Grid diagram for singular links

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Knots and Spatial Graphs 2020
(A workshop in memory of Choon Bae Jeon)

KAIST, Korea

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- $\mathcal{K}$ = { equivalent classes of topological knots }
- $\mathcal{G}$ = { grid diagrams }
- $\mathcal{B}$ = { equivalent classes of braids modulo conjugation and exchange }
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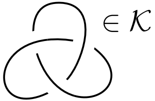

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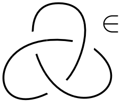
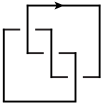
 $\in \mathcal{K}$

 $\in \mathcal{G}$

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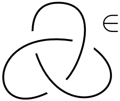
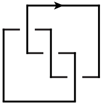
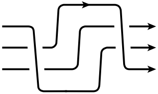
 $\in \mathcal{K}$

 $\in \mathcal{G}$

 $\in \mathcal{B}$

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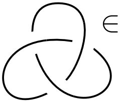
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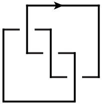
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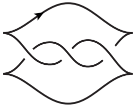
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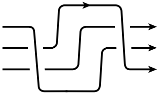
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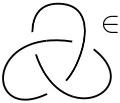
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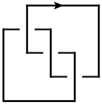
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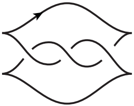
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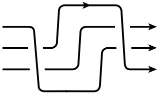
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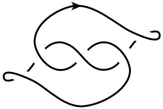
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Motivation(cont.)

Let $\overline{\mathcal{G}} = \mathcal{G}/\{(\mathrm{Cm}), (\mathrm{Tr})\}$.

Theorem 1

(1) [Khandhawit-Ng] The diagram (a) commutes,

(2) [Ozsváth-Szabó-D. Thurston, Ng-D. Thurston] For $\overline{\mathcal{G}}$, there are bijections induced by the canonical maps.

$$\mathcal{B} \longleftrightarrow \overline{\mathcal{G}}/\{(NE), (SE)\}$$

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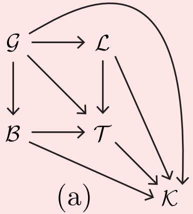

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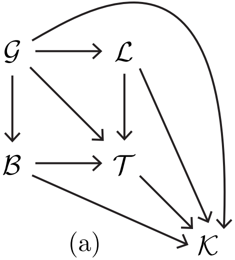

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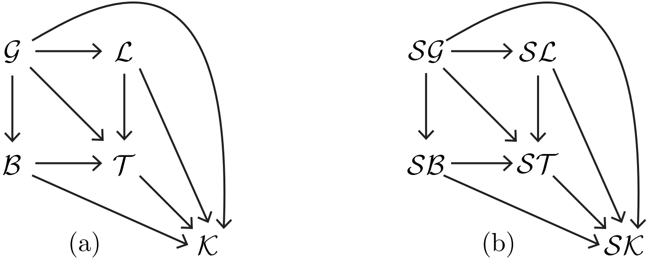

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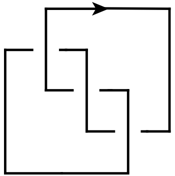

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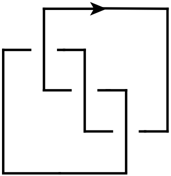

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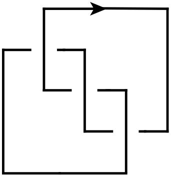

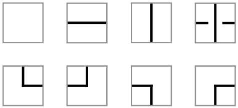

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Elementary Moves on Grid Diagrams

Cromwell(1995)

Two grid diagrams of the same link can be obtained from each other by a finite sequence of the following elementary moves.

- Stabilization and Destabilization;
- Commutation(Cm) : interchanging adjacent two non-interleaved vertical edges or horizontal edges, respectively;
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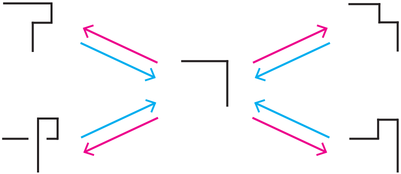

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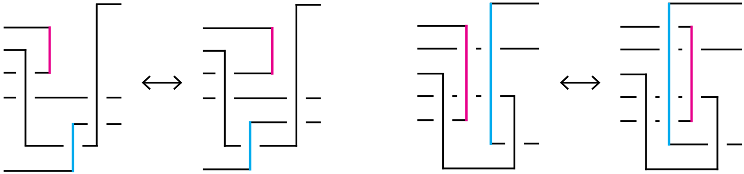

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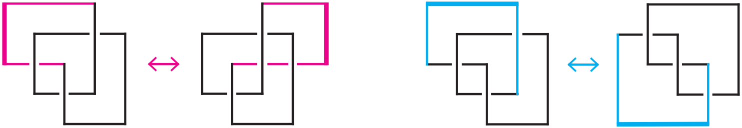

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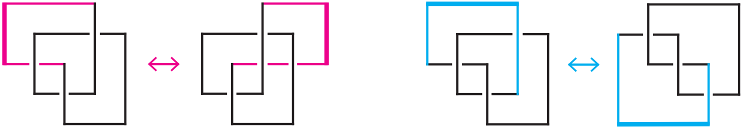

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$\overline{\mathcal{G}}/\{\text{(De)stabilization}\} \to \mathcal{K}$

- $\mathcal{G}$: the set of all grid diagrams
- $\overline{\mathcal{G}}$: $\mathcal{G}/\{\text{(Cm),(Tr)}\}$
- $\mathcal{K}$: the set of all equivalent classes of knots

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The map $\mathcal{G} \longrightarrow \mathcal{K}$ induces a bijection

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Four types of (De)stabilizations

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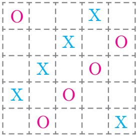

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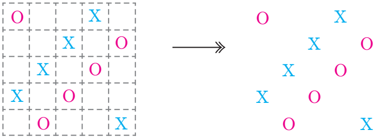

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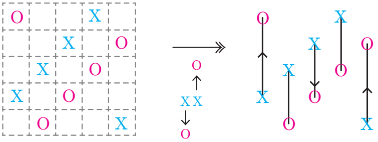

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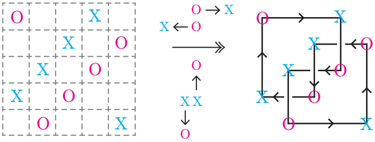

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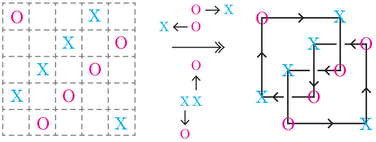
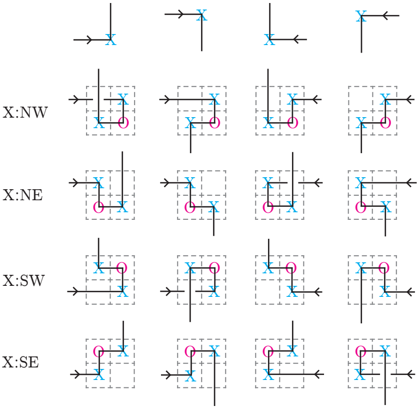
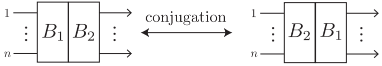
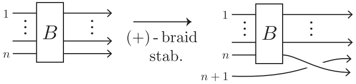




Braid

- **[Markov Theorem]** The closures of two braids B and B' represent the same link *if and only if* one braid can be deformed into the other by a sequence of braid isotopies and Markov moves





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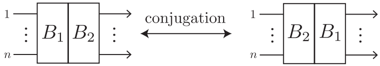

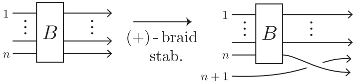

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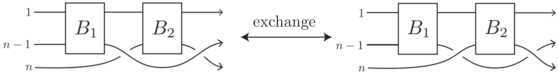

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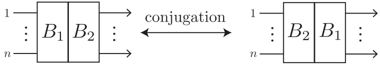

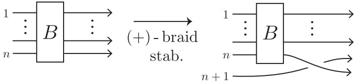

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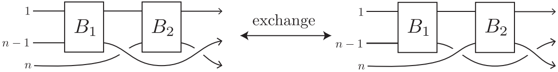

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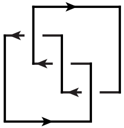

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The map $\mathcal{G} \longrightarrow \mathcal{B}$ induces a bijection

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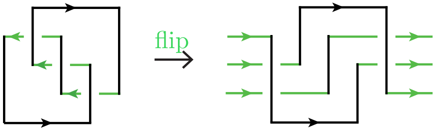

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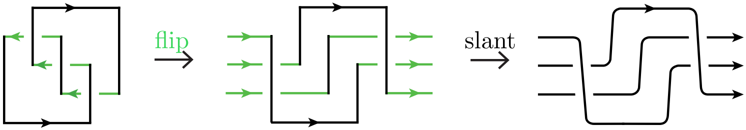

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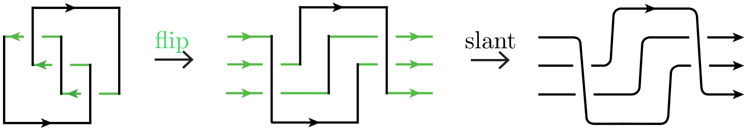

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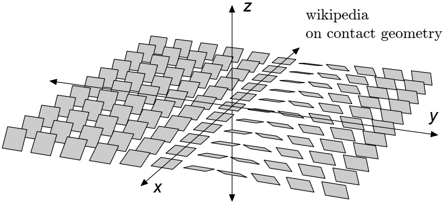
Legendrian and transverse knots

The *standard contact structure* assigns to the point $p = (x, y, z)$ the plane

$$\xi_p = \ker(dz - ydx),$$

where we orient $\mathbb{R}^3$ via the Right Hand Rule.



A *Legendrian knot* and a *transverse knot* are topological knots which are tangent and transverse to the standard contact structure, respectively.

Legendrian and transverse knots

The *standard contact structure* assigns to the point $p = (x, y, z)$ the plane

$$\xi_p = \ker(dz - y dx),$$

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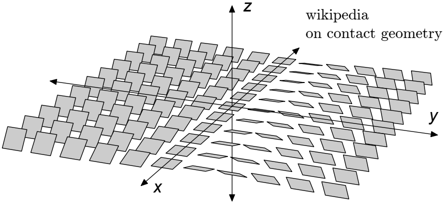

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xz-projections of Legendrian knots and transverse knots, called *front projections*, can be characterized;

- For Legendrian knots, (1) there are no vertical tangencies, and (2) an arc having lower slope is lying over an arc having higher slope at each crossing.
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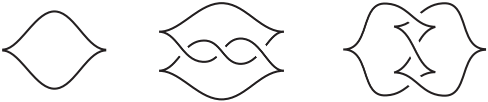

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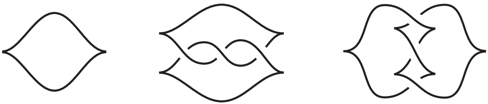

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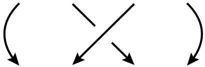

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$$\overline{\mathcal{G}}/\ \{\text{(NE),(SW)}\} \to \mathcal{L}$$
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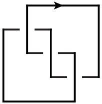

- $\mathcal{L}$: the set of all equivalent classes of Legendrian links in $(\mathbb{R}^3, \xi_0)$, where $\xi_0 = \ker(dz - ydx)$ is the *standard contact structure*.
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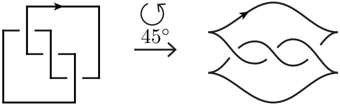

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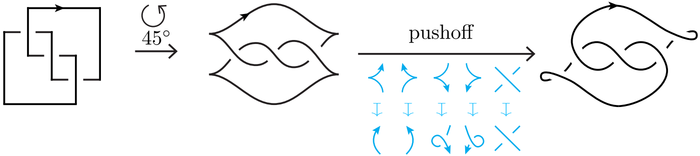

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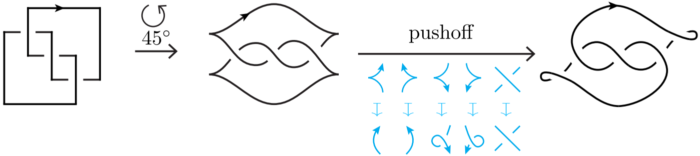

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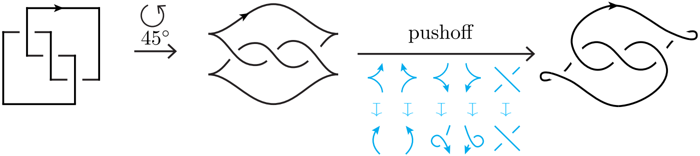

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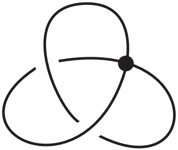

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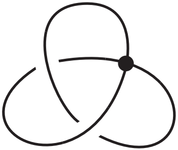

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The *front projectin* $\pi_F : \mathbb{R}^3 \to \mathbb{R}^2_{xz}$ is defined as the projection onto the xz-plane.

- For any $L \in \mathcal{SL}$, $\pi_F(L)$ near each singular point looks like because the same y-coordinate yields the same slope dz/dx.
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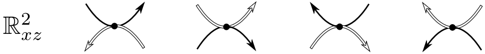

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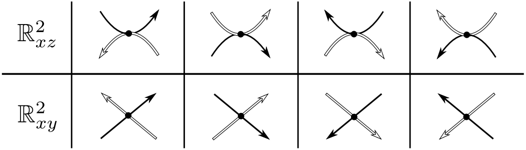

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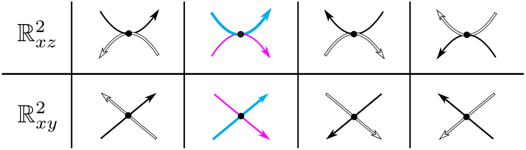

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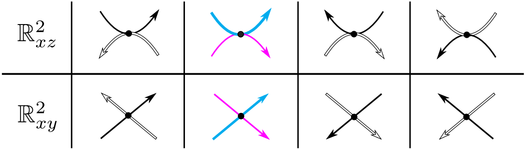

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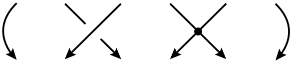

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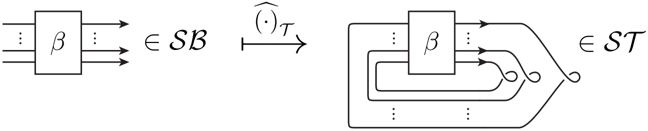

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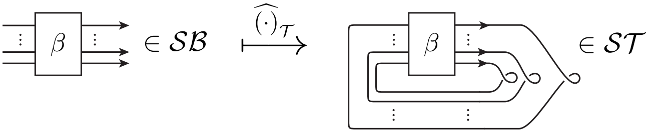

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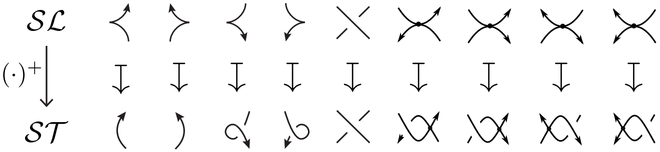

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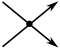
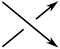
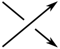
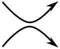
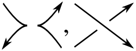
 $\in \mathcal{SK}, \mathcal{ST}, \mathcal{SB}$			
 $\in \mathcal{SL}$			

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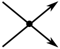
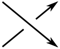
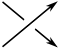
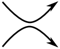
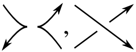
 $\in \mathcal{SK}, \mathcal{ST}, \mathcal{SB}$			
 $\in \mathcal{SL}$			

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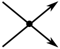
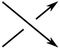
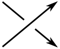
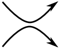
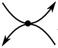
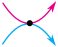
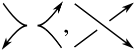
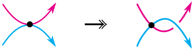
 $\in \mathcal{SK}, \mathcal{ST}, \mathcal{SB}$			
,  $\in \mathcal{SL}$			



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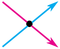
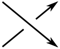
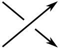
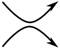
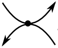
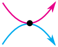
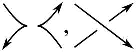
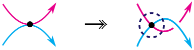
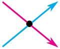
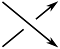
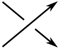
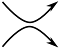
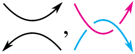
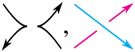
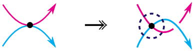
 $\in \mathcal{SK}, \mathcal{ST}, \mathcal{SB}$			
,  $\in \mathcal{SL}$			



Resolutions

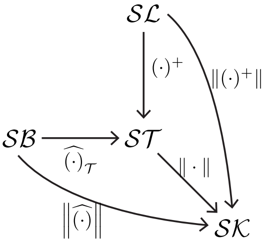
For each $K \in \mathcal{SB}, \mathcal{SL}, \mathcal{ST}$ or $\mathcal{SK}$, the *ϵ-resolution* $\mathcal{R}_\epsilon$ at a singular point p for each $\epsilon \in \{+, -, 0\}$ is defined as follows.

K	$\mathcal{R}_+(p)$	$\mathcal{R}_-(p)$	$\mathcal{R}_0(p)$
 $\in \mathcal{SK}, \mathcal{ST}, \mathcal{SB}$			
 $\in \mathcal{SL}$			



A commutative diagram

There is a commutative diagram (a) which extends the previous maps between non-singular objects.



(a)

We want to complete the diagram (b) in which all maps commute with *resolutions*. In this case, we say that *$\mathcal{SG}$ gives a unified description* for $\mathcal{SB}$, $\mathcal{SL}$, $\mathcal{ST}$, and $\mathcal{SK}$.

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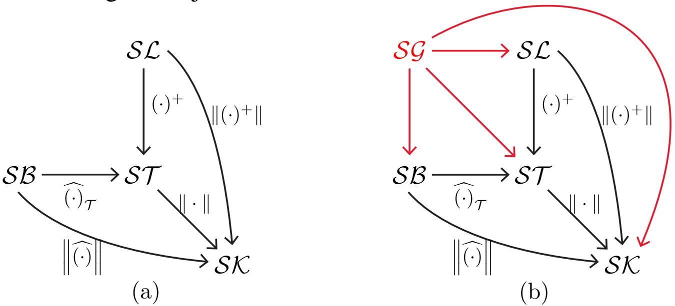

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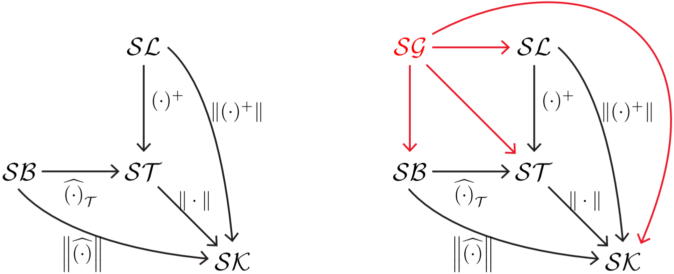

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Singular point tiles

Considering $\mathcal{SG}$, we can naturally consider the following two tiles $t_\times$ and $t_\bullet$ with transverse intersection and non-transverse intersection near the singular point, respectively:

$t_\times =$  and $t_\bullet =$ .

Let $\mathcal{SG}_\times$ and $\mathcal{SG}_\bullet$ be the set of all grid diagrams extended by $t_\times$ and $t_\bullet$, respectively.

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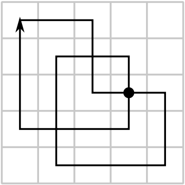 $\in \mathcal{SG}_\times$

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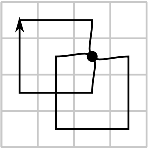 $\in \mathcal{SG}_\bullet$

(b)

On $\mathcal{SG}_\times$

Theorem 2. (An-L.)

$\mathcal{SG}_\times$ does not give a unified description whatever the resolutions and maps on $\mathcal{SG}_\times$ are defined.

We can define easily the maps from $\mathcal{SG}_\times$ to $\mathcal{SB}, \mathcal{ST}$ and $\mathcal{SK}$ which extend corresponding maps between non-singular objects and commute with resolutions.

However, it is NOT possible to define $\mathcal{SG}_\times \to \mathcal{SL}$ such that $\mathcal{SG}_\times$ gives a unified description.

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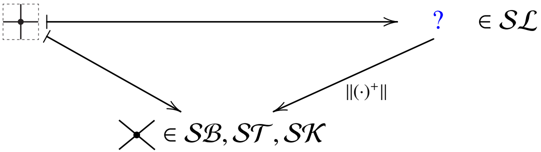
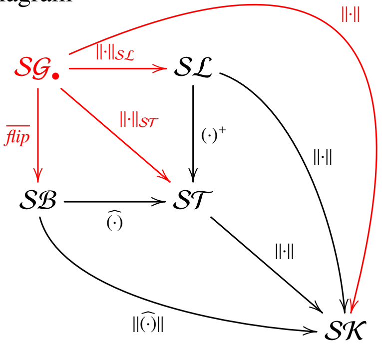


However, it is NOT possible to define $\mathcal{SG}_\times \to \mathcal{SL}$ such that $\mathcal{SG}_\times$ gives a unified description.

Theorem 3. (An-L.)

The set $\mathcal{SG}_\bullet$ gives a unified description for $\mathcal{SB}, \mathcal{SL}, \mathcal{ST}$ and $\mathcal{SK}$.

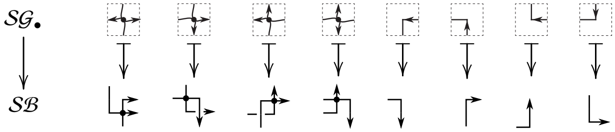
In other words, the diagram



is commutative and all maps commute with resolutions.

Sketch of the proof of Theorem 3.

- $\mathcal{SG}_\bullet \to \mathcal{SB}$



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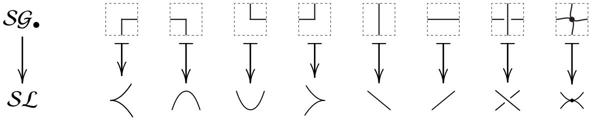

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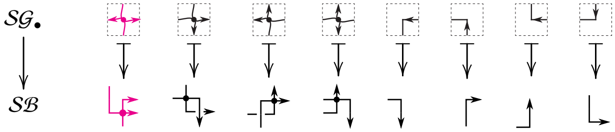

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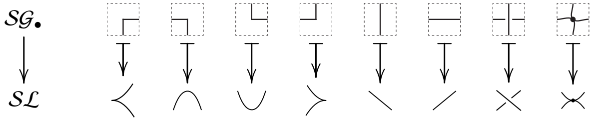

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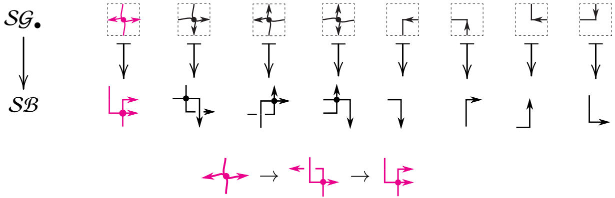

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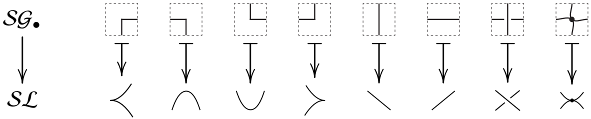

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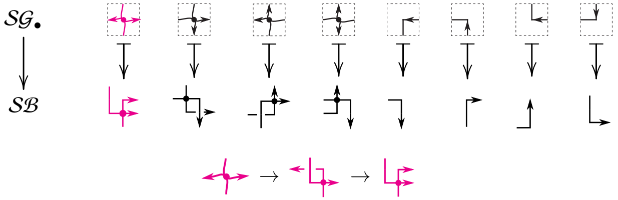

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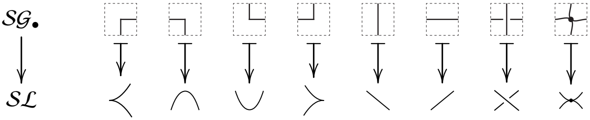

- $\mathcal{SG}_\bullet \to \mathcal{ST} := \mathcal{SG}_\bullet \to \mathcal{SL} \xrightarrow{(\cdot)^+} \mathcal{ST}$.
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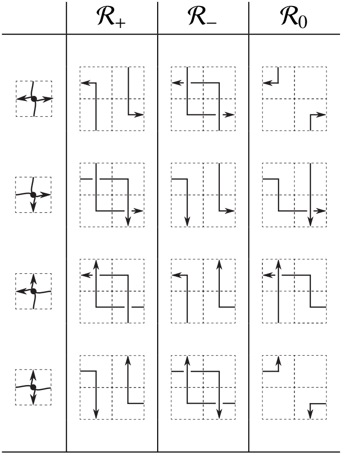

For example, $\mathcal{SG}_\bullet \to \mathcal{SB}$ commutes with (+)-resolutions since

From now on, we use $\mathcal{SG}$ instead of $\mathcal{SG}_\bullet$.

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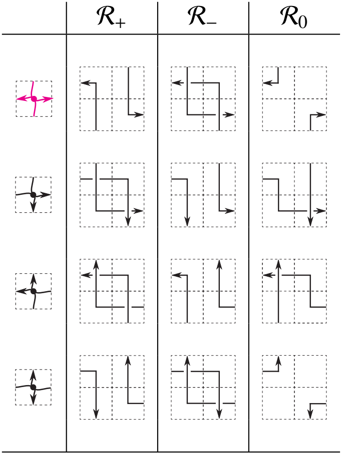

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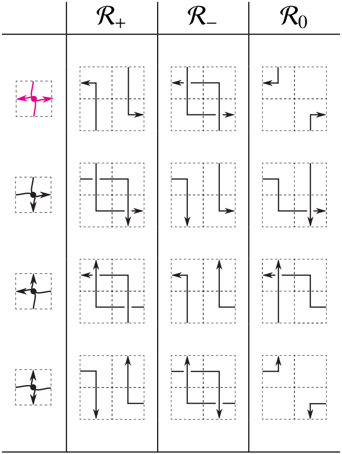

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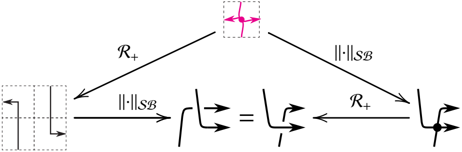

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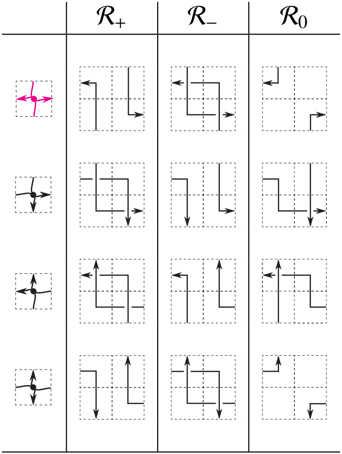

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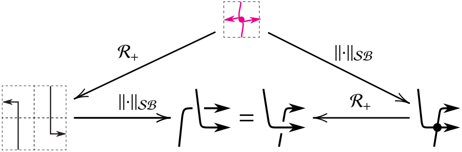

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Singular grid moves – Translation

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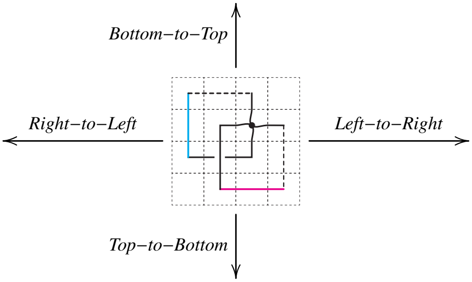

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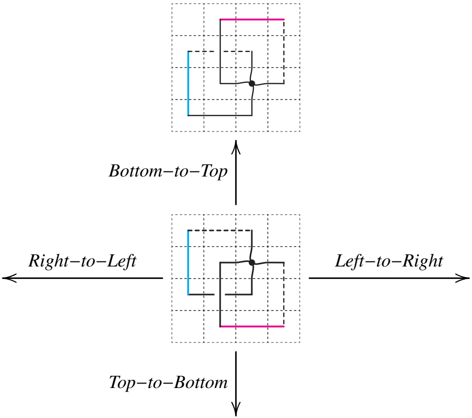

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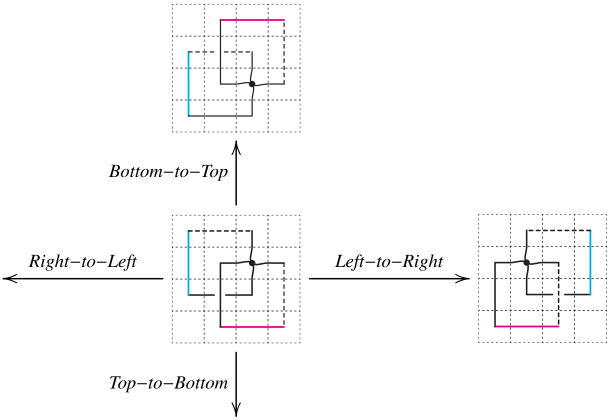

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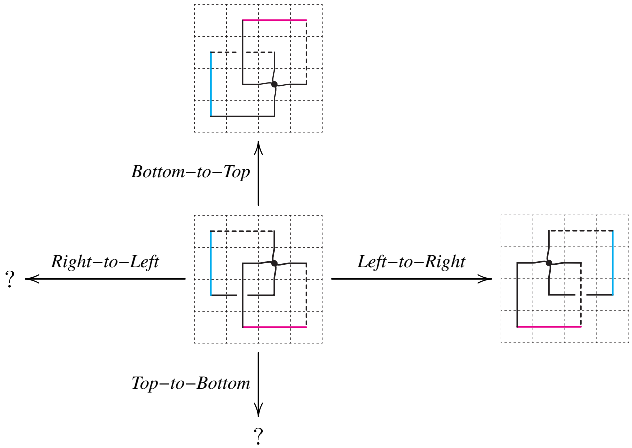
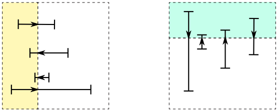
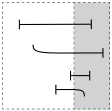


Singular grid moves – Translation

So, we define generalized translations for *admissible decompositions*, which are horizontal or vertical decompositions of a singular grid diagram into two parts such that all segments connecting two parts end only at corners.



Admissible



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The *translation* (Tr) on an admissible decomposition is defined as follows.

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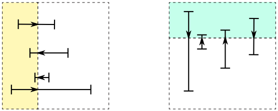

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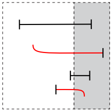

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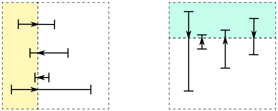

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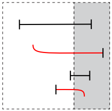

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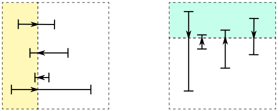

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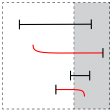

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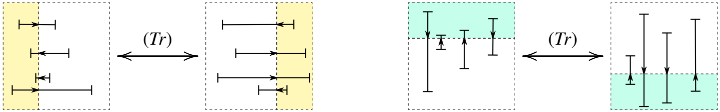
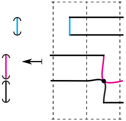
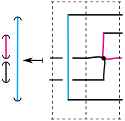
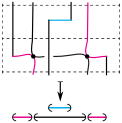


Singular grid moves – Commutation

We say that two contiguous columns (or rows) in G are *non-interleaving* if their vertical (or horizontal) segments are non-interleaving, otherwise they are *interleaving*.







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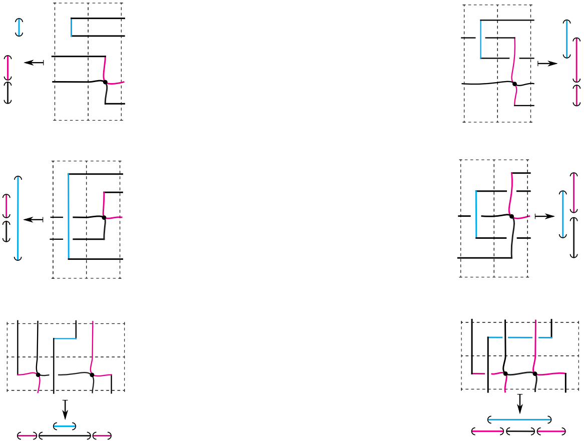

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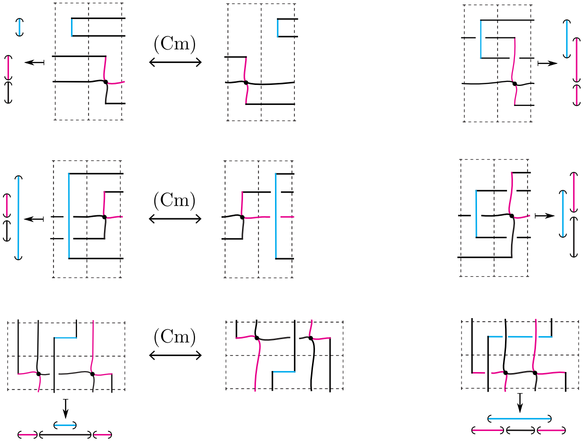
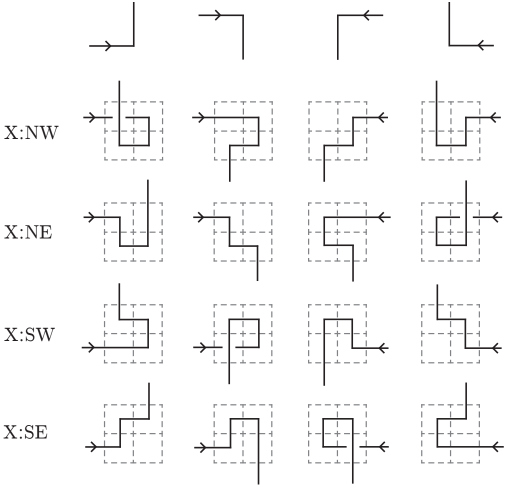
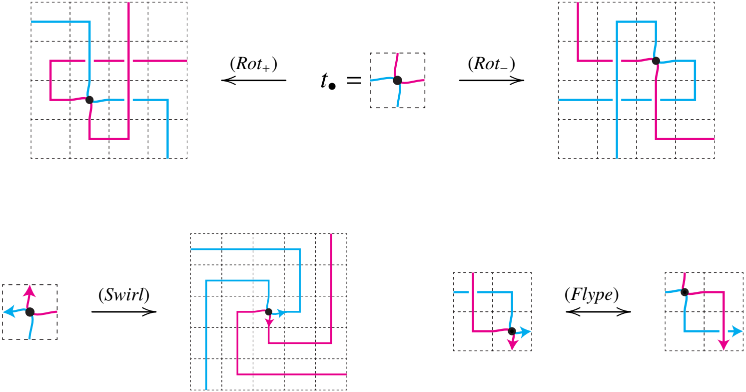


Singular grid moves – (De)Stabilizations



Singular grid moves – Rotations, Swirl and Flype



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Main Result

Main Theorem. (An-L.)

Let $\overline{\mathcal{SG}} := \mathcal{SG}/\{(\mathrm{Tr}),(\mathrm{Cm})\}$. Then the following holds.

$$\mathcal{SB} = \overline{\mathcal{SG}}/\{(NE), (SE), (Flype), (Swirl), (Rot^{*}_{\pm})\}$$
$$\mathcal{SL} = \overline{\mathcal{SG}}/\{(NE), (SW), (Rot_{\pm})\}$$
$$\mathcal{ST} = \overline{\mathcal{SG}}/\{(NE), (SW), (SE), (Flype), (Rot_{\pm})\}$$
$$\mathcal{SK} = \overline{\mathcal{SG}}/\{(NE), (NW), (SE), (SW), (Flype), (Rot_{\pm})\}$$

Thank you.