Disk surgery and the primitive disk complexes of the 3-sphere

Jung Hoon Lee
(joint with Sangbum Cho and Yuya Koda)

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Outline

- disk complex

- primitive disk complex

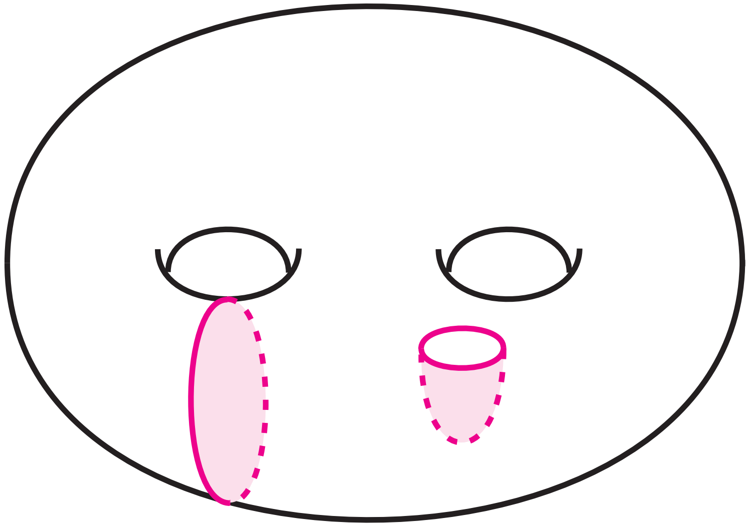
- disk surgery

- result (example)

- **compressing disk**

V : a handlebody of genus $g(\geq 2)$

A *compressing disk* D of V is
a properly embedded ($D \cap \partial V = \partial D$) disk in V such that
∂D does not bound a disk in ∂V.

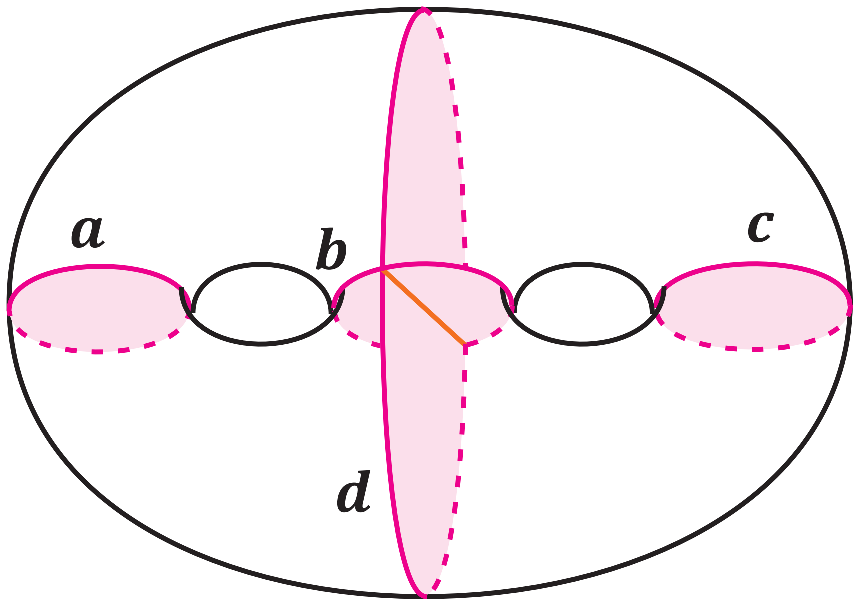


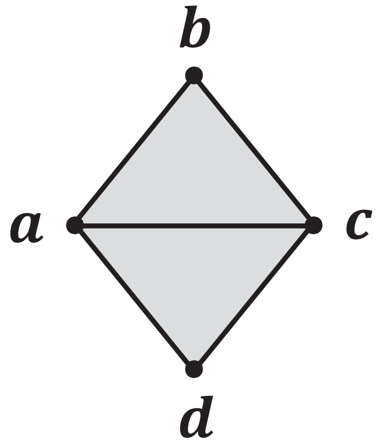
• disk complex

A *disk complex* $\mathcal{D}(V)$ is a simplicial complex defined as follows.

• Vertices of $\mathcal{D}(V)$ are isotopy classes of compressing disks of V.

• A collection of $k+1$ vertices forms a k-simplex if there are representatives for each that are pairwise disjoint.


a
b
c
d


b
a
c
d

• primitive disk

$M = V \cup_\Sigma W$: a Heegaard splitting

A disk $D \subset V$ is *primitive* if
there is a disk $\overline{D} \subset W$ such that $|D \cap \overline{D}| = 1$. (dual disk)

$\Longleftrightarrow$ ∂D is a primitive element of $\pi_1(W)$ ([Gordon]).

(An element w of a free group is *primitive* if
w can be a member of a basis.)

- **primitive elements of $F(x,y)$ ([Osborne-Zieschang])**

Suppose $0 \le m < n$, $(m,n)=1$. (mod $m+n$) : $1,2,\ldots,m+n$.
$f : \{1+im\}_{i=0}^{m+n-1} \longrightarrow \{x,y\}$

$$f(1+im) = \begin{cases} x & \text{if} \quad 1 \le 1+im \ (\text{mod } m+n) \le m, \\ y & \text{if} \quad m+1 \le 1+im \ (\text{mod } m+n) \le m+n. \end{cases}$$

$$W(x,y) = \prod_{i=0}^{m+n-1} f(1+im)$$

primitive elements $= W(x^{\pm 1}, y^{\pm 1})$ or their conjugates
example) $m=5$, $n=7$, $m+n-1=11$, (mod 12)

1	6	11	16	21	26	31	36	41	46	51	56	
1	6	11	4	9	2	7	12	5	10	3	8	(mod 12)
x	y	y	x	y	x	y	y	x	y	x	y	

$W(x,y) = xy^2xyxy^2xyxy$

- **primitive disk complex**

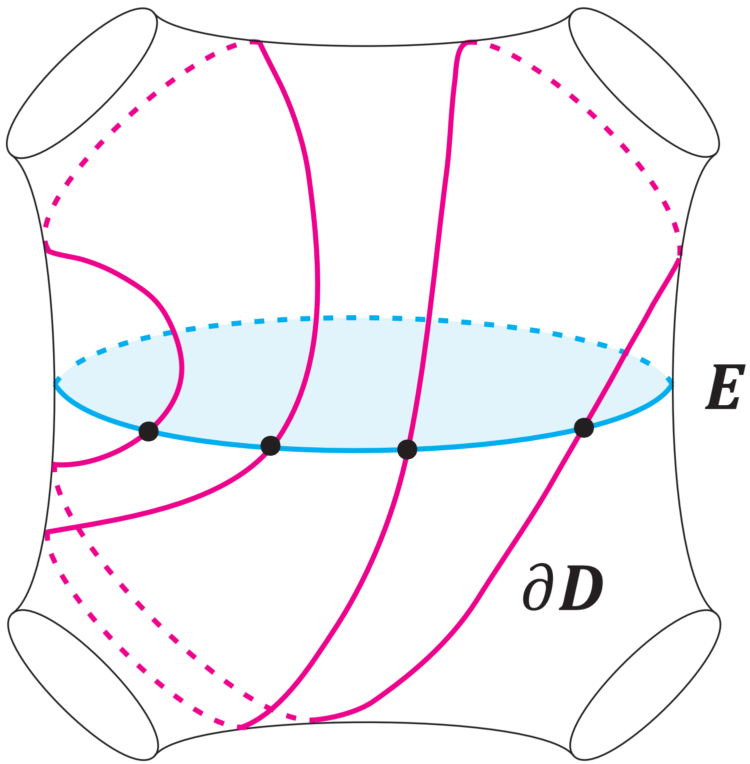
The *primitive disk complex* $\mathcal{P}(V)$ for V is a subcomplex of $\mathcal{D}(V)$ spanned by primitive disks.

$\longrightarrow$ mapping class group of the splitting:

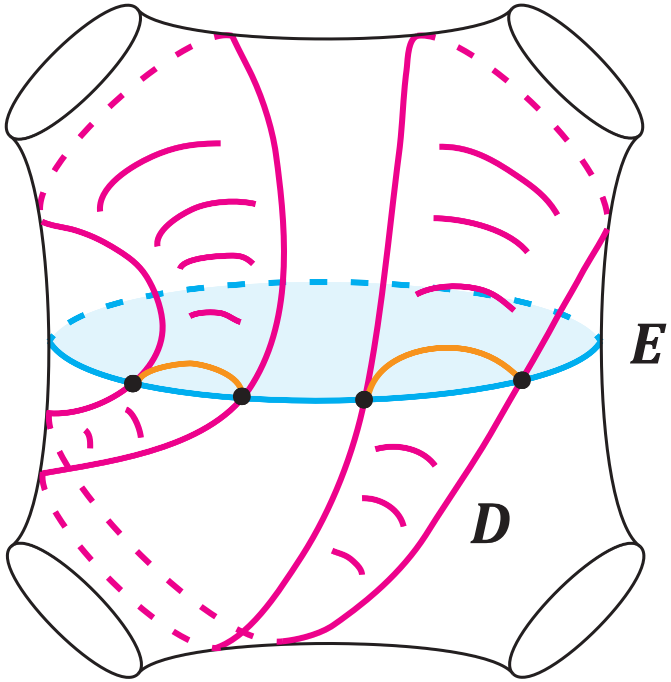
the group of isotopy classes of homeomorphisms of M that preserve V and W setwise

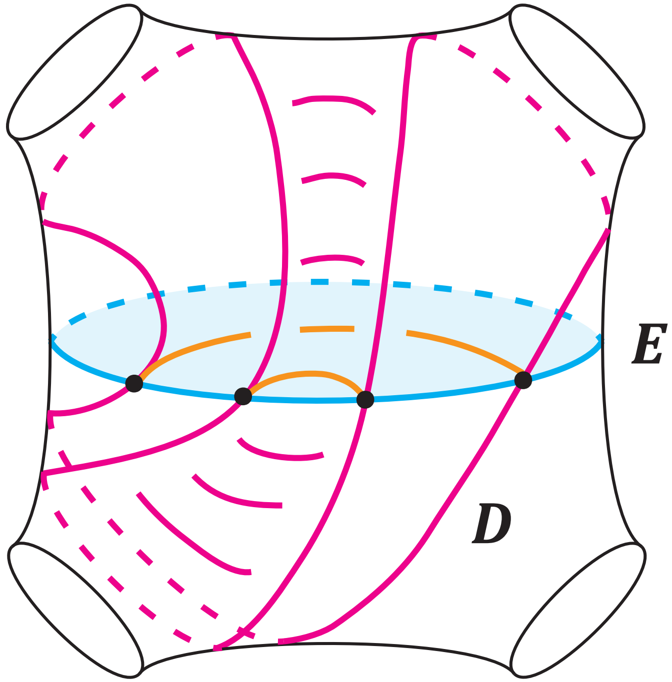
- **intersection pattern**

D, E : disks
$D \cap E \neq \emptyset$



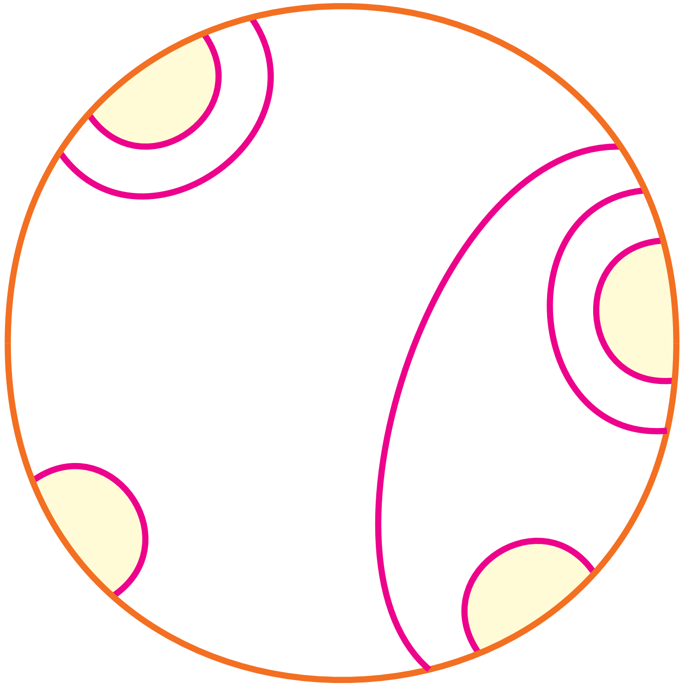
The intersection pattern of D and E may not be unique, by isotopy of D and E.





- **outermost disk**

$D \cap E \neq \emptyset$

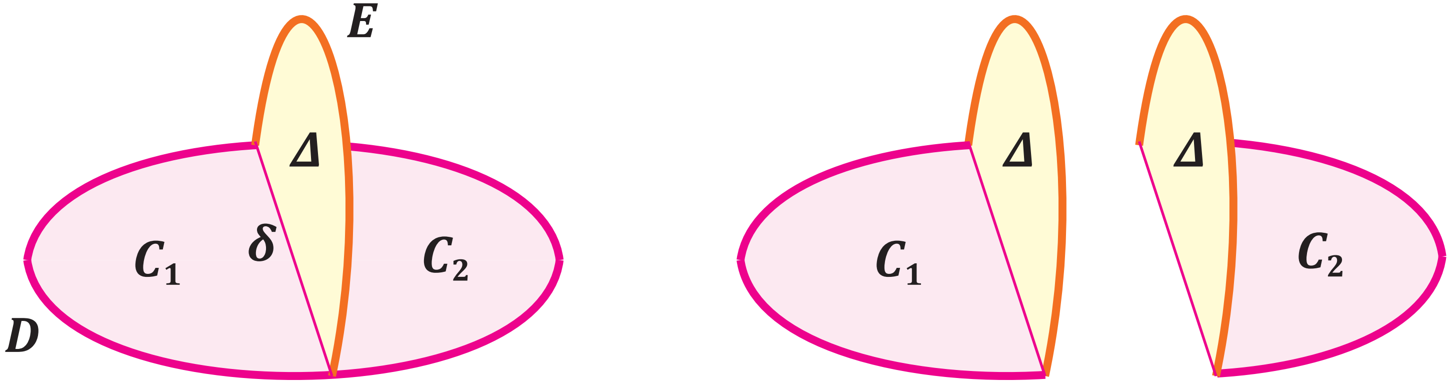


D*⋂*E* in *E

- **disk surgery**

Suppose $D \cap E \neq \emptyset$.
δ : an outermost arc of $D \cap E$ in E
Δ : the corresponding outermost disk in E cut off by δ

The arc δ cuts D into two disks C_1 and C_2.
We call $C_1 \cup \Delta$ and $C_2 \cup \Delta$ the disks obtained by a *disk surgery* on D along E.



$\mathcal{X}$: a subcomplex of $\mathcal{D}(V)$

(1) $\mathcal{X}$ is *closed under disk surgery operation* if
for any disks D and E with $D \cap E \neq \emptyset$ representing vertices of $\mathcal{X}$,
there exists an intersection pattern $D \cap E$ such that **every** surgery
on D along E always yields a disk representing a vertex of $\mathcal{X}$.

(2) $\mathcal{X}$ is *weakly closed under disk surgery operation* if
for any disks D and E with $D \cap E \neq \emptyset$ representing vertices of $\mathcal{X}$,
there exists an intersection pattern $D \cap E$ with **a** surgery on D
along E yielding a disk representing a vertex of $\mathcal{X}$.

Proposition.

(1) If $\mathcal{X}$ is weakly closed under disk surgery operation,
then $\mathcal{X}$ is connected.

(2) If $\mathcal{X}$ is closed under disk surgery operation,
then $\mathcal{X}$ is contractible ([McCullough], [Cho]).

For a genus-2 Heegaard splitting of S^3,
$\mathcal{P}(V)$ is closed under disk surgery operation ([Cho]),
hence it is contractible.

It is still an open question whether $\mathcal{P}(V)$ in the case of $g > 3$ is connected, contractible or not, and whether $\mathcal{P}(V)$ in the case of $g = 3$ is contractible or not.

Recently, it is shown that $\mathcal{P}(V)$ is connected in the case of $g = 3$ ([Freedman-Scharlemann], [Zupan]).

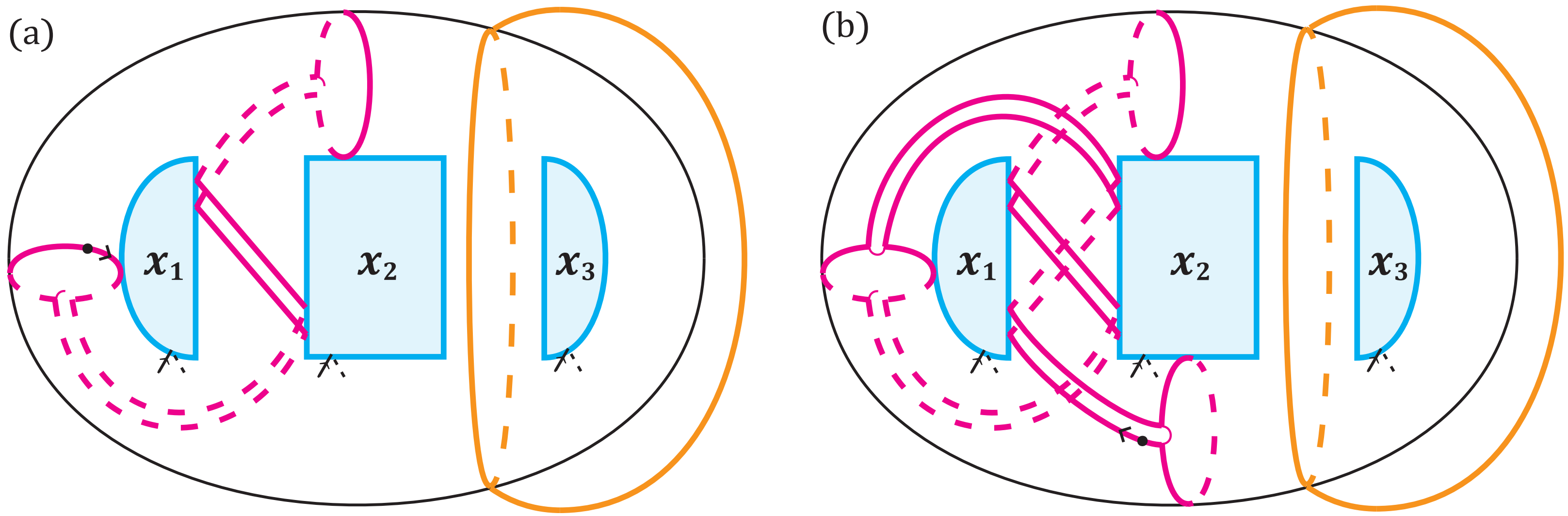
Theorem ([Cho-Koda-L.]).

Let $V \cup_{\Sigma} W$ be a genus-g Heegaard splitting of S^3 with $g \geq 3$.

Then $\mathcal{P}(V)$ is **not** weakly closed under disk surgery operation,

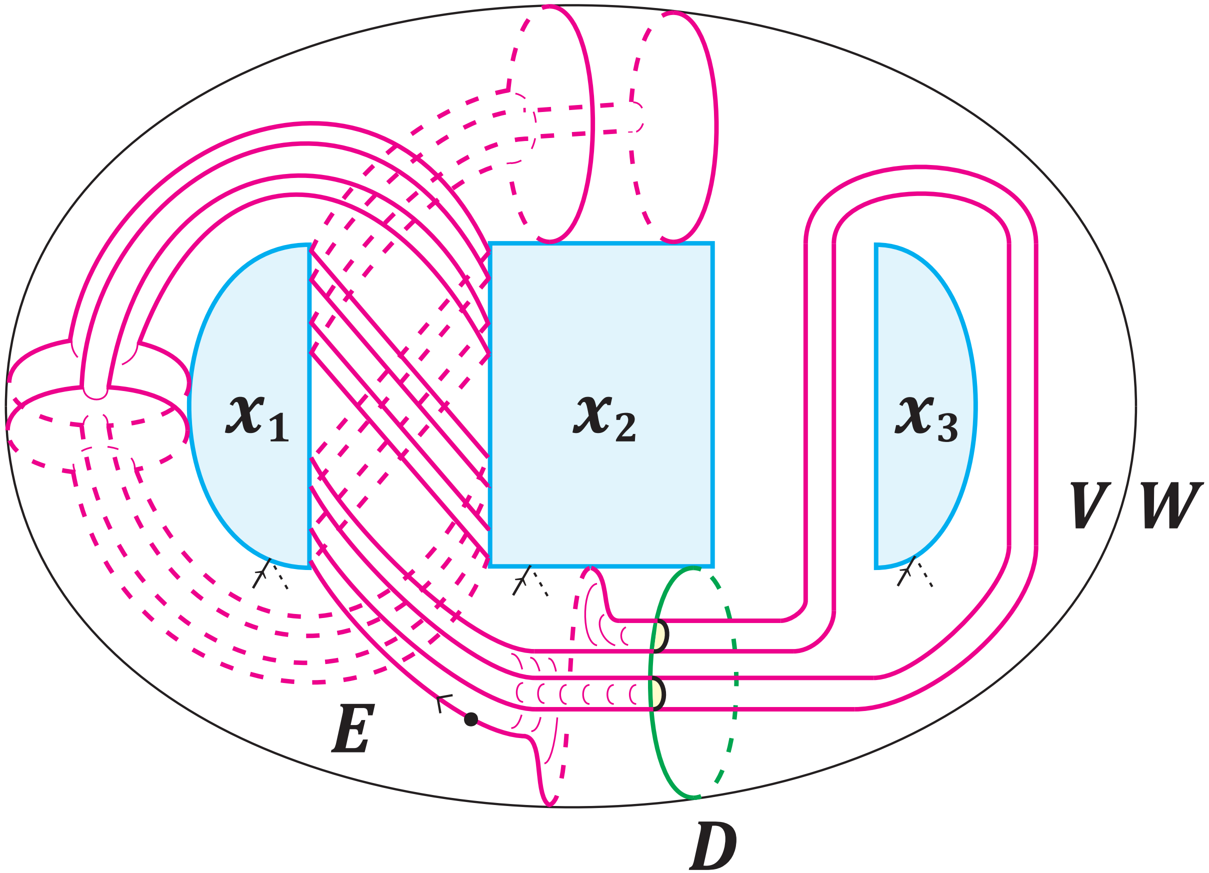
i.e. there exist two intersecting primitive disks in, say V, such that any disk surgery on one along the other yields **no** primitive disks.

• non-primitive disks in genus three



(a) : $x_1 x_2^{-1} x_1 x_2 x_1^{-1} x_2$, not primitive
(b) : $x_1 x_2^{-1} x_1 x_2^{-1} x_1 x_2 x_1^{-1} x_2 x_2 x_1^{-1} x_2$, not primitive

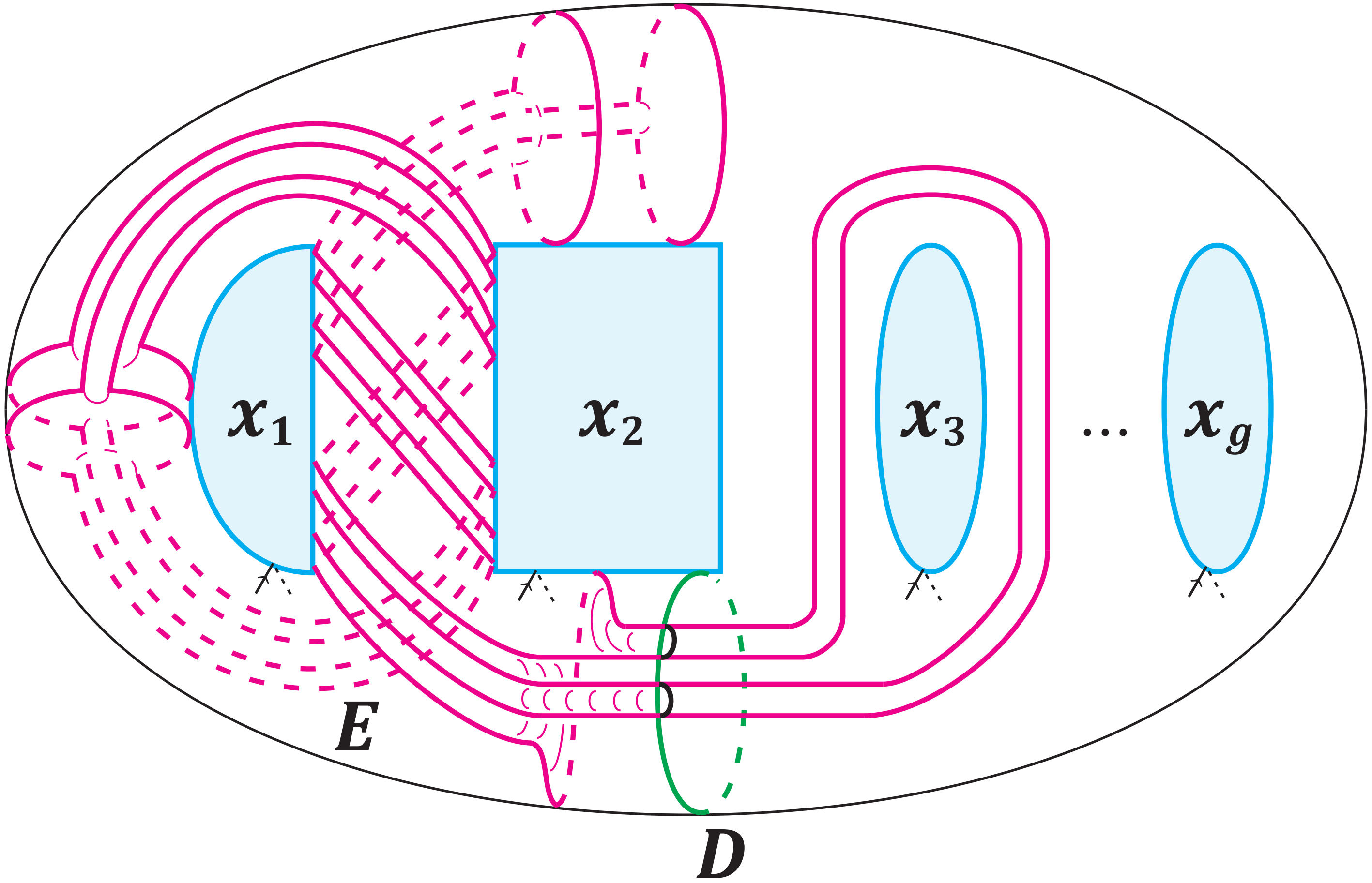
∵ In a free group of rank two, if a cyclically reduced word has both x_i and x_i^{-1}, then it is not primitive ([Osborne-Zieschang]).



D and E are primitive disks.

E : $(x_1x_2^{-1}x_1x_2^{-1}x_1x_2x_1^{-1}x_2x_2x_1^{-1})(x_1x_2^{-1}x_2^{-1}x_1x_2^{-1}x_1^{-1}x_2x_1^{-1}x_2x_1^{-1})x_2$

$\forall$ disk surgery on D and E $\longrightarrow$ disks (a) or (b), not primitive



x_1
x_2
x_3
…
x_g
E
D

Thank you for your attention.