Jeon's work on superbridge index

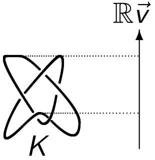
February 17, 2020

Knots and Spatial Graphs 2020

KAIST

Bridge number and bridge index (Schubert 1954)

Given a knot K and a unit vector $\vec{v}$ in $\mathbb{R}^3$, we define $b_{\vec{v}}(K)$ as the number of connected components of the preimage of the set of local maximum values of the orthogonal projection $K \to \mathbb{R}\vec{v}$.
The figure illustrates an example of $b_{\vec{v}}(K) = 3$.



The *bridge number* of K is defined by the formula

$$b(K) = \min_{\|\vec{v}\|=1} b_{\vec{v}}(K).$$

The *bridge index* of a knot K is defined by the formula

$$b[K] = \min_{K' \in [K]} b(K') = \min_{K' \in [K]} \min_{\|\vec{v}\|=1} b_{\vec{v}}(K').$$

Superbridge number and superbridge index (Kuiper 1987)

The *superbridge number* of K is defined by the formula

$$s(K) = \max_{\|\vec{v}\|=1} b_{\vec{v}}(K).$$

The *superbridge index* of a knot K is defined by the formula

$$s[K] = \min_{K' \in [K]} s(K') = \min_{K' \in [K]} \max_{\|\vec{v}\|=1} b_{\vec{v}}(K').$$

Theorem (1)

For any nontrivial knot K, $b[K] < s[K]$.

Theorem (2)

For any two coprime integers p and q, satisfying $2 \leq p < q$, the superbridge index of the torus knot of type (p, q) is $\min\{2p, q\}$.

2-bridge knots and 3-superbridge knots

We know that nontrivial knots have bridge index at least 2 and that there are infinitely many 2-bridge knots.

Theorem (1) implies that nontrivial knots have superbridge index at least 3 and that all 3-superbridge knots are 2-bridge knots.

Theorem (2) implies that there are infinitely many $2p$-superbridge knots for $p \geq 2$.

Proposition

The trefoil knot and the figure eight knot have superbridge index 3.

Question

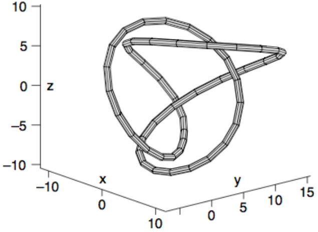
Are there infinitely many 3-superbridge knots?

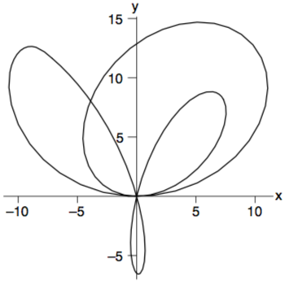
Quadrisecant of a knot

Theorem (Pannwitz, Kuperberg, Morton-Mond)

Every nontrivial knot has a quadrisecent.

This figure shows a figure eight knot which has the *z*-axis as a quadrisecant.



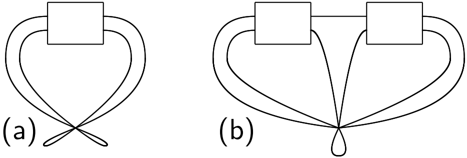


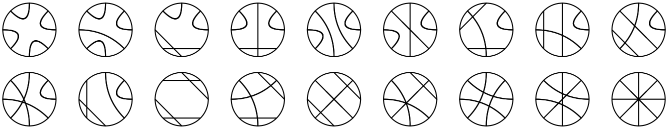
3-superbridge knots

Theorem (Jeon-J, JKTR 10(2001) no.2)

There are only finitely many 3-superbridge knots.

Proof. A 3-superbridge knot has a very simple projection in $\mathcal{Q}^{\perp}$ where $\mathcal{Q}$ is a quadrisecant. It consists of four simple loops with a common base point as in the figure (a) and (b) with each box containing half twists of up to 3 crossings. At the projection of $\mathcal{Q}$, there are 18 possible patterns as shown at the bottom. □





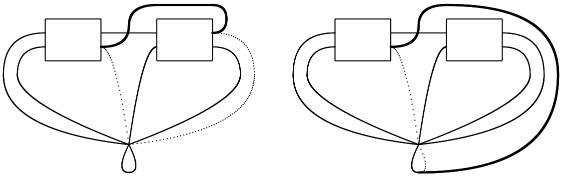
3-superbridge knots are 2-bridge knots up to 9 crossings

The projection of figure (a) gives diagrams of up to 9 crossings.

The projection of figure (b) can be deformed as shown below to have no more than 10 crossings without changing the knot type.

The result gives nonalternating diagrams.

As 2-bridge knots are alternating knots, the diagrams are not of minimal crossings.



3-superbridge knot candidates

Theorem (Jeon-J, JKTR 11(2002) no.3)

No knots other than 3_1, 4_1, 5_2, 6_1, 6_2, 6_3, 7_2, 7_3, 7_4, 8_4, 8_7 *and* 8_9 *have superbridge index 3.*

Theorem (Adams 2011)

$s[8_7] = 4$

Proof. In the proof of (Jeon-J, 2002), each possibility of 8_7 shows that the quadrisecant used is not alternating. Therefore $s[8_7] > 3$. On the other hand $s[8_7] \leq 2b[8_7] = 4$. $\square$

Proof of (Jeon-J, 2002)

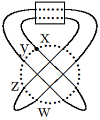
C	Arc levels	[0]	[2]
	xyzw, xywz, xzwy, xwzy, yxzw, yxwz, yzwx, ywzx, zxyw, zyxw, zwxy, zwyx, wxyz, wyxz, wzxy, wzyx	○	○
	xzyw, ywxz, wxzy	○	$\star 4_1$
	xwyz	$\star 3_1$	$\star 5_2$
	yzxw, zxwy, wyzx	○	3_1
	zywx	3_1	○

TABLE C

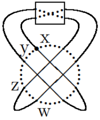
D	Arc levels	[1]	[3]
	xyzw, xywz, xzwy, xwzy, yxzw, yxwz, yzwx, ywzx, zxyw, zyxw, zwxy, zwyx, wxyz, wyxz, wzxy, wzyx	○	○
	xzyw, ywxz, wxzy	3_1	$\rightleftarrows$
	xwyz	4_1	$\rightleftarrows$
	yzxw, zxwy, zywx, wyzx	○	$\rightleftarrows$

TABLE D

Proof of (Jeon-J, 2002) – continued

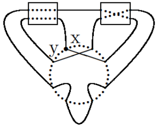
E	Arc levels	[0,1]	[0,3]	[2,1]	[2,−1]	[2,3]	[2,−3]
	xy	○	3_1	○	3_1	○	$\not{5}_1$
	yx	○	3_1	4_1	○	$\star 6_2$	5_2

TABLE E

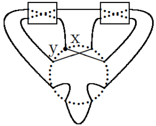
F	Arc levels	[1,1]	[1,−1]	[1,±3]	[3,±3]
	xy	○	○	$\rightleftarrows$	$\rightleftarrows$
	yx	3_1	○	$\rightleftarrows$	$\rightleftarrows$

TABLE F

Proof of (Jeon-J, 2002) – continued

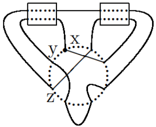
G	Arc levels	[0,0]	[0,2]	[2,2]	[2,−2]
	xyz, yxz	○	○	○	5_2
	xzy	○	3_1	○	○
	yzx	○	4_1	6_2	$\not{5}_1$
	zxy, zyx	○	○	4_1	3_1

TABLE G

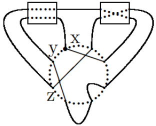
H	Arc levels	[0,1]	[0,3]	[2,1]	[2,−1]	[2,±3]
	xyz, yxz	○	○	○	4_1	⇄
	xzy	○	4_1	○	○	⇄
	yzx	3_1	5_2	5_2	○	⇄
	zxy, zyx	○	○	3_1	○	⇄

TABLE H

Proof of (Jeon-J, 2002) – continued

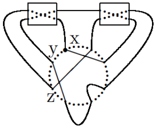
I	Arc levels	[1,1]	[1,−1]	[1,3]	[1,−3]	[3,±3]
	xyz, yxz	◯	3_1	◯	$\not{5}_1$	⇄
	xzy, zxy, zyx	◯	◯	3_1	3_1	⇄
	yzx	4_1	◯	6_2	5_2	⇄

TABLE I

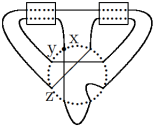
J	Arc levels	[0,0]	[0,2]	[2,2]	[2,−2]
	xyz, zyx	◯	◯	3_1	4_1
	xzy	◯	4_1	◯	◯
	yxz	◯	◯	◯	$\star 6_1$
	yzx	◯	3_1	5_2	6_1
	zxy	◯	◯	5_2	◯

TABLE J

Proof of (Jeon-J, 2002) – continued

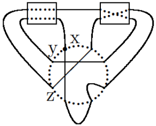
K	Arc levels	[0,1]	[0,3]	[2,1]	[2,−1]	[2,±3]
	xyz, zyx	○	○	○	3_1	⇄
	xzy	3_1	5_2	○	○	⇄
	yxz	○	○	3_1	5_2	⇄
	yzx	○	4_1	3_1	$\not{5}_1$	⇄
	zxy	○	○	4_1	○	⇄

TABLE K

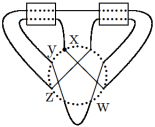
L	Arc levels	[0,0]	[0,2]	[2,2]	[2,−2]
	xyzw, xywz, xwyz, yxzw, yxwz, zywx, zwxy, zwyx, wzxy, wzyx	○	○	3_1	4_1
	xzyw, xzwy, xwzy	○	4_1	○	○
	yzxw	3_1	5_2	$\star 7_4$	○
	yzwx, ywzx, wyzx	○	3_1	5_2	6_1
	ywxz, wxyz, wyxz	○	○	○	6_1
	zxyw, zxwy, zyxw	○	○	5_2	○
	wxzy	3_1	○	○	○

TABLE L

Proof of (Jeon-J, 2002) – continued

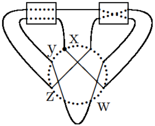
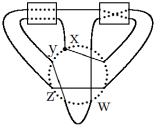
M	Arc levels	[0,1]	[0,3]	[2,1]	[2,−1]	[2,±3]
	xyzw, xywz, xwyz, yxzw, yxwz, zywx, zwxy, zwyx, wzxy, wzyx	○	○	○	3_1	$\rightleftarrows$
	xzyw, xzwy, xwzy	3_1	5_2	○	○	$\rightleftarrows$
	yzxw	4_1	6_1	6_2	3_1	$\rightleftarrows$
	yzwx, ywzx, wyzx	○	4_1	3_1	$\not{5}_1$	$\rightleftarrows$
	ywxz, wxyz, wyxz	○	○	3_1	5_2	$\rightleftarrows$
	zxyw, zxwy, zyxw	○	○	4_1	○	$\rightleftarrows$
	wxzy	○	3_1	○	○	$\rightleftarrows$

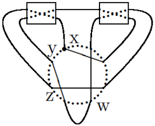
Table M

Proof of (Jeon-J, 2002) – continued

N	Arc levels	[0,1]	[0,3]	[2,1]	[2,−1]	[2,3]	[2,−3]
	xyzw, yxzw, yzxw	○	○	○	5_2	4_1	$\star 7_3$
	xwzy	○	5_2	○	○	3_1	3_1
	yzwx	4_1	6_2	6_1	○	$\star 8_4$	$\star 7_2$
	wxzy, wzxy, wzyx	3_1	$\not{5}_1$	○	○	3_1	3_1

Reducible to "xy" in Table E : xywz, xzyw, xzwy, xwyz, yxwz, zxyw, zxwy, zyxw
Reducible to "yx" in Table E : ywxz, ywzx, zywx, zwxy, zwyx, wxyz, wyxz, wyzx

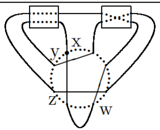
TABLE N

O	Arc levels	[1,1]	[1,−1]	[1,±3]	[3,±3]
	xyzw, yxzw, yzxw	○	4_1	$\rightleftarrows$	$\rightleftarrows$
	xwzy	○	3_1	$\rightleftarrows$	$\rightleftarrows$
	yzwx	5_2	○	$\rightleftarrows$	$\rightleftarrows$
	wxzy,wzxy,wzyx	○	○	$\rightleftarrows$	$\rightleftarrows$

Reducible to "xy" in Table F : xywz, xzyw, xzwy, xwyz, yxwz, zxyw, zxwy, zyxw
Reducible to "yx" in Table F : ywxz, ywzx, zywx, zwxy, zwyx, wxyz, wyxz, wyzx

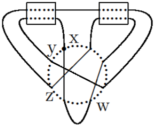
TABLE O

Proof of (Jeon-J, 2002) – continued

P	Arc levels	[0,1]	[0,3]	[2,1]	[2,−1]	[2,3]	[2,−3]
	xyzw, xzyw, xzwy	3_1	$\not{5}_1$	$\bigcirc$	$\bigcirc$	$\Supset$	3_1
	yxzw	3_1	$\not{5}_1$	6_2	4_1	$\Supset$	$\bigcirc$
	ywzx, wyzx, wzyx	$\bigcirc$	$\bigcirc$	$\bigcirc$	5_2	4_1	$\Supset$
	wzxy	$\bigcirc$	$\bigcirc$	3_1	$\not{5}_1$	$\bigcirc$	$\Supset$

Reducible to "xy" in Table E : xywz, xwyz, xwzy, zxyw, zxwy, zwxy, wxyz, wxzy
Reducible to "yx" in Table E : yxwz, yzxw, yzwx, ywxz, zyxw, zywx, zwyx, wyxz

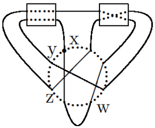
TABLE P

Q	Arc levels	[0,0]	[0,2]	[2,2]	[2,−2]
	xyzw, xywz	$\bigcirc$	$\bigcirc$	4_1	3_1
	xzyw	3_1	5_2	$\bigcirc$	$\bigcirc$
	zxyw	$\bigcirc$	$\bigcirc$	6_1	$\bigcirc$
	zwyx, wzyx	$\bigcirc$	$\bigcirc$	$\bigcirc$	5_2
	wyxz	$\bigcirc$	$\bigcirc$	3_1	7_2
	wyzx	3_1	$\bigcirc$	3_1	7_3

Reducible to "xyz" in Table G : xwyz, yxzw, yxwz, ywxz, wxyz
Reducible to "xzy" in Table G : xzwy, xwzy, wxzy
Reducible to "yzx" in Table G : yzxw, yzwx, ywzx
Reducible to "zxy" in Table G : zxwy, zyxw, zywx, zwxy, wzxy

TABLE Q

Proof of (Jeon-J, 2002) – continued

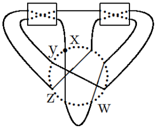


R	Arc levels	[0,1]	[0,3]	[2,1]	[2,−1]	[2,±3]
	xyzw, xywz	○	○	3_1	○	⇄
	xzyw	4_1	6_1	○	○	⇄
	zxyw	○	○	5_2	3_1	⇄
	zwyx, wzyx	○	○	○	4_1	⇄
	wyxz	○	○	4_1	6_1	⇄
	wyzx	○	3_1	○	6_1	⇄

Reducible to “xyz” in Table H : xwyz, yxzw, yxwz, ywxz, wxyz
Reducible to “xzy” in Table H : xzwy, xwzy, wxzy
Reducible to “yzx” in Table H : yzxw, yzwx, ywzx
Reducible to “zxy” in Table H : zxwy, zyxw, zywx, zwxy, wzxy

TABLE R

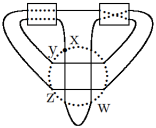
Proof of (Jeon-J, 2002) – continued

S	Arc levels	[1,1]	[1,−1]	[1,3]	[1,−3]	[3,±3]
	xyzw, xywz	○	○	3_1	3_1	$\rightleftarrows$
	xzyw, zxyw	3_1	○	$\not{5}_1$	○	$\rightleftarrows$
	zwyx, wzyx	○	3_1	○	$\not{5}_1$	$\rightleftarrows$
	wyxz	3_1	$\not{5}_1$	○	$\not{7}_1$	$\rightleftarrows$
	wyzx	○	5_2	4_1	7_3	$\rightleftarrows$

Reducible to “xyz” in Table I : xwyz, yxzw, yxwz, ywxz, wxyz
Reducible to “xzy” in Table I : xzwy, xwzy, wxzy, zxwy, zyxw, zywx, zwxy, wzxy
Reducible to “yzx” in Table I : yzxw, yzwx, ywzx

TABLE S

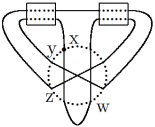
Proof of (Jeon-J, 2002) – continued

T	Arc levels	[0,1]	[0,3]	[2,1]	[2,−1]	[2,3]	[2,−3]
	xyzw, xzyw	○	○	○	○	$\Supset$	3_1
	xywz, zxyw	○	3_1	○	4_1	$\Supset$	6_2
	xzwy	4_1	6_2	○	○	$\Supset$	3_1
	yxzw	○	○	3_1	$\star 6_3$	$\Supset$	$\star 8_7$
	ywzx	○	5_2	3_1	6_2	$\Supset$	$\star 8_9$
	zwyx, wyxz	○	3_1	5_2	3_1	$\Supset$	6_3
	wyzx, wzyx	3_1	$\not{5}_1$	$\not{5}_1$	3_1	$\Supset$	○
	wzxy	3_1	$\not{5}_1$	5_2	○	$\Supset$	4_1

⊚ or ○ : xwyz, xwzy, yxwz, yzxw, yzwx, ywxz, zxwy, zyxw, zywx, zwxy, wxyz, wxzy

TABLE T

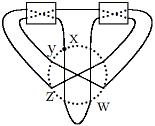
Proof of (Jeon-J, 2002) – continued

U	Arc levels	[0,0]	[0,2]	[2,2]	[2,−2]
	xyzw	◯	◯	$\Supset$	◯
	xywz	◯	3_1	$\Supset$	4_1
	xzyw	3_1	◯	◯	3_1
	xzwy	◯	4_1	◯	3_1
	yxzw	◯	◯	$\Supset$	6_3
	ywzx	◯	5_2	$\Supset$	6_2
	zxyw	◯	◯	◯	6_2
	zwyx	◯	◯	5_2	6_3
	wyxz	◯	3_1	$\Supset$	3_1
	wyzx	3_1	$\not{5}_1$	$\Supset$	3_1
	wzxy	◯	3_1	5_2	4_1
	wzyx	◯	3_1	$\not{5}_1$	◯

⊚ or ◯ : xwyz, xwzy, yxwz, yzxw, yzwx, ywxz, zxwy, zyxw, zywx, zwxy, wxyz, wxzy

TABLE U

Proof of (Jeon-J, 2002) – continued

V	Arc levels	[1,1]	[1,−1]	[1,3]	[1,−3]	[3,3]	[3,−3]
	xyzw	◯	◯	$\Supset$	3_1	$\Supset$	◯
	xywz	◯	◯	$\Supset$	3_1	$\Supset$	6_3
	xzyw	◯	3_1	◯	$\not 5_1$	$\Supset$	3_1
	xzwy	◯	◯	3_1	3_1	$\Supset$	6_2
	yxzw	◯	4_1	$\Supset$	6_2	$\Supset$	8_9
	ywzx	5_2	3_1	$\Supset$	6_3	$\Supset$	8_7
	zxyw	◯	3_1	◯	$\not 5_1$	$\Supset$	8_7
	zwyx	◯	4_1	5_2	6_2	$\Supset$	8_9
	wyxz	5_2	◯	$\Supset$	4_1	$\Supset$	6_2
	wyzx	$\not 5_1$	3_1	$\Supset$	◯	$\Supset$	3_1
	wzxy	◯	◯	3_1	3_1	$\Supset$	6_3
	wzyx	3_1	◯	$\not 5_1$	◯	$\Supset$	◯

⊚ or ◯ : xwyz, xwzy, yxwz, yzxw, yzwx, ywxz, zxwy, zyxw, zywx, zwxy, wxyz, wxzy

TABLE V

Thank you