Linear spatial complete graphs and generalized Conway-Gordon theorems

NIKKUNI Ryo
Tokyo Woman's Christian University

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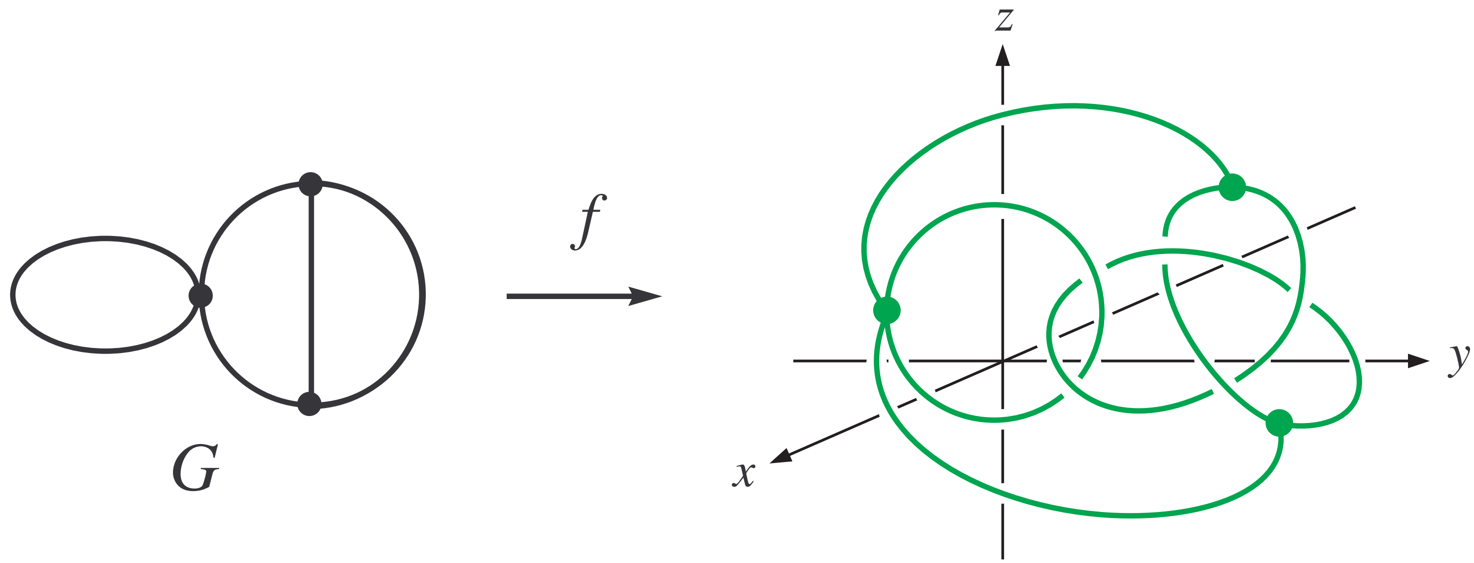
§1. Conway-Gordon theorems

Spatial graph = The image of a spatial embedding f of a finite graph G into $\mathbb{R}^3$



For a (disjoint union of) *cycle(s)* λ of G, $f(\lambda)$ is called a *constituent knot (link)* of the spatial graph.

$$\mathrm{SE}(G) \overset{\text{def.}}{=} \{\text{embedding } f : G \to \mathbb{R}^3\}$$

$$\Gamma_k(G) \overset{\text{def.}}{=} \{k\text{-cycles of } G\}$$

$$\Gamma_{k,l}(G) \overset{\text{def.}}{=} \{\text{a disjoint pair of } k\text{-cycle and } l\text{-cycle of G}\}$$

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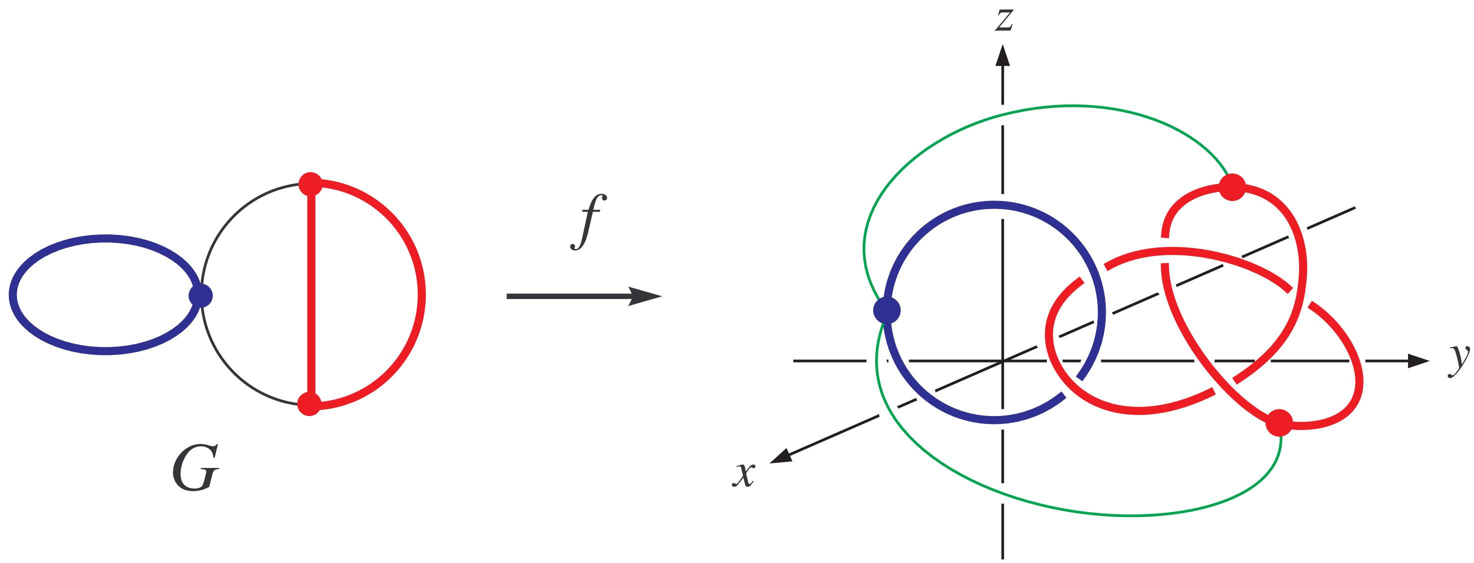

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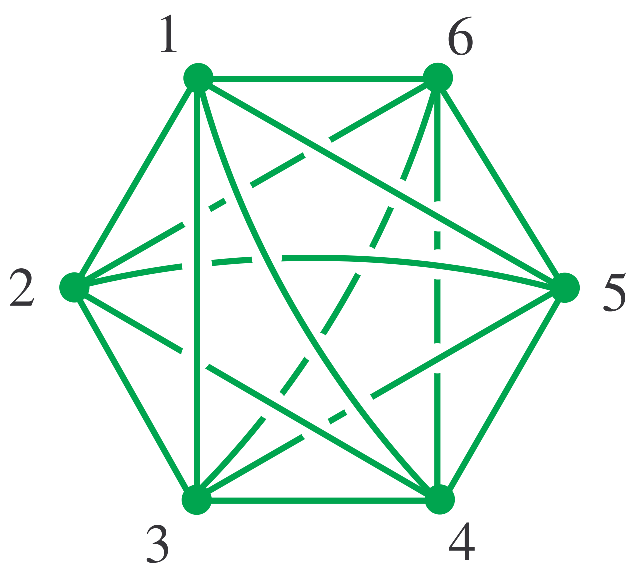
K_n : *complete graph* on n vertices

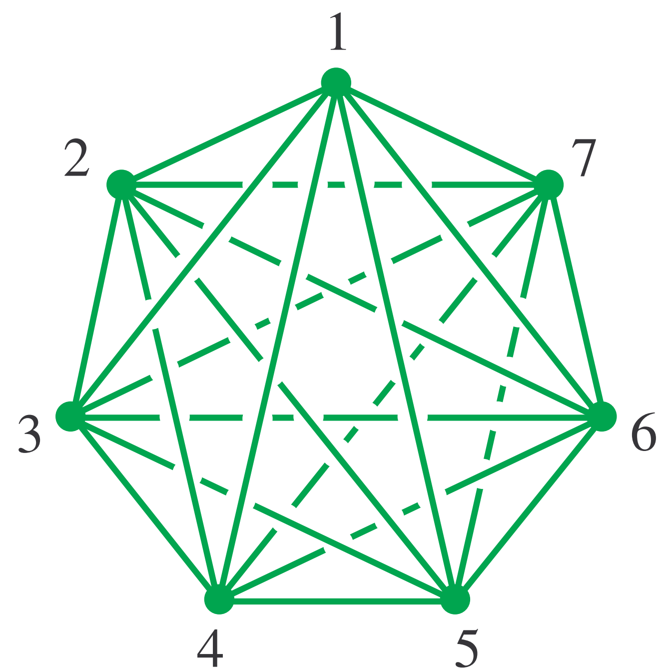
Theorem 1.1. [Conway-Gordon '83]

(1) $\forall f \in \mathrm{SE}\,(K_6)$, $\displaystyle\sum_{\lambda \in \Gamma_{3,3}(K_6)} \mathrm{lk}(f(\lambda)) \equiv 1 \pmod 2$.

(2) $\forall f \in \mathrm{SE}\,(K_7)$, $\displaystyle\sum_{\gamma \in \Gamma_7(K_7)} a_2\,(f\,(\gamma)) \equiv 1 \pmod 2$.

Here, lk: *linking number*, a_2: 2nd coefficient of $\nabla(z)$.





$\therefore\ \forall f\,(K_6) \supset$ nonsplittable link, $\quad \forall f\,(K_7) \supset$ nontrivial knot.

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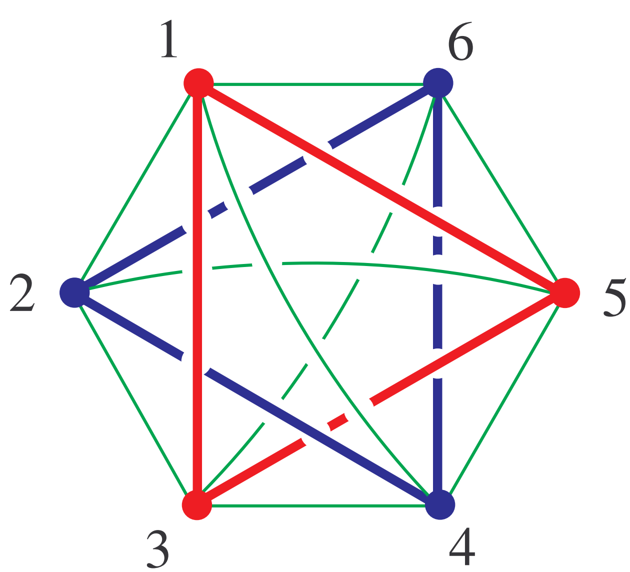

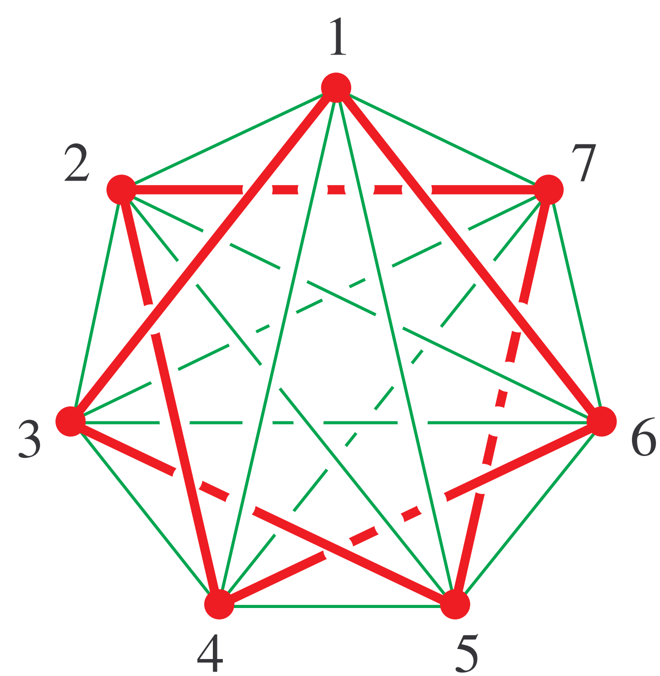

$\therefore$ $\forall f(K_6) \supset$ nonsplittable link, $\forall f(K_7) \supset$ nontrivial knot.

Integral lifts of the Conway-Gordon theorems are known:

Theorem 1.2.

(1) [Nikkuni '09] $\forall f \in \mathsf{SE}(K_6)$,

$$\sum_{\gamma \in \Gamma_6(K_6)} a_2(f(\gamma)) - \sum_{\gamma \in \Gamma_5(K_6)} a_2(f(\gamma)) = \frac{1}{2} \sum_{\lambda \in \Gamma_{3,3}(K_6)} \mathrm{lk}(f(\lambda))^2 - \frac{1}{2}.$$

(2) [Nikkuni '09] $\forall f \in \mathsf{SE}(K_7)$,

$$\sum_{\gamma \in \Gamma_7(K_7)} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(K_7)} a_2(f(\gamma))$$
$$= \frac{1}{7}\left(2 \sum_{\lambda \in \Gamma_{3,4}(K_7)} \mathrm{lk}(f(\lambda))^2 + 3 \sum_{\lambda \in \Gamma_{3,3}(K_7)} \mathrm{lk}(f(\lambda))^2\right) - 6.$$

Remark. Thm 1.2 $\xrightarrow{\text{mod } 2}$ Conway-Gordon theorems

Our purposes are to give "integral Conway-Gordon theorem" for K_n with arbitrary $n \geq 6$ and investigate the behavior of the "Hamiltonian" constituent knots and links.

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(2) [Morishita-Nikkuni '19] $\forall f \in \mathrm{SE}(K_7)$,

$$\sum_{\gamma \in \Gamma_7(K_7)} a_2(f(\gamma)) - 2 \sum_{\gamma \in \Gamma_5(K_7)} a_2(f(\gamma))$$

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Remark. **Thm 1.2** $\xrightarrow{\text{mod } 2}$ Conway-Gordon theorems

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§2. Generalizations of the Conway-Gordon theorems

Theorem 2.1. [Morishita-Nikkuni '19]

For $n \geq 6$, $\forall f \in \mathrm{SE}(K_n)$,

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) - (n-5)! \sum_{\gamma \in \Gamma_5(K_n)} a_2(f(\gamma))$$
$$= \frac{(n-5)!}{2}\left(\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 - \binom{n-1}{5} \right).$$

$n=6$: $\sum_6 a_2 - \sum_5 a_2 = \frac{1}{2}\sum_{3,3} \mathrm{lk}^2 - \frac{1}{2}$. (**Thm. 1.2** (1))

$n=7$: $\sum_7 a_2 - 2\sum_5 a_2 = \sum_{3,3} \mathrm{lk}^2 - 6$. (**Thm. 1.2** (2))

$n=8$: $\sum_8 a_2 - 6\sum_5 a_2 = 3\sum_{3,3} \mathrm{lk}^2 - 63$.

Note: $\nexists\lambda \in \Gamma_{3,3}(K_n)$ s.t. λ is shared by two distinct subgraphs of K_n isomorphic to K_6.

$$\overset{\text{Thm. 1.1 (1)}}{\Longrightarrow} \quad \forall f \in \mathrm{SE}(K_n), \quad \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}\,(f(\lambda))^2 \geq \binom{n}{6}.$$

Corollary 2.2. For $n \geq 6$, $\forall f \in \mathrm{SE}\,(K_n)$,

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2\,(f(\gamma)) - (n-5)! \sum_{\gamma \in \Gamma_5(K_n)} a_2\,(f(\gamma))$$
$$\geq \frac{(n-5)(n-6)(n-1)!}{2 \cdot 6!}.$$

Remark. [Otsuki '96] For $n \geq 6$, $\exists f_{\mathrm{b}} \in \mathrm{SE}(K_n)$ s.t.

$f_{\mathrm{b}}\,(K_n) \supset$ exactly $\binom{n}{6}$ triangle-triangle Hopf links.

(f_{b}: *canonical book presentation* of K_n [Endo-Otsuki '94])

Thus the lower bound of **Cor. 2.2** is sharp.

By **Thm 2.1**, $\sum_n a_2(f(\gamma)) \equiv \sum_n a_2(g(\gamma)) \pmod{(n-5)!}$ for $\forall f, g \in \mathrm{SE}(K_n)$. Moreover we have:

Corollary 2.3. For $n \geq 7$, $\forall f \in \mathrm{SE}(K_n)$, we have the following congruence modulo $(n-5)!$:

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f(\gamma)) \equiv \begin{cases} -\dfrac{(n-5)!}{2}\dbinom{n-1}{5} & (n \equiv 0 \pmod 8) \\ 0 & (n \not\equiv 0, 7 \pmod 8) \\ \dfrac{(n-5)!}{2}\dbinom{n}{6} & (n \equiv 7 \pmod 8). \end{cases}$$

$n = 7$: $\sum_7 a_2 \equiv 7 \equiv 1 \pmod 2$. (**Thm. 1.1** (1))

$n = 8$: $\sum_8 a_2 \equiv -63 \equiv 3 \pmod 6$. [Foisy '08]+[Hirano '10]

$n = 9$: $\sum_9 a_2 \equiv 0 \pmod{24}$.

For "Hamiltonian" 2-component constituent links, we also have the following formula:

Theorem 2.4. [Morishita-Nikkuni '19]

(1) For $n = p + q$ $(p, q \geq 3)$, $\forall f \in \mathrm{SE}(K_n)$,

$$\sum_{\lambda \in \Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 = \begin{cases} (n-6)! \displaystyle\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 & (p = q) \\ 2(n-6)! \displaystyle\sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 & (p \neq q). \end{cases}$$

(2) For $n \geq 6$, $\forall f \in \mathrm{SE}(K_n)$,

$$\sum_{p+q=n} \sum_{\lambda \in \Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 = (n-5)! \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2.$$

$n = 7$: $\displaystyle\sum_{3,4} \mathrm{lk}^2 = 2 \sum_{3,3} \mathrm{lk}^2.$

$n = 8$: $\displaystyle\sum_{3,5} \mathrm{lk}^2 = 2 \sum_{4,4} \mathrm{lk}^2 = 4 \sum_{3,3} \mathrm{lk}^2.$

Since $\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f(\lambda))^2 \geq \binom{n}{6}$, we have the following:

Corollary 2.5.

(1) For $n = p + q$ $(p, q \geq 3)$, $\forall f \in \mathrm{SE}\,(K_n)$,

$$\sum_{\lambda\in\Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 \geq \begin{cases} n!/6! & (p = q) \\ 2\cdot n!/6! & (p \neq q) \end{cases}.$$

(2) For $n \geq 6$, $\forall f \in \mathrm{SE}\,(K_n)$,

$$\sum_{p+q=n} \sum_{\lambda\in\Gamma_{p,q}(K_n)} \mathrm{lk}(f(\lambda))^2 \geq (n-5)\cdot\frac{n!}{6!}.$$

$n = 7$: $\sum_{3,4} \mathrm{lk}^2 \geq 2\cdot 7 = 14$. [Fleming-Mellor '09]

$n = 8$: $\sum_{3,5} \mathrm{lk}^2 \geq 2\cdot 8\cdot 7 = 112$, $\sum_{4,4} \mathrm{lk}^2 \geq 8\cdot 7 = 56$.

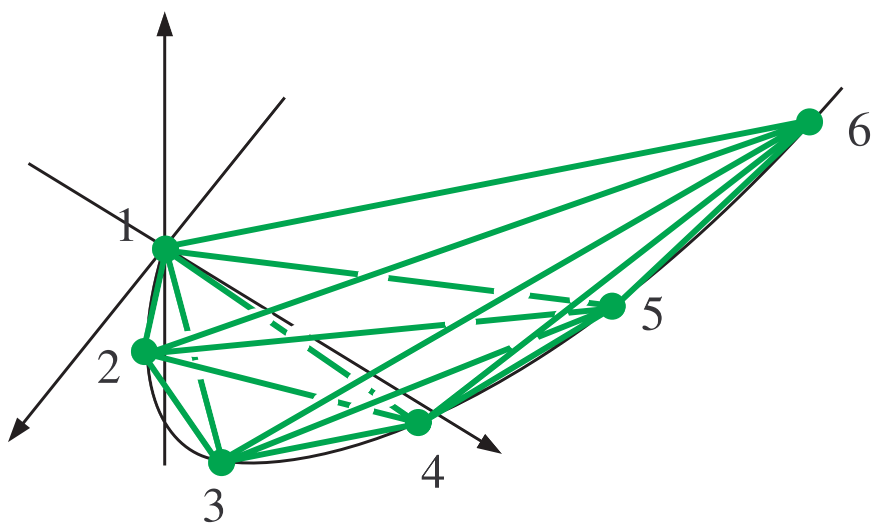
Remark. The lower bounds in **Cor. 2.5** are sharp.

§3. (Recti)linear spatial complete graphs

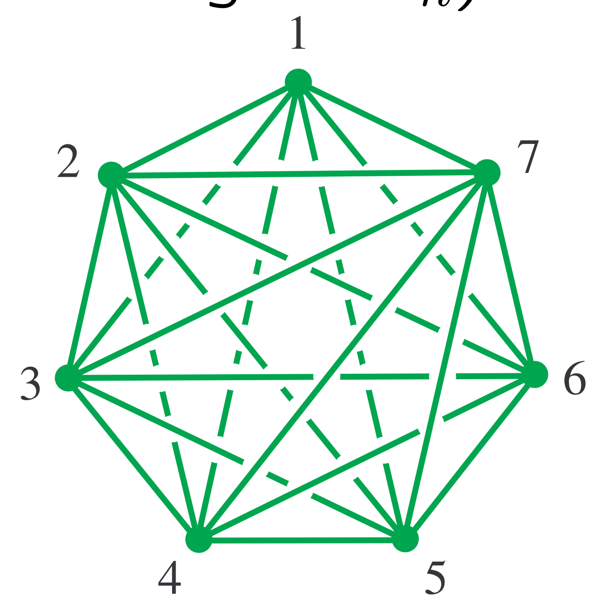
A spatial embedding f_{r} of G is *rectilinear* (or linear)
$\overset{\text{def.}}{\Leftrightarrow}$ $\forall$ edge e of G, $f_{\mathrm{r}}(e)$ is a straight line segment in $\mathbb{R}^3$

$$\mathrm{RSE}\,(G) \overset{\text{def.}}{=} \left\{\text{rectilinear embedding } f_{\mathrm{r}} : G \to \mathbb{R}^3\right\}$$

Example. (*Standard* rectilinear embedding of K_n)



$f_{\mathrm{r}}(K_6)$



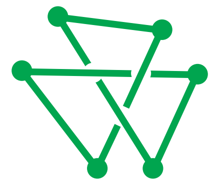
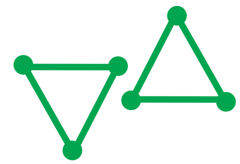
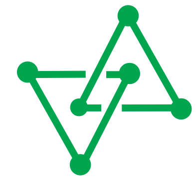
$f_{\mathrm{r}}(K_7)$

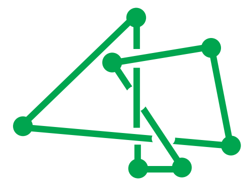
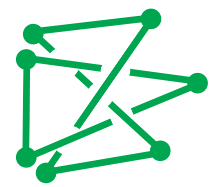
Take n vertices on the curve (t, t^2, t^3) and connect every pair of distinct vertices by a straight line segment.

Stick number $s(L)$ of a link (knot) L :
$s(L) =$ min. # of edges in a polygon which represents L

Proposition 3.1.

(1) L is a nontrivial knot $\Longrightarrow s(L) \geq 6$.

(2) $s(L) = 6 \iff L \cong$  ,  or  .

(3) $s(L) = 7 \iff L \cong$  or  .

Theorem 3.2. [M-N '19] For $n \geq 6$, $\forall f_r \in$ RSE (K_n),

$$\sum_{\gamma \in \Gamma_n(K_n)} a_2(f_r(\gamma)) = \frac{(n-5)!}{2}\left(\sum_{\lambda \in \Gamma_{3,3}(K_n)} \text{lk}(f_r(\lambda))^2 - \binom{n-1}{5}\right).$$

$$\sum_{\lambda \in \Gamma_{3,3}(K_n)} \text{lk}(f_r(\lambda))^2 = \# \text{ of}$$  's

Proposition 3.3. [Hughes '06] [Huh-Jeon '06] [N '09]

$\forall f_{\mathsf{r}} \in \mathsf{RSE}\,(K_6),\ f_{\mathsf{r}}(K_6) \supset$ at most 3 Hopf links.

$$\Longrightarrow\ \forall f_{\mathsf{r}} \in \mathsf{RSE}(K_n),\ \binom{n}{6} \leq \sum_{\lambda \in \Gamma_{3,3}(K_n)} \mathsf{lk}\,(f_{\mathsf{r}}(\lambda))^2 \leq 3\binom{n}{6}.$$

Corollary 3.4. For $n \geq 6$, $\forall f_{\mathsf{r}} \in \mathsf{RSE}\,(K_n)$,

$$\frac{(n-5)(n-6)(n-1)!}{2\cdot 6!} \leq \sum_n a_2 \leq \frac{3(n-2)(n-5)(n-1)!}{2\cdot 6!}.$$

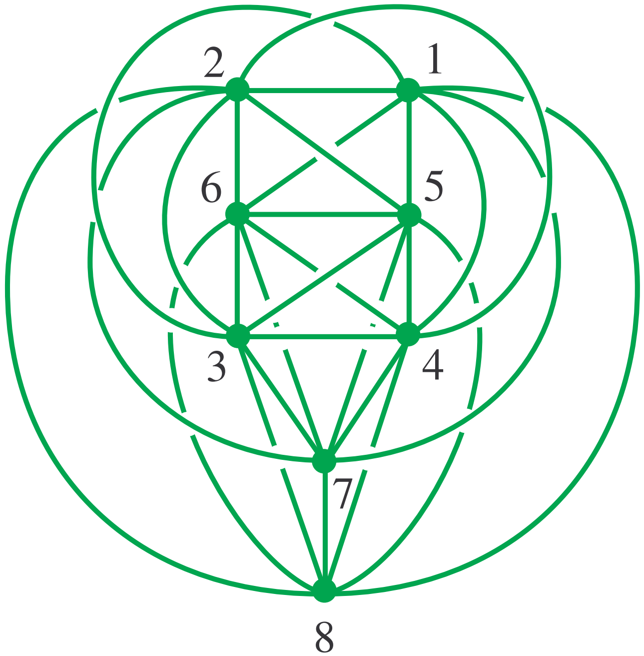
$n=6$: $0 \leq \sum_6 a_2 \leq 1$. ($\Longrightarrow$ $\exists$ at most one trefoil knot)

$n=7$: $1 \leq \sum_7 a_2 \leq 15$, $\sum_7 a_2 \equiv 1 \pmod 2$. ($\Longrightarrow$ $\exists$ trefoil)

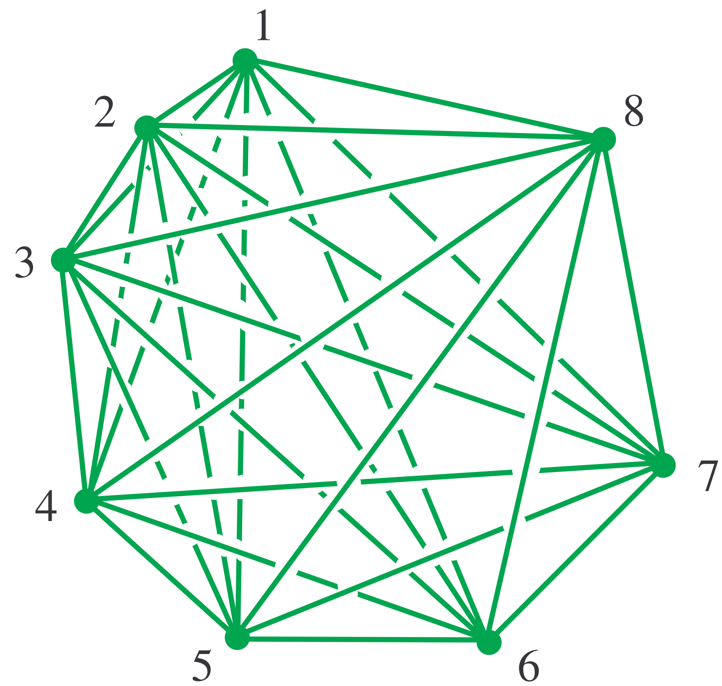
$n=8$: $21 \leq \sum_8 a_2 \leq 189$, $\sum_8 a_2 \equiv 3 \pmod 6$.

Remark. The lower bound in **Cor. 3.4** is sharp (realized by a standard rectilinear embedding), but the upper bound is not expected to be sharp if $n \geq 7$.

Example. $(n = 8)$ Each of the following spatial K_8 contains exactly 21 trefoils as nontrivial Hamiltonian knots:



[BBFHL '07]



[Alfonsín '08]

$\forall$5-cycle knots are trivial $\overset{\text{Cor.2.2}}{\Longrightarrow} \sum_8 a_2 \geq 21.$

Remark. The above mentioned spatial graphs of K_8 are NOT equivalent. (Observe the link with $\mathrm{lk} = 2$)

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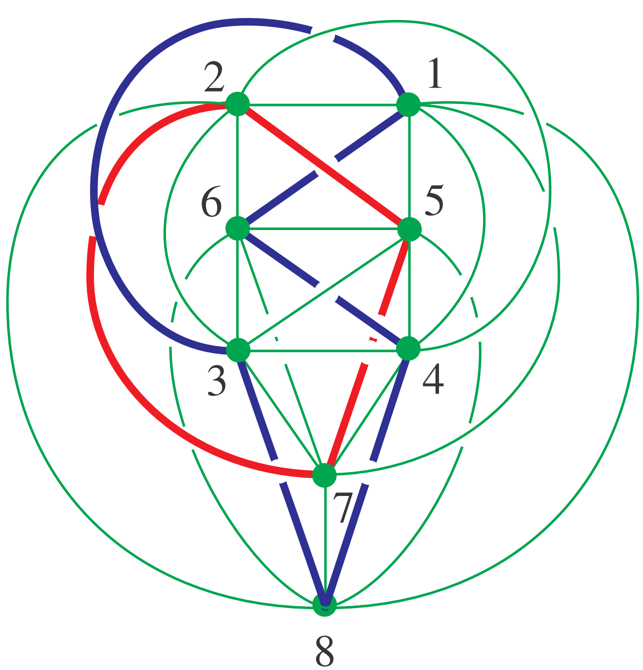

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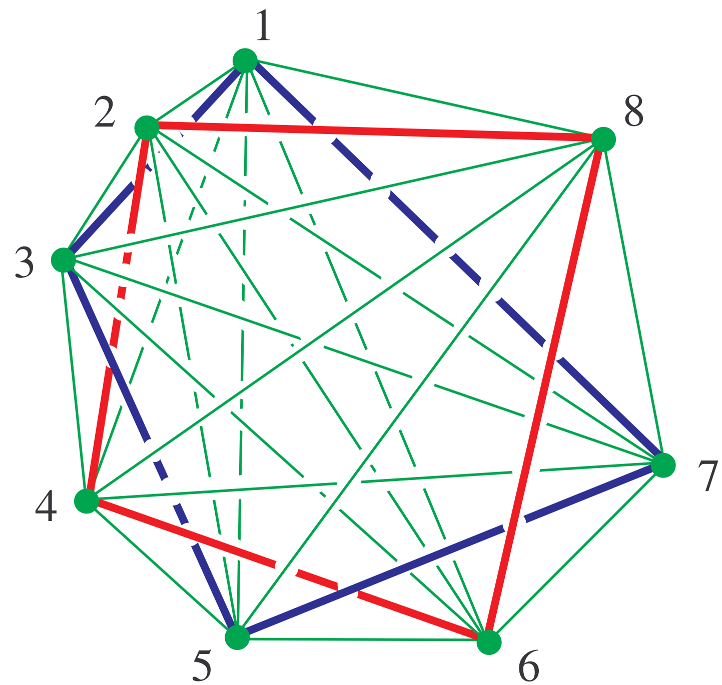

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§4. Jeon's work on linear spatial embeddings of K_7, K_8 and several open problems

According to a computer search with the help of oriented matroid theory in [Jeon et al. IWSG2010] (unpublished) :

The number of knots and links in rectilinear K_7					
6-trefoil	7-trefoil	4_1	$(3,3)$-Hopf	$(3,4)$-Hopf	4_1^2
0	1	0	7	14	0
1	3	0	9	18	0
2	5	0	11	22	0
3	7	0	13	22	1
3	7	0	13	26	0
3	8	1	13	26	0
4	9	0	15	26	1
4	11	2	15	30	0
4	12	3	15	30	0
5	11	0	17	30	1

$$\nexists f_{\mathrm{r}} \in \mathrm{RSE}(K_7) \quad \text{s.t} \quad \sum_{\lambda \in \Gamma_{3,3}(K_7)} \mathrm{lk}(f_{\mathrm{r}}(\lambda))^2 = 19,\ 21$$

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3	7	0	13	26	0
3	8	1	13	26	0
4	9	0	15	26	1
4	11	2	15	30	0
4	12	3	15	30	0
5	11	0	17	30	1

$$\nexists f_{\mathrm{r}} \in \mathrm{RSE}(K_7) \quad \text{s.t} \quad \sum_{\lambda \in \Gamma_{3,3}(K_7)} \mathrm{lk}(f_{\mathrm{r}}(\lambda))^2 = 19,\ 21$$

Note: If this classification is correct, we have

$$(n-6)\sum_{\lambda\in\Gamma_{3,3}(K_n)} \mathrm{lk}(f_{\mathrm{r}}(\lambda))^2 \le \sum_i \Bigg(\sum_{\lambda\in\Gamma_{3,3}(K_7^{(i)})} \mathrm{lk}(f_{\mathrm{r}}(\lambda))^2\Bigg) \le 17\binom{n}{7}.$$

Corollary 4.1. For $n \geq 7$, $\forall f_{\mathrm{r}} \in \mathrm{RSE}\,(K_n)$,

$$\sum_{\gamma\in\Gamma_n(K_n)} a_2(f_{\mathrm{r}}(\gamma)) \;\le\; \frac{(17n-42)(n-5)(n-1)!}{2\cdot 7!}.$$

Problem 4.2. Determine the sharp upper bound of $\sum_n a_2$ for all rect. emb. $f_{\mathrm{r}} \in \mathrm{RSE}(K_n)$ for each $n \geq 7$.

Problem 4.2 is equivalent to the following problem.

Problem 4.3. Determine the maximum number of triangle-triangle Hopf links in $f_{\mathrm{r}}(K_n)$ for each $n \geq 7$.

Conjecture. (given in [Jeon et al. IWSG2013])
$\forall f_{\mathrm{r}} \in \mathrm{RSE}(K_8)$, $f_{\mathrm{r}}(K_8)$ contains a figure eight knot 4_1 or a $(2,4)$-torus link 4_1^2.

- They examined over 50,000,000 random rectilinear spatial embeddings of K_8 and could not find a counterexample.
- By **Cor. 2.3**, **Cor. 2.5** and **Cor. 3.4**,

$$\sum_{8} a_2 = 3 \sum_{3,3} \mathrm{lk}^2 - 63, \quad \sum_{8} a_2 \equiv 3 \pmod 6,$$

$$21 \leq \sum_{8} a_2 \leq 189,$$

$$112 \leq \sum_{3,5} \mathrm{lk}^2 = 2 \sum_{4,4} \mathrm{lk}^2 = \sum_{3,4} \mathrm{lk}^2 = 4 \sum_{3,3} \mathrm{lk}^2 \leq 336.$$

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$$\sum_{8} a_2 = 3 \sum_{3,3} \mathrm{lk}^2 - 63, \quad \sum_{8} a_2 \equiv 3 \pmod 6,$$

$$21 \leq \sum_{8} a_2 \leq 141,$$

$$112 \leq \sum_{3,5} \mathrm{lk}^2 = 2 \sum_{4,4} \mathrm{lk}^2 = \sum_{3,4} \mathrm{lk}^2 = 4 \sum_{3,3} \mathrm{lk}^2 \leq 272.$$