Twisted torus knots which are torus knots

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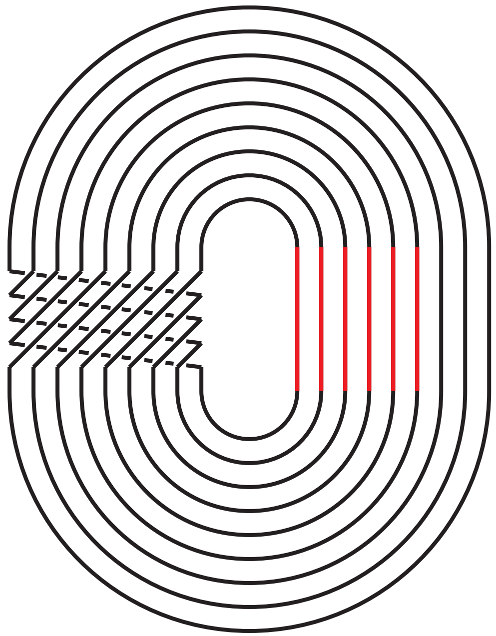
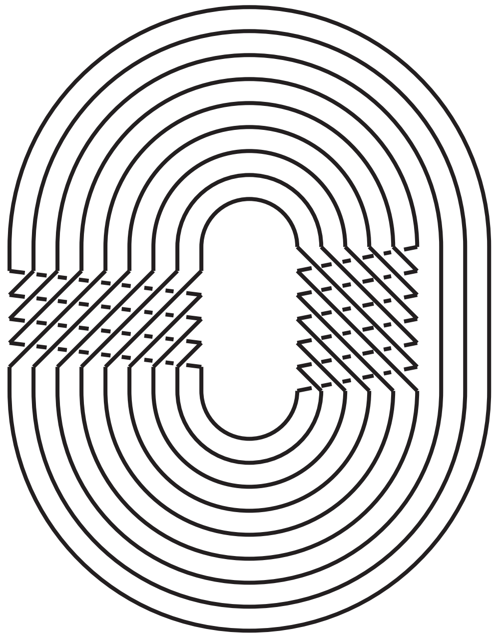
Twisted torus knots $T(p, q, r, s)$

p, q : coprime integers with $1 < q < p$
r : an integer with $1 \leq r \leq p+q$
s : a nonzero integer
$T(p, q)$: the torus knot of type (p, q)

$T(p, q, r, s)$ is a knot obtained from $T(p, q)$ by twisting r adjacent strands fully s times.



T(9, 4)



T(9, 4, 6, −1)

Question. Can we determine the parameters p, q, r, s for which $T(p, q, r, s)$ is a torus knot?

Known

Easy to see: $T(p, q, p, s) = T(p, q + ps)$

Theorem 1. $T(p, q, kq, s)$ *is the* $(q, p+k^2qs)$*-cable knot on* $T(k, ks+1)$.

Theorem 2. *Assume* $r \neq p$ *and* r *is not a multiple of* q.
If $T(p, q, r, s)$ *is a torus knot, then* $s = -2, -1,$ *or* 1.

$s = -2$: $T(p, q, r, s) = T(2n + \varepsilon, n, n + \varepsilon, -2) = T(2n + \varepsilon, -2\varepsilon)$,
$n \geq 1, \varepsilon = \pm 1, (n, \varepsilon) \neq (1, -1)$

$s = 1$: $T(p, q, r, s) = T(mn + 1, n, n - 1, 1) = T(mn + n - 1, n)$,
$m \geq 1, n \geq 3$

$s = -1$: $T(p, q, r, s) =$?

What if $r = p - kq$ for some integer k?

Main Results

$p - kq = r < q$	Theorem 3	
$q < r = p - kq$	Theorem 4(2)	Theorem 4(1)
	$k = 1$	$k \geq 2$

Theorem 3. *Let p, q, k be positive integers such that p and q are co-prime, $1 < q < p$ and $2 \leq p - kq$. Suppose that $p - kq < q$. Then*

$T(p, q, p - kq, -1)$ *is a torus knot*
$\iff (p, q, p - kq) = (mn + n - 1, n, n - 1)$

for some integers $m \geq 1$ and $n \geq 3$. In this case,

$$T(p, q, p - kq, -1) = T(mn + 1, n).$$

Theorem 4. *Let* p, q, k *be positive integers such that* p *and* q *are coprime,* $1 < q < p$ *and* $2 \leq p - kq$*. Suppose that* $q < p - kq$*. Then*

$T(p, q, p - kq, -1)$ *is a torus knot* $\iff$ *one of the following holds:*

(1) if $k \geq 2$*, then either*

(i) $(p, q, p - kq) = (mn + n + 1, n, n + 1)$ *for some integers* $m \geq 2$ *and* $n \geq 2$*; or*

(ii) $(p, q, p - kq) = (4n + 3\varepsilon, n + \varepsilon, 2n + \varepsilon)$ *for some integers* $n \geq 1$ *and* $\varepsilon = \pm 1$ *such that* $(n, \varepsilon) \neq (1, -1)$ *and* $(n, \varepsilon) \neq (2, -1)$*.*

In the former, $T(p, q, p - kq, -1) = T(mn - 1, n)$*, and in the latter,* $T(p, q, p - kq, -1) = T(2n + \varepsilon, 2\varepsilon)$*.*

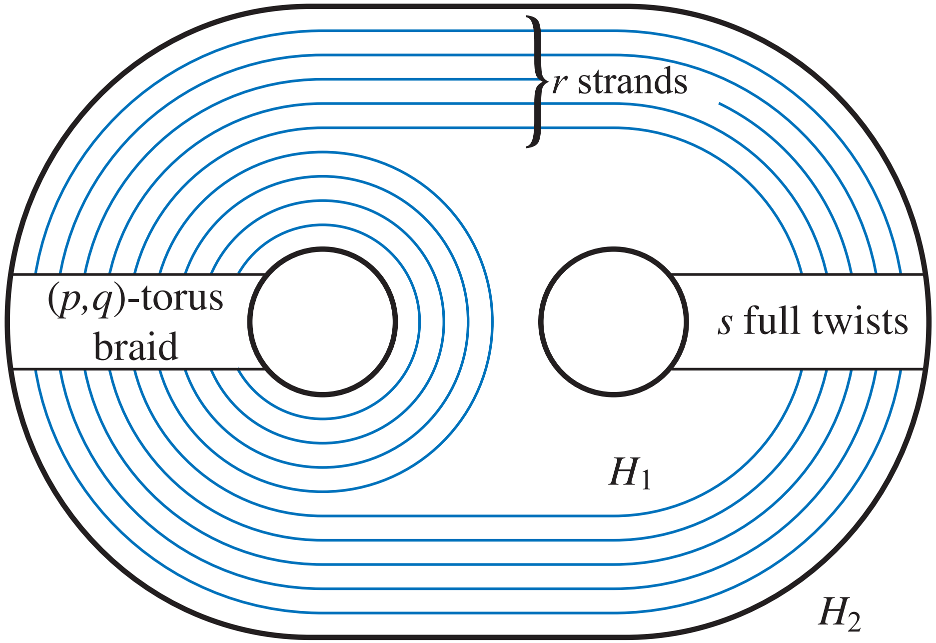
(2) *if* $k = 1$, *then either*

(i) $(p, q, p-q) = (nf_{m+3} - f_{m+2}, nf_{m+1} - f_m, nf_{m+2} - f_{m+1})$ *for some integers* $m \geq 1$ *and* $n \geq 2$ *such that* $(m, n) \neq (1, 2)$; *or*

(ii) $(p, q, p-q) = (nf_{m+1} + f_{m+2}, nf_{m-1} + f_m, nf_m + f_{m+1})$ *for some integers* $m \geq 2$ *and* $n \geq 2$,

where f_m *is the* m*th Fibonacci number defined by the recurrence relation* $f_{m+1} = f_{m-1} + f_m$ *with seed values* $f_1 = 1$ *and* $f_2 = 1$. *In this case,* $T(p, q, r, -1) = T(n + 1, (-1)^m n)$.

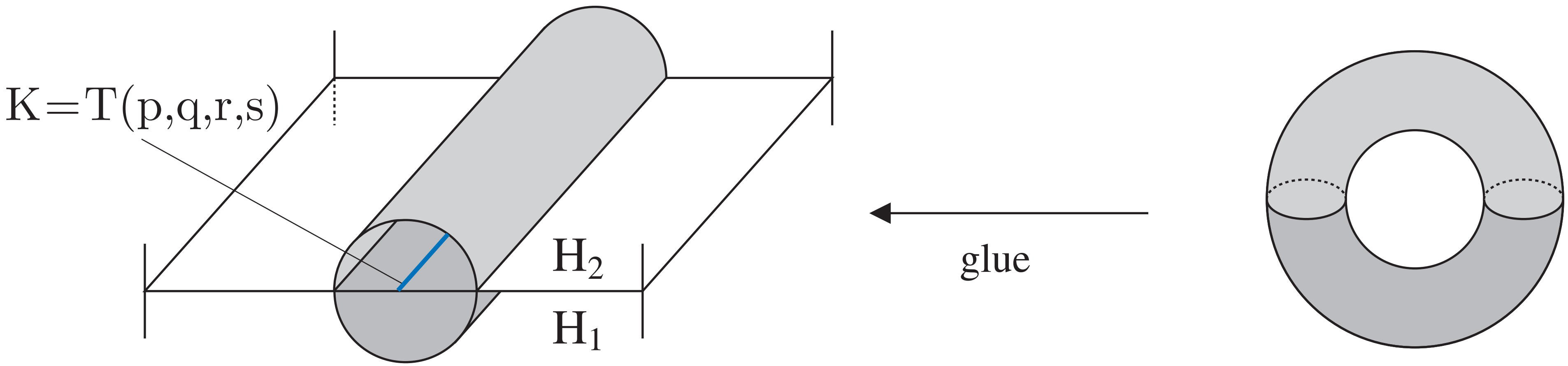
H_1, H_2 : genus two handlebodies such that $\partial H_1 = \partial H_2$ contains $T(p,q,r,s)$



$T(p,q,r,s)$ defines two elements $w_{p,q,r,s} \in \pi_1 H_1$ and $w'_{p,q,r,s} \in \pi_1 H_2$

Lemma (John Dean).
(1) $w_{p,q,r,s}$ is primitive in $\pi_1 H_1 \iff r \equiv \pm 1$ or $\pm q \mod p$.
(2) $w'_{p,q,r,s}$ is primitive in $\pi_1 H_2 \iff r \equiv \pm 1$ or $\pm p \mod q$ and $|s| = 1$.

Dehn surgery along surface slope



Lemma (John Dean).
The surface slope of $T(p,q,r,s)$ with respect to ∂H_1 is $pq + r^2 s$.

$$K(pq + r^2 s) = H_1[K] \cup_\partial H_2[K]$$

Lemma (Franks-Williams).

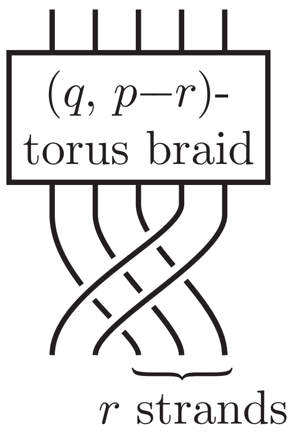
Let β be a positive braid on n strands with $\beta = a\Delta^2$, where Δ^2 is the full twist word. Then n is the braid index of $\widehat{\beta}$.

Lemma 1. *Suppose that $p \geq q + r$ and $q > r$. Then*

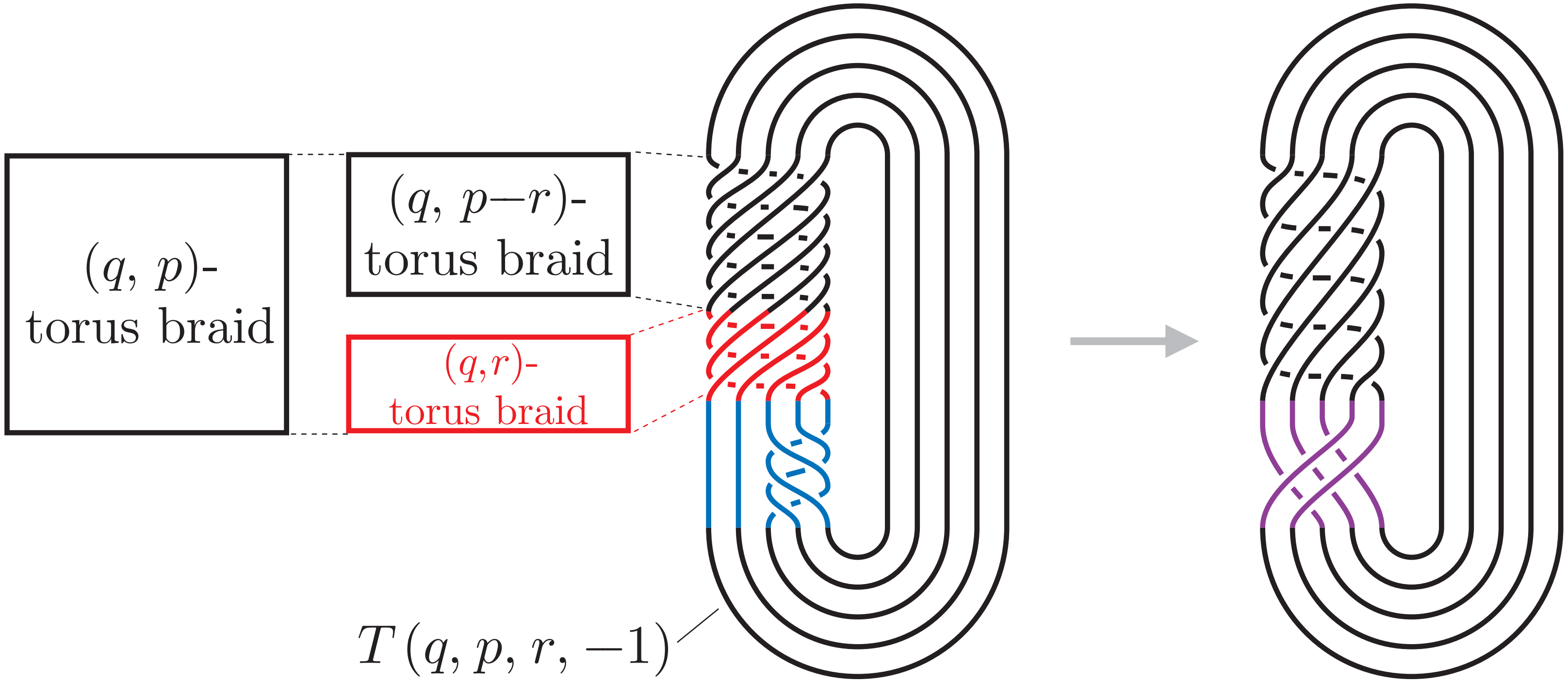
$$\mathrm{br}(T(p, q, r, -1)) = q,$$

$$2\mathrm{g}(T(p, q, r, -1)) = (p - 1)(q - 1) - r(r - 1).$$

Proof. Dean: $T(p, q, r, -1) \sim T(q, p, r, -1)$.
$T(q, p, r, -1)$ is isotopic to the closure of the positive q-braid.



The case where $(p,q,r) = (8,5,3)$ is shown below. Since $p - r \geq q$, the resulting positive braid knot contains a full twist and it follows from a result of Franks-Williams that the knot has braid index q.

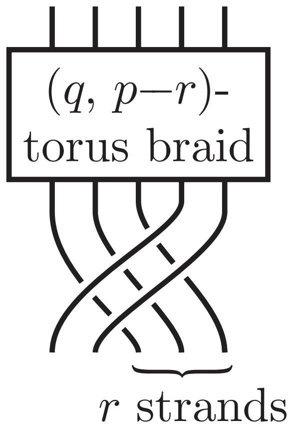


Known: K = a positive b-braid knot with c crossings

$$\Rightarrow \mathrm{g}(K) = \frac{1-b+c}{2}.$$

$T(q,p,r,-1)$: q strands, #crossing $= (q-1)(p-r) + r(q-r)$, so

$$2\mathrm{g}(T(q,p,r,-1)) = (p-1)(q-1) - r(r-1).$$



□

Lemma 1. $p \geq q + r$ and $r < q \;\Rightarrow\;$ br$(T(p, q, r, -1)) = q$, and

$$2g(T(p, q, r, -1)) = (p - 1)(q - 1) - r(r - 1).$$

Lemma 2. *Suppose $T(p, q, p - q, -1)$ is a torus knot and $1 \leq p - q < q$. Then $p = q + 1$ or $p = 2q - 1$.*

Proof. Suppose $T(p, q, p - q, -1) = T(a, b)$, where $a < b$.
Since br$(T(a, b)) = a$ and $2g(T(a, b)) = (a - 1)(b - 1)$,
by Lemma 1 we obtain

$$a = q$$

and

$$(a - 1)(b - 1) = (p - 1)(q - 1) - (p - q)(p - q - 1).$$

$$p - q \equiv -q \mod p$$

and

$$p - q \equiv p \mod q,$$

so Dean's Lemmas show that Dehn surgery on $T(p, q, p-q, -1)$ with slope $pq - (p-q)^2$ gives a lens space with order $pq - (p-q)^2 (\geq 2)$.

Lemma (John Dean).
(1) $w_{p,q,r,s}$ is primitive in $\pi_1 H_1 \iff r \equiv \pm 1$ or $\pm q \mod p$.
(2) $w'_{p,q,r,s}$ is primitive in $\pi_1 H_2 \iff r \equiv \pm 1$ or $\pm p \mod q$ and $|s| = 1$.

$$K(pq + r^2 s) = H_1[K] \cup_\partial H_2[K]$$

By the classification of the manifolds obtained by Dehn surgery on a torus knot, we must have an integer $\varepsilon = \pm 1$ such that

$$ab + \varepsilon = pq - (p - q)^2.$$

Thus we have

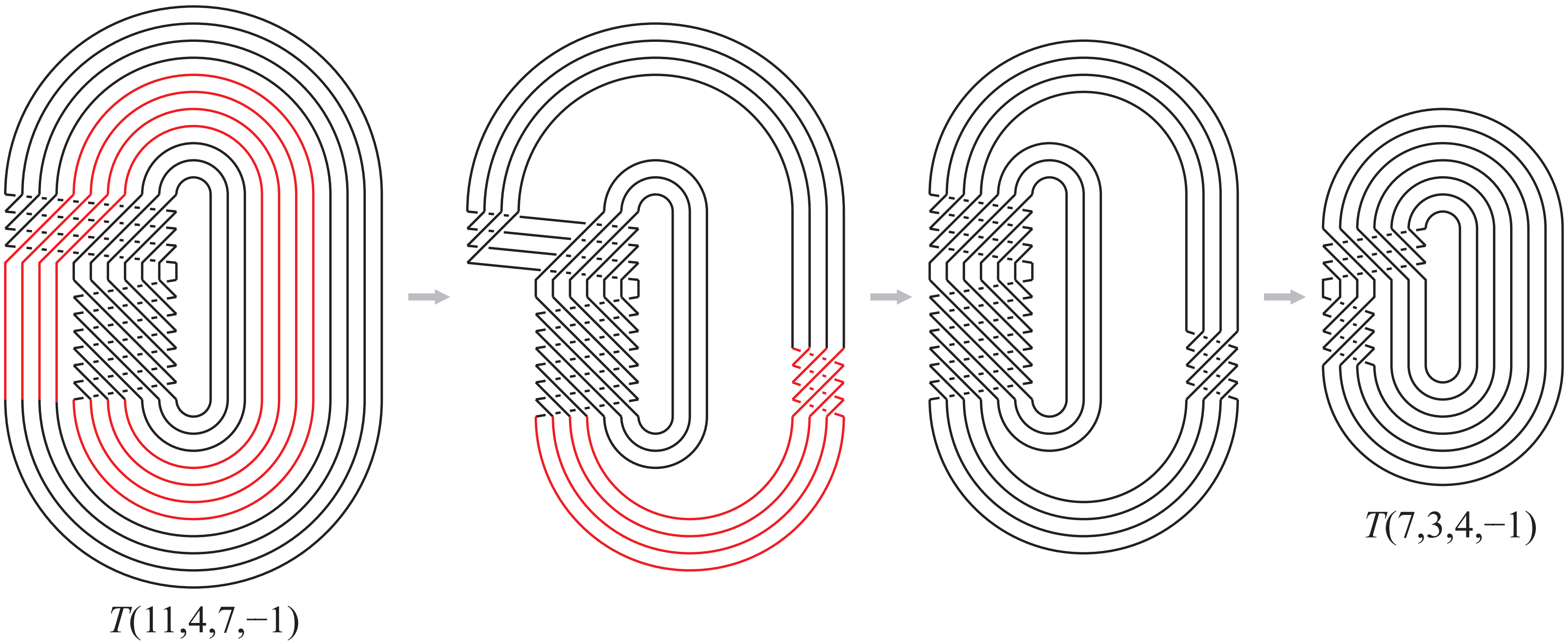
$$
\begin{aligned}
b &= (ab+\varepsilon) - (a-1)(b-1) - a - \varepsilon + 1 \\
&= [pq - (p-q)^2] - [(p-1)(q-1) - (p-q)(p-q-1)] - q - \varepsilon + 1 \\
&= q - \varepsilon.
\end{aligned}
$$

Since $a < b$, we have $\varepsilon = -1$ and $b = q+1$. Putting $a = q, b = q+1$, and $\varepsilon = -1$ in the equation $ab + \varepsilon = pq - (p-q)^2$, we get

$$q(q+1) - 1 = pq - (p-q)^2.$$

Solving the resulting quadratic equation, we obtain $p = q+1$ or $p = 2q-1$ as desired. $\square$

Lemma 3. *Suppose* $q < p-q$. *Then* $T(p,q,p-q,-1)$ *is isotopic to the mirror image of* $T(p-q,p-2q,q,-1)$.



Theorem 4. Suppose $T(p,q,p-q,-1)$ is a torus knot and $2 \leq q < p-q$. Then $T(p,q,p-q,-1) = T(n+1,(-1)^m n)$ and either

(i) $(p,q,p-q) = (nf_{m+3} - f_{m+2}, nf_{m+1} - f_m, nf_{m+2} - f_{m+1})$ for some integers $m \geq 1$ and $n \geq 2$ such that $(m,n) \neq (1,2)$; or

(ii) $(p,q,p-q) = (nf_{m+1} + f_{m+2}, nf_{m-1} + f_m, nf_m + f_{m+1})$ for some integers $m \geq 2$ and $n \geq 2$,

where f_m is the mth Fibonacci number defined by the recurrence relation $f_{m+1} = f_{m-1} + f_m$ with seed values $f_1 = 1$ and $f_2 = 1$.

Lemma 3. $q < p - q \Rightarrow T(p, q, p - q, -1) = T(p - q, p - 2q, q, -1)^!$.

$$\begin{aligned} q < p - q \quad &\Rightarrow \quad T(p, q, p - q, -1) &= \quad T(p - q, p - 2q, q, -1)^! \\ p - 2q < q \quad &\Rightarrow \quad T(p - q, p - 2q, q, -1) &= \quad T(q, 3q - p, p - 2q, -1)^! \end{aligned}$$

$$\vdots$$

As long as $q_m < r_m = p_m - q_m$,

$$
\begin{aligned}
& T(1p + 0q, 0p + 1q, 1p - 1q, -1) \\
= \; & T(1p - 1q, 1p - 2q, 0p + 1q, -1)^{!} \\
= \; & T(0p + 1q, -1p + 3q, 1p - 2q, -1) \\
= \; & T(1p - 2q, 2p - 5q, -1p + 3q, -1)^{!} \\
= \; & T(-1p + 3q, -3p + 8q, 2p - 5q, -1) \\
= \; & T(2p - 5q, 5p - 13q, -3p + 8q, -1)^{!} \\
= \; & T(-3p + 8q, -8p + 21q, 5p - 13q, -1) \\
& \quad \vdots \\
= \; & T(p_m, q_m, r_m, -1) \\
& \quad \vdots
\end{aligned}
$$

We can prove: $p_m - q_m = r_m < q_m$ for some m

Lemma 2. $T(p, q, p-q, -1)$ is a torus knot and $1 \leq p-q < q$.
$\Rightarrow p = q + 1$ or $p = 2q - 1$

By Lemma 2, $p_m = q_m + 1$ or $p_m = 2q_m - 1$.

Suppose $p_m = q_m + 1$ and let $q_m = n$. Then

$$T(p_m, q_m, r_m, -1) = T(n+1, n, 1, -1) = T(n+1, n).$$

$$\begin{aligned} T(p_m, q_m, r_m, -1) &= T(n+1, n) \\ T(p_{m-1}, q_{m-1}, r_{m-1}, -1) &= T(n+1, -n) \\ T(p_{m-2}, q_{m-2}, r_{m-2}, -1) &= T(n+1, n) \\ &\vdots \\ T(p, q, r, -1) = T(p_0, q_0, r_0, -1) &= T(n+1, (-1)^m n) \end{aligned}$$

$$
\begin{aligned}
& T(1p+0q, 0p+1q, 1p-1q, -1) \\
= \ & T(1p-1q, 1p-2q, 0p+1q, -1)^{!} \\
= \ & T(0p+1q, -1p+3q, 1p-2q, -1) \\
= \ & T(1p-2q, 2p-5q, -1p+3q, -1)^{!} \\
= \ & T(-1p+3q, -3p+8q, 2p-5q, -1) \\
= \ & T(2p-5q, 5p-13q, -3p+8q, -1)^{!} \\
& \quad \vdots \\
= \ & T(p_m, q_m, r_m, -1) = T(n+1, n, 1, -1)
\end{aligned}
$$

$$
\Rightarrow \quad \begin{cases} p = p_0 = f_{m+1}n + f_{m+2} \\ q = q_0 = f_{m-1}n + f_m \\ r = r_0 = f_m n + f_{m+1}. \end{cases}
$$

Suppose $p_m = 2q_m - 1$.

By a similar argument as in the case that $p_m = q_m + 1$, we can show

$$\begin{cases} p = p_0 = f_{m+3}n - f_{m+2} \\ q = q_0 = f_{m+1}n - f_m \\ r = r_0 = f_{m+2}n - f_{m+1}. \end{cases}$$

$$T(p, q, r, -1) = T(n + 1, (-1)^m n)$$

Theorem 5. *Let* m *are* n *be positive integers. Then we have the following:*

(1) $T(mn+m+1, mn+1, mn, -1) = T(mn+n+1, m+1)$*;*

(2) $T(mn+m+1, mn+1, mn+m+2, -1) = T(mn+m+n+2, -m-1)$*;*

(3) $T(mn+m+1, mn+1, mn+m, -1) = T(mn+m-n, -m+1)$*;*

(4) $T(mn+m+1, mn+1, mn+2, -1) = T(mn-n+1, m-1)$*;*

(5) $T(mn+m-1, mn-1, mn+m-2, -1) = T(mn+m-n-2, -m+1)$*;*

(6) $T(mn+m-1, mn-1, mn, -1) = T(mn-n-1, m-1)$*;*

(7) $T(2n+1, n, 2n-1, -1) = T(2n-3, -n+1)$*; and*

(8) $T(n+1, n, 2n-1, -1) = T(3n-2, -n+1)$*.*

Question. Are there any other examples?

THANK YOU