joint with Hosung Kim & Yongnam Lee 7/24

Positivity of Tangent Bundles of Fano Threefolds

- X: smooth projective variety / $\mathbb{C}$ of dim n
  $\Rightarrow T_X$: vector bundle of rank n

§1. Positivity of Vector Bundles

Def L: line bundle on X

L is ample if $\varphi_{|L^{\otimes m}|}: X \to \mathbb{P}^N$ is an embedding

nef if $L.C \geq 0$ for every curve C on X

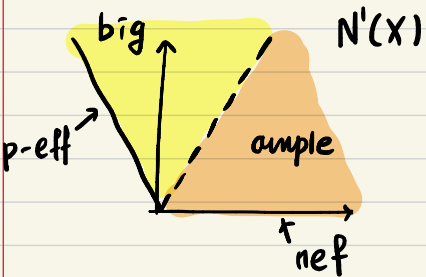
big if $\varphi_{|L^{\otimes m}|}: X \dashrightarrow \mathbb{P}^N$ is birat'l

pseudoeffective if L is a limit of "effective" L_t in $N^1(X) = \{\text{line bdls}\}/\equiv_{num}$

Rmk (1)

$$\begin{tikzcd} \text{ample} \arrow[r, Rightarrow, "\text{limit}"] \arrow[d, Rightarrow] & \text{nef} \arrow[d, Rightarrow] \arrow[dl, Rightarrow, shift left, "\times" description] \\ \text{big} \arrow[r, Rightarrow, "\text{limit}"] \arrow[ur, Rightarrow, shift left, "\times" description] & \text{p-eff} \end{tikzcd}$$

(2) $\rho = \mathrm{rk}\, N^1(X) = 2$ (Picard no.)



(3) $h^0(L^{\otimes m}) \sim n^m \Leftrightarrow L$: big

(4) $h^0(L^{\otimes m}) \neq 0$ for some $m > 0 \Rightarrow L$: p-eff

Def V: vector bundle on X

$\Rightarrow \pi : \mathbb{P}(V) \to X$: projectivized bundle (Grothendieck's notion)

i.e., $\{ p \in \mathbb{P}(V|_x) \} \leftrightarrow \{$ hyberplanes of $V|_x \}$

or $\{$ quotient line bdls $V \to L \} \leftrightarrow \{$ sections of $\pi : V \to X \}$

$\xi = \mathcal{O}_{\mathbb{P}(V)}(1)$: tautological line bundle on $\mathbb{P}(V)$

$\Rightarrow \pi_* \mathcal{O}_{\mathbb{P}(V)}(m) = S^m V$

V is <u>ample/nef/big/p-eff</u> if so is ξ.

Rmk $N^1(\mathbb{P}(T_X))_{\mathbb{Q}} \cong \mathbb{Q}[\xi] \oplus \pi^* N^1(X)_{\mathbb{Q}}$

Thm (Mori '79, Siu-Yau '80) X : sm. proj.

T_X : ample $\Rightarrow X \approx \mathbb{P}^n$

Conj (Campana-Peternell '91) X : sm. Fano ($-K_X = \Lambda^n T_X$: ample)

T_X : nef $\Rightarrow X$: rat'l homogeneous

and confirmed for $n = 3$.

$\Rightarrow X \simeq \mathbb{P}^3, Q, \mathbb{P}^2 \times \mathbb{P}^1, \mathbb{P}(T_{\mathbb{P}^2}), \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Thm (Solá Conde - Wiśniewski '04) T_X : big & 1-ample $\Rightarrow$ nef

$\varphi_{|mL|} : Y \longrightarrow \mathbb{P}^N$ is morphism & $\dim \varphi^{-1}(z) \leq 1$

$\Rightarrow X \simeq \mathbb{P}^n$, $X \simeq Q^n$ or $X \simeq \mathbb{P}(T_{\mathbb{P}^2})$ (only when $n = 3$)

Thm (Hsiao '15) X : toric or partial flag $\Rightarrow T_X$: big

Thm (Höring-Liu-Shao '20) X : Fano 2-fold

T_X is big $\Leftrightarrow X \simeq Bl_d \mathbb{P}^2$ for $d \leq 4$ or $X \approx \mathbb{P}^1 \times \mathbb{P}^1$

$\Leftrightarrow (-K_X)^2 \geq 5$

"+" (Höring-Liu '21) X : Fano 3-fold with $\rho = 1$

linear section of $Gr(2,5) \subseteq \mathbb{P}^9$

T_X is big $\Leftrightarrow X \simeq \mathbb{P}^3$ or Q or V_5

$\Leftrightarrow (-K_X)^3 \geq 40$

§2. Smooth Fano Varieties: $-K_X = \Lambda^n T_X$ is ample

$n=1 \Rightarrow X \simeq \mathbb{P}^1$

$n=2 \Rightarrow$ del Pezzo surfaces, $1 \le (-K_X)^2 \le 9$

$X \simeq \mathrm{Bl}_m \mathbb{P}^2 \ (0 \le m \le 8)$ or $X = \mathbb{P}^1 \times \mathbb{P}^1$

blow-up at m points in general position

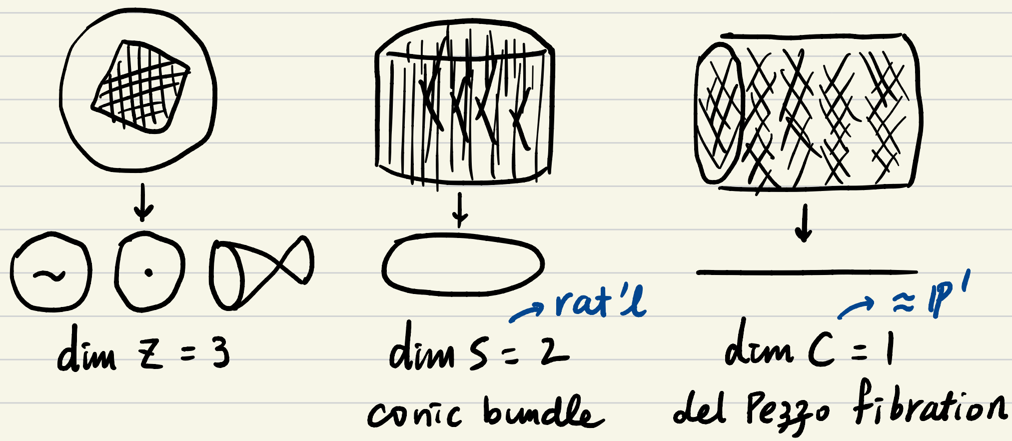
$n=3 \Rightarrow$ (Mori-Mukai '81) 105 deformation types, $1 \le \rho \le 10$.

<u>Rmk</u> A sm. Fano variety is covered by rat'l curves.

$\rightsquigarrow \exists$ families $K_1, K_2, \ldots \subseteq \mathrm{RatCurves}(X)$

$\rightsquigarrow$ "extremal" rat'l curves are contracted

For $n=3$, the <u>extremal contractions</u> are classified.



Ex ($\rho=1$) 17 deform. types $2 \leq (-K_X)^3 \leq 64$

$\Rightarrow$ extremal contraction: $X \to \{*\}$

$\mathbb{P}^3, Q, V_5, V_4 = Q \cap Q', V_3, \dots$ cubic hypersurface

($\rho=2$) 36 deform. types, $4 \leq (-K_X)^3 \leq 62$

$\Rightarrow \exists!$ two nontrivial extremal contractions

$$\begin{tikzcd}[column sep=small]
& (2.2) \arrow[dl] \arrow[dr] & & & (2.19) \arrow[dl] \arrow[dr] & & & (2.20) \arrow[dl] \arrow[dr] & \\
\mathbb{P}^2 & & \mathbb{P}^1 & \mathbb{P}^3 & & V_4 & V_5 & & \mathbb{P}^2 \\
\text{conic} & & dP_z & \text{curve} & & \text{curve} & \text{curve} & & \text{conic}
\end{tikzcd}$$

Prop (Kim-K.-Lee) (1) $X \to Z$: blow-up along a smooth curve

T_X : big $\Rightarrow$ T_Z : big

$\Rightarrow T_{(2.19)}$: not big ($\because T_{V_4}$: not big (HLS))

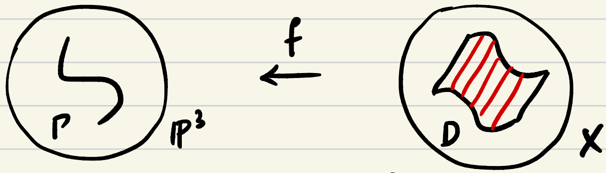
(2) $X \to C$: dP_z fibr'n of deg d

$d \leq 4 \Rightarrow T_X$: not big

$\Rightarrow T_{(2.2)}$: not big

§3. Total Dual VMRT (By Example)

EX $X = Bl_\Gamma \mathbb{P}^3$, Γ: sm. nondegenerate curve



$\Rightarrow$ X has some families of rat'l curves.

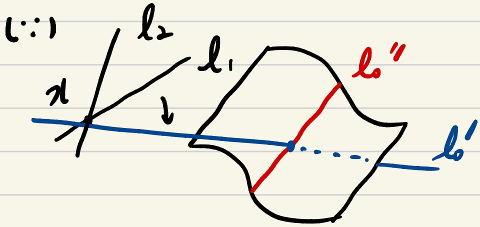
(1) $\mathcal{K}$: strict transforms of lines on $\mathbb{P}^3$

$\mathcal{K}_x$: " passing thru $x \in X$

$\Downarrow$

l_t has split limit as $t \to 0$: not good

($\because$)



$l_2 . D = 0$, $l_1 . D = 0$

$l_0' . D = 1$

the correct limit is $l_0 = \underbrace{l_0' \cup l_0''}_{\text{split}}$

(2) $\mathcal{K}$: rulings of $D \simeq \mathbb{P}_\Gamma(N_{\Gamma|\mathbb{P}^3})$

$\Rightarrow$ does not dominate X : not good

(3) $\mathcal{K}$: strict transforms of secant lines of Γ on $\mathbb{P}^3$

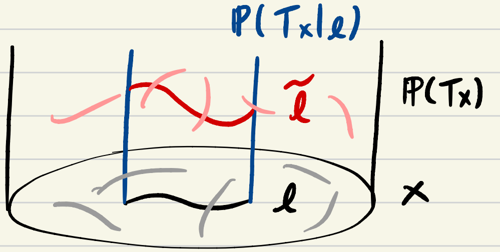
$\Rightarrow$ $\mathcal{K}$ dominates X

& $\mathcal{K}_x \ni l_t$ has unsplit limit for general x : good

Note that such $l \in \mathcal{K}$ on X has trivial normal bundle, i.e.

$$N_{l|X} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$$

Then we can consider trivial liftings $\tilde{\ell}$ of ℓ on $\mathbb{P}(T_X)$, i.e.



$\leftrightarrow \quad T_X|_\ell \to N_{\ell|X} \to \mathcal{O}_{\mathbb{P}^1}$

<u>Prop</u> (Occhetta - Solá Conde - Watanabe '16) Under "good" assumption,
$\exists$ effective divisor $\check{\mathcal{E}} = \overline{\cup \tilde{\ell}}$ on $\mathbb{P}(T_X)$ s.t.

$$\check{\mathcal{E}}|_x \subseteq \mathbb{P}(T_X|_x) \xleftrightarrow{\text{dual}} \mathcal{C}_x \subseteq \mathbb{P}(\Omega_X|_x) \text{ for general } x$$

& $\check{\mathcal{E}}$ is called <u>the total dual VMRT</u> associated to $\mathcal{K}$.

<u>Prop</u> (Höring - Liu - Shao) $X \to S$: conic bundle

$$[\check{\mathcal{E}}] \sim \xi + \pi^* K_{X/S}$$

conic fibers

<u>Prop</u> (Kim - K. - Lee) $X = Bl_\Gamma \mathbb{P}^3 \xrightarrow{f} \mathbb{P}^3$, Γ : sm. nondeg. w/o 4-sec.
deg. d & genus g.

$$[\check{\mathcal{E}}] \sim \left(\tfrac{1}{2}(d-1)(d-2) - g\right)\xi + (d+g-1)\,\pi^* H \quad = f^*\mathcal{O}_{\mathbb{P}^3}(1)$$

secant lines

$$- \left((d-1) - \tfrac{1}{2}(d-2)(d-3) + g\right)\pi^* D$$

f-excep'l

§4. Criteria for Bigness of T_X

Lemma If (divisorial) total dual VMRTs $\check{\mathcal{C}}_i$ satisfy

$$\sum [\check{\mathcal{C}}_i] \sim k\xi + \pi^* D$$

for some effective $D \in N^1(X)_{\mathbb{Q}}$, then T_X is not big.

Ex (2,2)-divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ $\qquad$ (1,1)-divisor on $\mathbb{P}^2 \times \mathbb{P}^2$

$$\begin{tikzcd} & (2.6) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^2 & & \mathbb{P}^2 \end{tikzcd} \quad H_i = \pi_i^* \mathcal{O}_{\mathbb{P}^2}(1). \quad \begin{tikzcd} & (2.32) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^2 & & \mathbb{P}^2 \end{tikzcd}$$

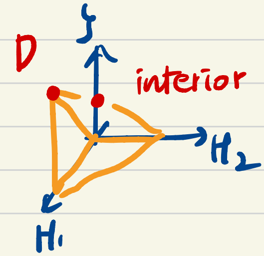
(2.6)

$$[\check{\mathcal{C}}_1] \sim \xi + 2\pi^* H_1 - \pi^* H_2$$
$$[\check{\mathcal{C}}_2] \sim \xi - \pi^* H_1 + 2\pi^* H_2$$
$$D \sim 2\xi + \pi^* H_1 + \pi^* H_2$$

$\Rightarrow T_{(2.6)}$: not big

(2.32)

$$[\check{\mathcal{C}}_1] \sim \xi + \pi^* H_1 - 2\pi^* H_2$$
$$[\check{\mathcal{C}}_2] \sim \xi - 2\pi^* H_1 + \pi^* H_2$$
$$D \sim 2\xi - \pi^* H_1 - \pi^* H_2$$



$\Rightarrow T_{(2.32)}$: big

(2.17) $X = Bl_\Gamma \mathbb{P}^3$, Γ: $d = 5$, $g = 1$

$$[\check{\mathcal{C}}] \sim 5\xi + 5\pi^* H - 2\pi^* D$$
$$= 5\xi + \pi^* D'$$

$\Rightarrow T_{(2.17)}$: not big

(2.22) $X = Bl_\Gamma \mathbb{P}^3$, Γ: $d=4$, $g=0$

$$[\check{C}] \sim 3\xi + 3\pi^*H - 2\pi^*D$$
$$= 3\xi + \pi^*D' - \pi^*D \rightarrow \text{indeterminate}$$

<u>Thm</u> (Kim-K.-Lee) $X = Bl_\Gamma \mathbb{P}^3$, Γ: sm. nondeg. w/o 4-secant deg. d

T_X is big $\Leftrightarrow$ $d = 3$

<u>Conclusion</u> X : Fano 3-folds w/ $\rho = 2$

T_X is big $\Leftrightarrow$ $(-K_X)^3 \geq 34$