{(2)}_{n/1}(K;1) = P_0^{(2)}(K^{(n,1)};1) = n^2 P_0^{(2)}(K;1).$$

□

Order ≤ 3 $\mathcal{V}_3 = < 1, a_2, P_0^{(3)}(1) >$. $P_0^{(2)}(1) = -8a_2$.

$(P_0)^{(3)}_{2/1}(1) = -48a_2 + 4P_0^{(3)}(1)$	$(P_0)^{(3)}_{2/1}(1) = 6P_0^{(2)}(1) + 4P_0^{(3)}(1)$
$(P_0)^{(3)}_{3/1}(1) = -192a_2 + 9P_0^{(3)}(1)$	$(P_0)^{(3)}_{3/1}(1) = 24P_0^{(2)}(1) + 9P_0^{(3)}(1)$
$(P_0)^{(3)}_{4/1}(1) = -480a_2 + 16P_0^{(3)}(1)$	$(P_0)^{(3)}_{4/1}(1) = 60P_0^{(2)}(1) + 16P_0^{(3)}(1)$
$(P_0)^{(3)}_{5/1}(1) = -960a_2 + 25P_0^{(3)}(1)$	$(P_0)^{(3)}_{5/1}(1) = 120P_0^{(2)}(1) + 25P_0^{(3)}(1)$
$(P_0)^{(3)}_{6/1}(1) = -1680a_2 + 36P_0^{(3)}(1)$	$(P_0)^{(3)}_{6/1}(1) = 210P_0^{(2)}(1) + 36P_0^{(3)}(1)$
$(P_0)^{(3)}_{7/1}(1) = -2688a_2 + 49P_0^{(3)}(1)$	$(P_0)^{(3)}_{7/1}(1) = 336P_0^{(2)}(1) + 49P_0^{(3)}(1)$

<u>Proposition</u> $(P_0)^{(3)}_{n/1}(1) = n(n^2-1)P_0^{(2)}(1) + n^2 P_0^{(3)}(1)$.

Proof. The following cabling formula holds for any knot K [Ohtsuki 2004]:

$$v_3(K^{(n,1)}) = n^2 v_3(K) - (1/6)n(n^2-1)v_2(K),$$

where v_2, v_3 are $\mathbb{Z}$-valued primitive Vassiliev invariants of order $2, 3$, respectively. Moreover, the following holds [Stanford 2002]:

$$v_2(K) = a_2(K),\ v_3(K) = (1/48)P_0^{(3)}(K;1) - (1/2)a_2(K).$$

By $a_2(K^{(n,1)}) = n^2 a_2(K)$ and $P_0^{(2)}(K;1) = -8a_2(K)$,

$$\begin{aligned} & (1/48)P_0^{(3)}(K^{(n,1)};1) - (1/2)n^2 a_2(K) \\ = \ & n^2\Big((1/48)P_0^{(3)}(K;1) - (1/2)a_2(K)\Big) \\ & -(1/6)n(n^2-1)\Big(-(1/8)P_0^{(2)}(K;1)\Big). \end{aligned}$$

□

Order ≤ 4 $\boldsymbol{\mathcal{V}_4 = < 1, a_2, P_0^{(3)}(1), a_2^2, a_4, P_0^{(4)}(1) >}$. $\boldsymbol{P_0^{(2)}(1) = -8a_2.}$

$$(P_0)^{(4)}_{2/1}(1) = -96a_2 + 8P_0^{(3)}(1) + 1920a_2^2 - 768a_4 + 4P_0^{(4)}(1)$$
$$(P_0)^{(4)}_{3/1}(1) = -768a_2 + 32P_0^{(3)}(1) + 11520a_2^2 - 4608a_4 + 9P_0^{(4)}(1)$$
$$(P_0)^{(4)}_{4/1}(1) = -2880a_2 + 80P_0^{(3)}(1) + 38400a_2^2 - 15360a_4 + 16P_0^{(4)}(1)$$
$$(P_0)^{(4)}_{5/1}(1) = -7680a_2 + 160P_0^{(3)}(1) + 96000a_2^2 - 38400a_4 + 25P_0^{(4)}(1)$$
$$(P_0)^{(4)}_{6/1}(1) = -16800a_2 + 280P_0^{(3)}(1) + 201600a_2^2 - 80640a_4 + 36P_0^{(4)}(1)$$

$$(P_0)^{(4)}_{2/1}(1) = 12P_0^{(2)}(1) + 8P_0^{(3)}(1) + 30P_0^{(2)}(1)^2 - 768a_4 + 4P_0^{(4)}(1)$$
$$(P_0)^{(4)}_{3/1}(1) = 96P_0^{(2)}(1) + 32P_0^{(3)}(1) + 180P_0^{(2)}(1)^2 - 4608a_4 + 9P_0^{(4)}(1)$$
$$(P_0)^{(4)}_{4/1}(1) = 360P_0^{(2)}(1) + 80P_0^{(3)}(1) + 600P_0^{(2)}(1)^2 - 15360a_4 + 16P_0^{(4)}(1)$$
$$(P_0)^{(4)}_{5/1}(1) = 960P_0^{(2)}(1) + 160P_0^{(3)}(1) + 1500P_0^{(2)}(1)^2 - 38400a_4 + 25P_0^{(4)}(1)$$
$$(P_0)^{(4)}_{6/1}(1) = 2100P_0^{(2)}(1) + 280P_0^{(3)}(1) + 3150P_0^{(2)}(1)^2 - 80640a_4 + 36P_0^{(4)}(1)$$

Question

$$\begin{aligned}(P_0)^{(4)}_{n/1}(1) &= 2(n-1)^2 n(n+1)P_0^{(2)}(1) + (4/3)n(n^2-1)P_0^{(3)}(1) \\ &\quad + (5/2)n^2(n^2-1)P_0^{(2)}(1)^2 - 64n^2(n^2-1)a_4 + n^2 P_0^{(4)}(1)?\end{aligned}$$

We have the following relation:

$$
\begin{aligned}
& a_4 \\
&= (1/768)(12P_0^{(2)}(1) + 8P_0^{(3)}(1) + 30P_0^{(2)}(1)^2 \\
&\quad + 4P_0^{(4)}(1) - (P_0)_{2/1}^{(4)}(1)) \\
&= (1/4608)(96P_0^{(2)}(1) + 32P_0^{(3)}(1) + 180P_0^{(2)}(1)^2 \\
&\quad + 9P_0^{(4)}(1) - (P_0)_{3/1}^{(4)}(1)) \\
&= (1/15360)(360P_0^{(2)}(1) + 80P_0^{(3)}(1) + 600P_0^{(2)}(1)^2 \\
&\quad + 16P_0^{(4)}(1) - (P_0)_{4/1}^{(4)}(1)) \\
&= (1/38400)(960P_0^{(2)}(1) + 160P_0^{(3)}(1) + 1500P_0^{(2)}(1)^2 \\
&\quad + 25P_0^{(4)}(1) - (P_0)_{5/1}^{(4)}(1)) \\
&= (1/80640)(2100P_0^{(2)}(1) + 280P_0^{(3)}(1) + 3150P_0^{(2)}(1)^2 \\
&\quad + 36P_0^{(4)}(1) - (P_0)_{6/1}^{(4)}(1)).
\end{aligned}
$$

Therefore, we see that

$$
\mathcal{V}_4 = < 1, P_0^{(2)}(1), P_0^{(3)}(1), P_0^{(2)}(1)^2, P_0^{(4)}(1), (P_0)_{n/1}^{(4)}(1) >
$$

for $2 \leq n \leq 6$.

Order ≤ 5 $\mathcal{V}_5 =< 1, a_2, P_0^{(3)}(1), a_2^2, a_4, P_0^{(4)}(1), a_2 P_0^{(3)}(1), P_0^{(5)}(1), P_4^{(1)}(1), F_4^{(1)}(\sqrt{-1})/\sqrt{-1} > .$

$$(P_0)^{(5)}_{2/1}(1) = -320a_2 + (460/9)P_0^{(3)}(1) + 10240a_2^2 + 3840a_4 + (170/9)P_0^{(4)}(1)$$
$$-(2720/3)a_2 P_0^{(3)}(1) + (44/9)P_0^{(5)}(1) - (24320/3)P_4^{(1)}(1) - (5120/3)F_4^{(1)}(\sqrt{-1})/\sqrt{-1}$$

$$(P_0)^{(5)}_{3/1}(1) = -3840a_2 + (1040/3)P_0^{(3)}(1) + 96000a_2^2 + 7680a_4 + (280/3)P_0^{(4)}(1)$$
$$-5440a_2 P_0^{(3)}(1) + (43/3)P_0^{(5)}(1) - 48640P_4^{(1)}(1) - 10240F_4^{(1)}(\sqrt{-1})/\sqrt{-1}$$

$$(P_0)^{(5)}_{4/1}(1) = -19840a_2 + (11000/9)P_0^{(3)}(1) + 427520a_2^2 - 23040a_4 + (2500/9)P_0^{(4)}(1)$$
$$-(54400/3)a_2 P_0^{(3)}(1) + (304/9)P_0^{(5)}(1) - (486400/3)P_4^{(1)}(1) - (102400/3)F_4^{(1)}(\sqrt{-1})/\sqrt{-1}$$

$$(P_0)^{(5)}_{5/1}(1) = -67840a_2 + (28400/9)P_0^{(3)}(1) + 1329920a_2^2 - 176640a_4 + (5800/9)P_0^{(4)}(1)$$
$$-(136000/3)a_2 P_0^{(3)}(1) + (625/9)P_0^{(5)}(1) - (1216000/3)P_4^{(1)}(1) - (256000/3)F_4^{(1)}(\sqrt{-1})/\sqrt{-1}$$

$$(P_0)^{(5)}_{6/1}(1) = -181440a_2 + (20300/3)P_0^{(3)}(1) + 3333120a_2^2 - 618240a_4 + (3850/3)P_0^{(4)}(1)$$
$$-95200a_2 P_0^{(3)}(1) + (388/3)P_0^{(5)}(1) - 851200P_4^{(1)}(1) - 179200F_4^{(1)}(\sqrt{-1})/\sqrt{-1}$$

We have the following relation:

$$
\begin{aligned}
&51072000P_4^{(1)}(1)+10752000F_4^{(1)}(\sqrt{-1})/\sqrt{-1}-714000P_0^{(2)}(1)P_0^{(3)}(1)\\
&=9144576252000P_0^{(2)}(1)+6096384322000P_0^{(3)}(1)\\
&\quad+22861441008000P_0^{(2)}(1)^2+3048192119000P_0^{(4)}(1)-762048000000(P_0)_{2/1}^{(4)}(1)\\
&\quad+30800P_0^{(5)}(1)-6300(P_0)_{2/1}^{(5)}(1)\\
&=1354752504000P_0^{(2)}(1)+451584364000P_0^{(3)}(1)\\
&\quad+2540161575000P_0^{(2)}(1)^2+127008098000P_0^{(4)}(1)-14112000000(P_0)_{3/1}^{(4)}(1)\\
&\quad+15050P_0^{(5)}(1)-1050(P_0)_{3/1}^{(5)}(1)\\
&=1234518541200P_0^{(2)}(1)+274337665000P_0^{(3)}(1)\\
&\quad+2057531704200P_0^{(2)}(1)^2+54867543500P_0^{(4)}(1)-3429216000(P_0)_{4/1}^{(4)}(1)\\
&\quad+10640P_0^{(5)}(1)-315(P_0)_{4/1}^{(5)}(1)\\
&=12383951670720P_0^{(2)}(1)+2063992164640P_0^{(3)}(1)\\
&\quad+19349925434280P_0^{(2)}(1)^2+322498794800P_0^{(4)}(1)-22256640(P_0)_{5/1}^{(4)}(1)\\
&\quad+8750P_0^{(5)}(1)-126(P_0)_{5/1}^{(5)}(1)\\
&=35833191760800P_0^{(2)}(1)+4777759126000P_0^{(3)}(1)\\
&\quad+53749788724800P_0^{(2)}(1)^2+614283341000P_0^{(4)}(1)-17063424000(P_0)_{6/1}^{(4)}(1)\\
&\quad+7760P_0^{(5)}(1)-60(P_0)_{6/1}^{(5)}(1).
\end{aligned}
$$

We see that

$\mathcal{V}_5$

$$=< 1, P_0^{(2)}(1), P_0^{(3)}(1), P_0^{(2)}(1)^2, P_0^{(4)}(1), (P_0)_{l/1}^{(4)}(1), P_0^{(2)}(1)P_0^{(3)}(1), P_0^{(5)}(1), P_4^{(1)}(1), F_4^{(1)}(\sqrt{-1})/\sqrt{-1} >$$

$$=< 1, P_0^{(2)}(1), P_0^{(3)}(1), P_0^{(2)}(1)^2, P_0^{(4)}(1), (P_0)_{l/1}^{(4)}(1), P_0^{(2)}(1)P_0^{(3)}(1), P_0^{(5)}(1), (P_0)_{m/1}^{(5)}(1), F_4^{(1)}(\sqrt{-1})/\sqrt{-1} >$$

$$=< 1, P_0^{(2)}(1), P_0^{(3)}(1), P_0^{(2)}(1)^2, P_0^{(4)}(1), (P_0)_{l/1}^{(4)}(1), P_0^{(2)}(1)P_0^{(3)}(1), P_0^{(5)}(1), P_4^{(1)}(1), (P_0)_{n/1}^{(5)}(1) >$$

for $2 \leq l, m, n \leq 6$.

$1, P_0^{(2)}(1), P_0^{(3)}(1), P_0^{(2)}(1)^2, P_0^{(4)}(1), (P_0)_{l/1}^{(4)}(1), P_0^{(2)}(1)P_0^{(3)}(1), P_0^{(5)}(1), (P_0)_{m/1}^{(5)}(1), (P_0)_{n/1}^{(5)}(1)$ are not linearly independent for $2 \leq l, m, n \leq 6, m \neq n$.

<u>Question</u> $\mathcal{V}_5$ is determined by $(P_0)_{p/q}$?

$$\boxed{\textsf{Order} \le 6}\ \mathcal{V}_6 = < 1, a_2, P_0^{(3)}(1), a_2^2, a_4, P_0^{(4)}(1), a_2P_0^{(3)}(1), P_0^{(5)}(1), P_4^{(1)}(1), F_4^{(1)}(\sqrt{-1})/\sqrt{-1}, a_2^3, a_2a_4, a_2P_0^{(4)}(1), P_0^{(3)}(1)^2, P_0^{(6)}(1), P_4^{(2)}(1), a_6, F_4^{(2)}(\sqrt{-1}), F_5^{(1)}(\sqrt{-1}) > .$$

$$(P_0)^{(6)}_{2/1}(1) = -160a_2 + (1360/3)P_0^{(3)}(1) + 21760a_2^2 - 69120a_4 + (950/3)P_0^{(4)}(1) -6080a_2P_0^{(3)}(1) + (232/3)P_0^{(5)}(1) + 25600P_4^{(1)}(1) + 23040F_4^{(1)}(\sqrt{-1})/\sqrt{-1} -15360a_2a_4 - 1520a_2P_0^{(4)}(1) + (380/3)P_0^{(3)}(1)^2 + (80/9)P_0^{(6)}(1) - 47360P_4^{(2)}(1) -404480a_6 + 7680F_4^{(2)}(\sqrt{-1}) - 40960F_5^{(1)}(\sqrt{-1})$$

$$(P_0)^{(6)}_{3/1}(1) = -11520a_2 + (11840/3)P_0^{(3)}(1) + 526080a_2^2 - 368640a_4 + (6520/3)P_0^{(4)}(1) -56320a_2P_0^{(3)}(1) + (1384/3)P_0^{(5)}(1) - 40960P_4^{(1)}(1) + 97280F_4^{(1)}(\sqrt{-1})/\sqrt{-1} - 1520640a_2^3 +1152000a_2a_4 - 9120a_2P_0^{(4)}(1) + 760P_0^{(3)}(1)^2 + (115/3)P_0^{(6)}(1) - 284160P_4^{(2)}(1) -2565120a_6 + 46080F_4^{(2)}(\sqrt{-1}) - 245760F_5^{(1)}(\sqrt{-1})$$

$$(P_0)^{(6)}_{4/1}(1) = -102080a_2 + 17280P_0^{(3)}(1) + 3668480a_2^2 - 1359360a_4 + 8020P_0^{(4)}(1) -249600a_2P_0^{(3)}(1) + (4696/3)P_0^{(5)}(1) - 752640P_4^{(1)}(1) + 194560F_4^{(1)}(\sqrt{-1})/\sqrt{-1} - 12165120a_2^3 +9646080a_2a_4 - 30400a_2P_0^{(4)}(1) + (7600/3)P_0^{(3)}(1)^2 + (1024/9)P_0^{(6)}(1) - 947200P_4^{(2)}(1) -9195520a_6 + 153600F_4^{(2)}(\sqrt{-1}) - 819200F_5^{(1)}(\sqrt{-1})$$

$$(P_0)^{(6)}_{5/1}(1) = -488960a_2 + (160640/3)P_0^{(3)}(1) + 15430400a_2^2 - 4377600a_4 + (65560/3)P_0^{(4)}(1) -774400a_2P_0^{(3)}(1) + (12056/3)P_0^{(5)}(1) - 3389440P_4^{(1)}(1) + 168960F_4^{(1)}(\sqrt{-1})/\sqrt{-1} - 53222400a_2^3 +42777600a_2a_4 - 76000a_2P_0^{(4)}(1) + (19000/3)P_0^{(3)}(1)^2 + (2425/9)P_0^{(6)}(1) - 2368000P_4^{(2)}(1) -25062400a_6 + 384000F_4^{(2)}(\sqrt{-1}) - 2048000F_5^{(1)}(\sqrt{-1})$$

Theorem We have the following relation:

$$
\begin{aligned}
&-342000P_0^{(2)}(1)P_0^{(4)}(1)-228000P_0^{(3)}(1)^2+85248000P_4^{(2)}(1)\\
&-13824000F_4^{(2)}(\sqrt{-1})+73728000F_5^{(1)}(\sqrt{-1})\\
=&-1908000P_0^{(2)}(1)-9702000P_0^{(2)}(1)^2+135000P_0^{(2)}(1)^3+1404000P_0^{(2)}(1)P_0^{(3)}(1)\\
&-480000P_0^{(3)}(1)-78000P_0^{(4)}(1)+18000P_0^{(2)}(1)P_0^{(4)}(1)+139200P_0^{(5)}(1)\\
&+46080000P_4^{(1)}(1)+41472000F_4^{(1)}(\sqrt{-1})/\sqrt{-1}+16000P_0^{(6)}(1)-728064000a_6\\
&+162000(P_0)_{2/1}^{(4)}(1)-4500P_0^{(2)}(1)(P_0)_{2/1}^{(4)}(1)-1800(P_0)_{2/1}^{(6)}(1)\\
=&-1872000P_0^{(2)}(1)-24948000P_0^{(2)}(1)^2+55336500P_0^{(2)}(1)^3+1812000P_0^{(2)}(1)P_0^{(3)}(1)\\
&+416000P_0^{(3)}(1)+436000P_0^{(4)}(1)-84375P_0^{(2)}(1)P_0^{(4)}(1)+138400P_0^{(5)}(1)\\
&-12288000P_4^{(1)}(1)+29184000F_4^{(1)}(\sqrt{-1})/\sqrt{-1}+11500P_0^{(6)}(1)-769536000a_6\\
&+24000(P_0)_{3/1}^{(4)}(1)+9375P_0^{(2)}(1)(P_0)_{3/1}^{(4)}(1)-300(P_0)_{3/1}^{(6)}(1)\\
=&-1719000P_0^{(2)}(1)-48592800P_0^{(2)}(1)^2+132618600P_0^{(2)}(1)^3+2242800P_0^{(2)}(1)P_0^{(3)}(1)\\
&+918000P_0^{(3)}(1)+594360P_0^{(4)}(1)-113040P_0^{(2)}(1)P_0^{(4)}(1)+140880P_0^{(5)}(1)\\
&-67737600P_4^{(1)}(1)+17510400F_4^{(1)}(\sqrt{-1})/\sqrt{-1}+10240P_0^{(6)}(1)-827596800a_6\\
&+7965(P_0)_{4/1}^{(4)}(1)+7065P_0^{(2)}(1)(P_0)_{4/1}^{(4)}(1)-90(P_0)_{4/1}^{(6)}(1)\\
=&-1739520P_0^{(2)}(1)-80405280P_0^{(2)}(1)^2+231981300P_0^{(2)}(1)^3+2682720P_0^{(2)}(1)P_0^{(3)}(1)\\
&+1271040P_0^{(3)}(1)+684120P_0^{(4)}(1)-125325P_0^{(2)}(1)P_0^{(4)}(1)+144672P_0^{(5)}(1)\\
&-122019840P_4^{(1)}(1)+6082560F_4^{(1)}(\sqrt{-1})/\sqrt{-1}+9700P_0^{(6)}(1)-902246400a_6\\
&+4104(P_0)_{5/1}^{(4)}(1)+5013P_0^{(2)}(1)(P_0)_{5/1}^{(4)}(1)-36(P_0)_{5/1}^{(6)}(1).
\end{aligned}
$$

Thank you.