May 13

Stability of Symmetric Powers of Vector Bundles over a Curve $/\mathbb{C}$

C: smooth projective curve of genus g

E: vector bundle on C of rank r & degree $d := \deg \Lambda^r E$

§1. Introduction

Def E is (semi) stable if

$\forall F \subseteq E$: proper nonzero subbundle, $\frac{\deg F}{\operatorname{rk} F} \underset{(=)}{<} \frac{\deg E}{\operatorname{rk} E}$. → slope $\mu(E)$

Equivalently, E is (semi) stable iff

$\forall E \to Q$: proper nonzero quotient bundle, $\frac{\deg Q}{\operatorname{rk} Q} \underset{(=)}{>} \frac{\deg E}{\operatorname{rk} E}$.

Rmk
- E: stable $\Leftrightarrow$ $E^\vee$: stable — dual
- E: stable $\Leftrightarrow$ $E \otimes L$: stable for L: line bundle

Prop (Hartshorne's ASAV) Let E be stable. Then
- $S^k E$ is semistable $\forall k \geq 2$
- $S^k E$ is stable $\forall k \geq 2$ for sufficiently general E when $g \geq 2$.

The proof uses the correspondence
(stability of E) $\leftrightarrow$ (irreducibility of $\rho_E : \pi_1(C) \to U(r)$)
introduced by Narasimhan & Seshadri ('65).

Q. Classify stable E w/ strictly semistable $S^k E$?
↳ semistable but not stable

From now on, let E be a vector bundle of rank $r=2$ & degree $d=0$. After a normalization.

$$\det(E\otimes L) = (\det E)\otimes L^2,$$
$$S^k(E\otimes L) = S^kE\otimes L^k,$$

<u>WMA</u> $\det E = \mathcal{O}_C$.

In this case,

$$\mathrm{rk}\, S^kE = k+1 \quad \& \quad \det S^kE = (\det E)^{\binom{k+1}{2}} = \mathcal{O}_C,$$

and

"self-dualities"

$$E^\vee = E\otimes(\det E)^{-1} = E \quad \& \quad (S^kE)^\vee = S^kE^\vee = S^kE.$$

<u>Rmk</u> S^kE is not stable $\Leftrightarrow$ $\exists$ destabilizing subbundle $V\to S^kE$ of rank $\leq \frac{k+1}{2}$

($\because$) Destabilizing $V\to S^kE$ yields the exact seq.

$$0\to V\to S^kE\to W\to 0$$

of vector bundles w/ $\mathrm{rk}\,V+\mathrm{rk}\,W = k+1$ & $\deg V = \deg W = 0$.

$\Rightarrow$ $\exists$ destabilizing $W^\vee\to(S^kE)^\vee = S^kE$.

§2. Destabilization of S^kE by Rank 1

X: the ruled surface $\pi : \mathbb{P}_C(E) \to C$

C_1: a unisecant divisor on X w/ $\pi_* \mathcal{O}_X(C_1) = E$

$\Rightarrow$ $C_1^2 = \deg E = 0$

$N_1(X)$: the set of divisors on X / $\equiv$ (num. equiv.)

$= \{ D \equiv kC_1 + bf \mid k, b \in \mathbb{Z} \}$ realized as a curve

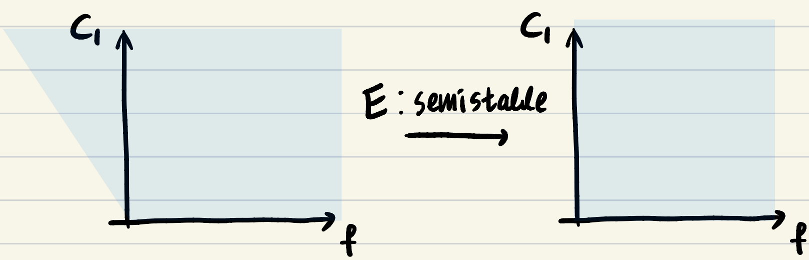
↳ k-secant divisor, k-section if effective

$\Rightarrow$ $N_1(X) \otimes \mathbb{Q}$: 2-dim'l vector space.

$NE(X)$: the cone of curves on X

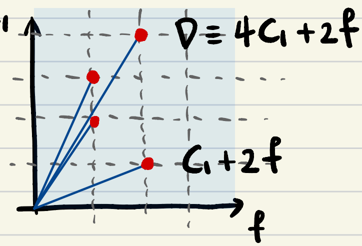
$= \{ D \in N_1(X)_{\mathbb{Q}} \mid D : \mathbb{Q}\text{-effective} \}$

If E is semistable, then $\overline{NE(X)} = \mathbb{Q}_{\geq 0} \cdot C_1 + \mathbb{Q}_{\geq 0} \cdot f$.



In particular, X has no curve D of $D^2 < 0$ if E is ss.

Rmk $NE(X)$ is not necessarily closed.



$D^2 = 16\cancel{C_1^2}^{\,0} + 16 C_1.f + 4\cancel{f^2}^{\,0} = 16$

Def A <u>minimal section</u> is a section (1-section) C_0 with the minimal self-intersection $\Leftrightarrow$ of min. num. class.

Then
$$E : \text{stable} \quad \Leftrightarrow \quad C_0^2 > 0$$
established by the correspondence

(line subbundle $L^{-1} \to S^k E$ of $\deg L = b$)
$$\longleftrightarrow \ (k\text{-section } D \equiv kC_1 + bf \text{ on } X)$$

When E is semistable, TFAE.

- $S^k E$ is destabilized by a line subbdl
- $\exists k$-section D on X of $D^2 = 0$
- $NE(X)$ is closed

Q. Classify stable E w/ $\overline{NE(X)} = NE(X)$?

§3. Stable E w/ sss S^2E

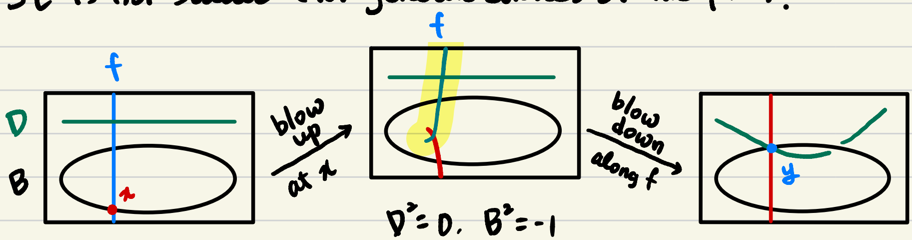
Recall that

S^2E: not stable $\Leftrightarrow \exists$ line subbundle $L^{-1} \to S^2E$ of $\deg L = 0$
↳ rk 3 $\qquad\qquad \Leftrightarrow \exists$ 2-section B on $X = \mathbb{P}_C(E)$ of $B^2 = 0$

__Ex__ (Choi, Park '14)

Let $E_0 = \mathcal{O}_C \oplus M$ for some $M \in J_2(C) \setminus \{\mathcal{O}_C\}$, i.e. $M^2 = \mathcal{O}_C$ but $M \neq \mathcal{O}_C$, and $X_0 = \mathbb{P}_C(E_0)$. Then $\exists$ an irreducible & reduced 2-section B on X_0 of $B^2 = 0$.

Let $X = \mathbb{P}_C(E)$ be the ruled surface obtained by taking elementary transforms at points of B from X. Then E is stable but S^2E is not stable (for general choices of the pt.s).



$D^2 = 0$, $B^2 = 0$

⟨ elementary transform at x ⟩

$D^2 = 1$, $B^2 = 0$
gives a desired example after checking D is a minimal section

On the other hand.

S^2E: not stable $\Leftrightarrow \exists$ quotient line bundle $S^2E \to L$ of $\deg L = 0$

$\Leftrightarrow$ E is orthogonal ($\Rightarrow$ requires the stability of E)

<u>Def</u> E is <u>orthogonal</u> w/ its values in M if
$\exists$ a nondegenerate symmetric bilinear form $E \otimes E \to M$.

<u>Thm</u> (Mumford's classification '71) E is orthogonal iff

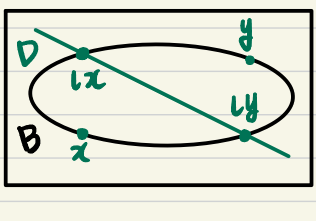
(i) $V \approx A \oplus A^{-1}$ (= up to normalization) for any line bundle A on C

(ii) $V \approx \pi_* R$ for an étale 2-covering $\pi: B \to C$
& a line bundle R on B

Note that B: irr. & red. 2-section on X of $B^2 = 0$
$\Rightarrow \pi = \Pi|_B : B \to C$ is uramified.

Let X = (elem. transf. at $x, y \in B$ from X_0).

$X = \mathbb{P}_C(E)$
$\Pi \downarrow$
C



$D \sim C_1 + \mathfrak{b}f$

$\Rightarrow \Pi_* \mathcal{O}_X(D) = \Pi_* \mathcal{O}_X(C_1) \otimes L$
for $L = \mathcal{O}_C(\mathfrak{b})$

$\Rightarrow \Pi_* \mathcal{O}_X(D) \approx E$

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_X(D-B) & \to & \mathcal{O}_X(D) & \to & \mathcal{O}_B(D) & \to & 0 \\ & & & & & & \downarrow \Pi_* & & \\ & & 0 & \to & \Pi_* \mathcal{O}_X(D) & \to & \Pi_* \mathcal{O}_B(\ell x + \ell y) & \to & 0 \end{array}$$

Thus $E \approx \pi_* R$. That is, the elm. transf. at pt.s of B generates all the orthogonal bundles.

§4. sss S^kE w/ stable $S^{k-1}E$

Thm (K.) S^kE is stable $\forall k \geq 2$ unless one of the following cases occurs.

under the stability of smaller sym. pw.

① S^2E: destabilized by rk 1

② S^3E: " rk 2 $\rightsquigarrow$ S^6E: destabilized by rk 1

③ S^4E: " rk 2 $\rightsquigarrow$ S^8E: "

④ S^6E: " rk 3 $\rightsquigarrow$ $S^{12}E$: "

Cor Let E be a stable vector bundle of $\mathrm{rk}\, E = 2$ w/ $\det E = \mathcal{O}_C$.

(1) S^kE is stable $\forall k \geq 2$ for general E.

(2) S^kE is not stable for some $k \geq 1$ iff $NE(\mathbb{P}_C(E))$ is closed.

(idea of pf) The proof uses

$$0 \to S^{n-1}E \otimes S^{m-1}E \to S^nE \otimes S^mE \to S^{n+m}E \to 0.$$

For example, $n=1$, $m=3$ & dualizing & $\otimes L$ gives

$$0 \to S^3E \otimes L \to E \otimes S^2E \otimes L \to S^2E \otimes L \to 0$$

$$\Rightarrow \mathrm{Hom}(L^{-1}, S^3E) \subseteq \mathrm{Hom}(E^\vee \otimes L^{-1}, S^2E).$$

So, S^3E cannot be destab. by rk 1 if S^2E is stable.

Q. What are the candidates for ② ~ ④?

(A) Étale-trivial bundles: $\pi^*E = \mathcal{O}_D \oplus \mathcal{O}_D$ for some étale k-covering $\pi: D \to C$.

By the correspondence

$$(\text{Surjection } \pi^*E \to R \text{ on } D) \longleftrightarrow (C\text{-morphism } \phi: D \to X = \mathbb{P}_C(E))$$

with $R = \phi^*\mathcal{O}_X(C_1)$, X admits an m-section D' of $D'^2 = 0$ for some $m|k$.

Thm (K.) Assume that E is as before & S^2E is stable. Then,

$$S^kE: \text{not stable for some } k \geq 3 \iff E: \text{étale-trivial}.$$

Rmk (Nori '76) defines a vector bundle V to be <u>finite</u> if $\exists$ finite collection S of vector bundles s.t. $\forall k \geq 2$,

$$\underbrace{V \otimes \cdots \otimes V}_{k\text{-times}} = W_{k_1} \oplus \cdots \oplus W_{k_{m_k}} \text{ for some } W_{k_i} \in S.$$

When the base field is alg. closed of char 0,

$$V \text{ is finite} \iff V \text{ is étale-trivial}.$$