Ruled surfaces with a k-section of zero self-intersection ($k = 1, 2, 3, 4$)

김정섭

KAIST 수리과학과

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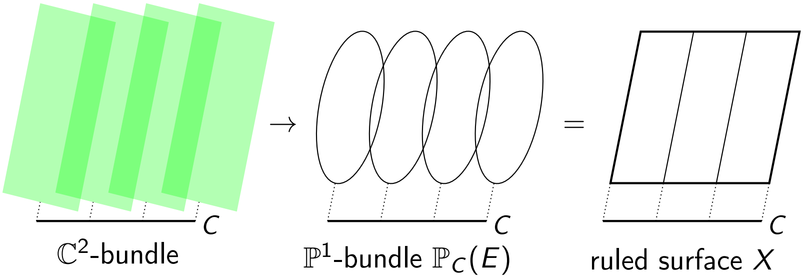
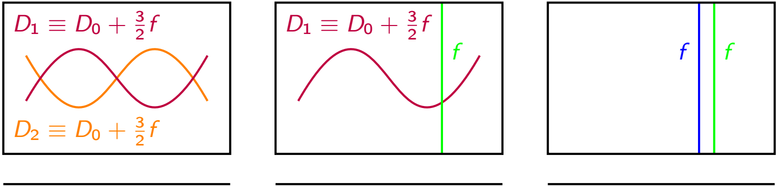
Introduction

- All varieties are defined over $\mathbb{C}$.
- C is a smooth projective curve of genus $g \geq 2$.
- E is a vector bundle on C of $\operatorname{rk} E = 2$ and $\deg E = \deg c_1(E) = d$.
- $X = \mathbb{P}_C(E)$ is the ruled surface with projection $\Pi : \mathbb{P}_C(E) \to C$.
- C_1 is a divisor on X satisfying $\Pi_* \mathcal{O}_X(C_1) = E$.



Introduction

- $\mathrm{Pic}(X) = \{D \sim_{\mathrm{lin}} kC_1 + \mathfrak{b}f \mid k \in \mathbb{Z},\ \mathfrak{b} \in \mathrm{Pic}(C)\}$
- $N_1(X) = \{D \equiv_{\mathrm{num}} kC_1 + bf \mid k,\ b \in \mathbb{Z}\}$
  - ▷ $N_1(X) \otimes \mathbb{Q}$ is a 2-dimensional $\mathbb{Q}$-vector space.
  - ▷ We choose a $\mathbb{Q}$-divisor $D_0 \equiv C_1 + bf$ as it to be ${D_0}^2 = 0$.
  - ▷ Note that $D_0.f = 1$ and $f^2 = 0$.

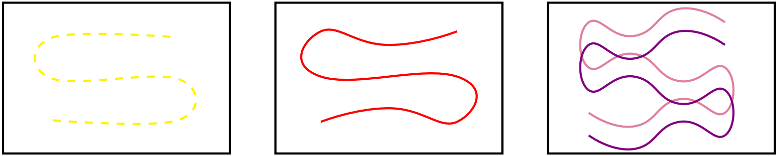


$D_1.D_2 = 3$ | $D_1.f = 1$ | $f.f = 0$

Introduction

For $k \geq 1$, a divisor $D \equiv kC_1 + bf$ on X is called

- a k-secant divisor,
- a *k-section* if D is effective, i.e. realized as a curve.
  - ▷ When $k = 1$, it is just called a *section*.
  - ▷ It is more likely to be effective as $b \to \infty$.



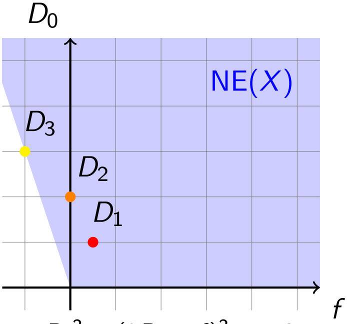
$D_1 \equiv 3D_0 - f$
D_1 is not effective

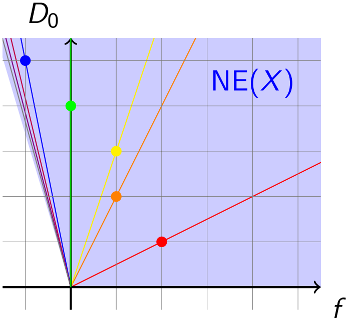
$D_2 \equiv 3D_0$
D_2 is effective

$D_3 \equiv 3D_0 + f$
D_3 is effective and
$\exists$ many 3-sections $\sim D_3$

Introduction

- $\mathrm{NE}(X) = \{D \in N_1(X) \otimes \mathbb{Q} \mid D \text{ is } \mathbb{Q}\text{-effective}\}$
  - ▷ The cone of curves $\mathrm{NE}(X)$ is convex.
  - ▷ A k-section may have negative self-intersection.
  - ▷ $\mathrm{NE}(X)$ is not necessarily closed.





- $D_3{}^2 = (3D_0 - f)^2 = -6$
- $D_2{}^2 = (2D_0)^2 = 0$
- $D_1{}^2 = (D_0 + \frac{1}{2}f)^2 = 1$

Introduction

There is a 1-to-1 correspondence between

$$\{k\text{-sections } D \sim kC_1 + \mathfrak{b}f \text{ on } X\}$$
$$\updownarrow \quad L = \mathcal{O}_C(\mathfrak{b})$$
$$\{\text{line subbundles } L^{-1} \to S^k E\}.$$

Then it follows that

$$D^2 = (kC_1 + bf)^2 = 0$$
$$\updownarrow$$
$$L^{-1} \to S^k E \text{ with } \deg L^{-1} = \frac{\deg S^k E}{\operatorname{rk} S^k E}$$
$$\Leftrightarrow \; L^{-1} \textit{ destabilizes } S^k E.$$

Introduction

A vector bundle V on C is *(semi-)stable* if every nontrivial proper subbundle $F \to V$ satisfies

$$\frac{\deg F}{\operatorname{rk} F} < \frac{\deg V}{\operatorname{rk} V} \ \left(\frac{\deg F}{\operatorname{rk} F} \leq \frac{\deg V}{\operatorname{rk} V} \right).$$

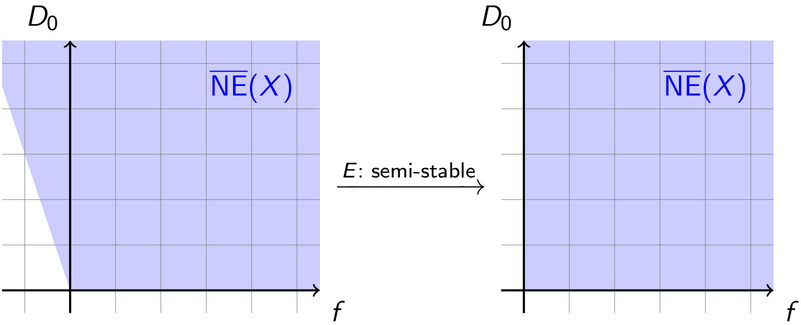
If V is semi-stable but not stable, then V is said to be *strictly semi-stable*.

If E is semi-stable, then it is known that

- $\mathrm{NE}(X) \subseteq \mathbb{Q}_{\geq 0} \cdot D_0 + \mathbb{Q}_{\geq 0} \cdot f$,
  - ▷ $\forall D \in \mathrm{NE}(X)$, $D^2 \geq 0$
- $\overline{\mathrm{NE}}(X) = \mathbb{Q}_{\geq 0} \cdot D_0 + \mathbb{Q}_{\geq 0} \cdot f$.

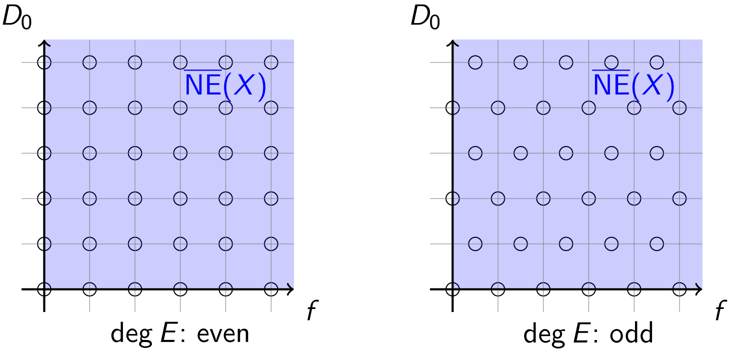
Introduction

From now on, E **is assumed to be semi-stable**.



Introduction

$$\mathrm{NE}(X) \text{ is closed}$$
$$\Updownarrow$$
$$\exists \text{ class of } k\text{-section } [D] \text{ at the boundary of } \mathrm{NE}(X)$$
$$\Updownarrow$$
$$\exists\ k\text{-section } D \text{ on } X \text{ with } D^2 = 0$$



$k = 1$

Recall the correspondence

$$\exists \text{ section } D \sim C_1 + \mathfrak{b} f \text{ with } D^2 = 0$$
$$\Updownarrow \quad L = \mathcal{O}_C(\mathfrak{b})$$
$$E \text{ is destabilized by a line subbundle } L^{-1} \to E.$$

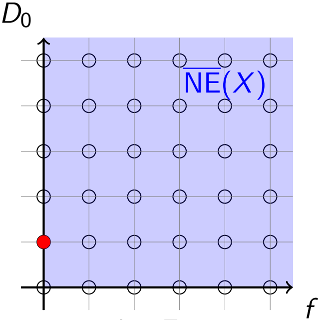
Since $\operatorname{rk} E = 2$, E is not stable if and only if

- E is destabilized by a line subbundle.

$k = 1$

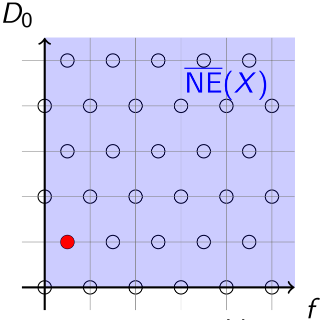
Therefore,

$$\exists \text{ section } D \text{ on } X \text{ with } D^2 = 0 \quad \Leftrightarrow \quad E \text{ is strictly semi-stable.}$$



deg E: even

E: strictly semi-stable



deg E: odd

$\nexists$ strictly semi-stable E

$k = 2$

Similarly from the correspondence

$$\exists \text{ 2-section } D \sim 2C_1 + \mathfrak{b}f \text{ with } D^2 = 0$$
$$\Updownarrow \quad L = \mathcal{O}_C(\mathfrak{b})$$
$$S^2E \text{ is destabilized by a line subbundle } L^{-1} \to S^2E.$$

Since $\operatorname{rk} S^2E = 3$, S^2E is not stable only if

- S^2E is destabilized by a line subbundle, or
- S^2E is destabilized by a subbundle of rank 2.

From some dualities, we can observe that

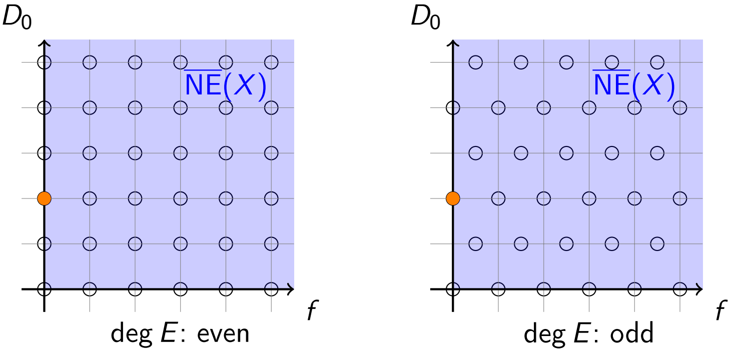
S^2E is destabilized by "rk 1" $\Leftrightarrow$ S^2E is destabilized by "rk 2".

That is, S^2E **is not stable if and only if it is destabilized by** "rk 1".

$k = 2$

Therefore,

$\exists$ 2-section D on X with $D^2 = 0 \quad \Leftrightarrow \quad S^2E$ is strictly semi-stable.



S^2E may be strictly semi-stable even if deg E is odd

$k = 2$

$$\exists \text{ section } D \text{ on } X \text{ with } D^2 = 0 \quad \Leftrightarrow \quad E \text{ is strictly semi-stable}$$

$$\Downarrow \quad B := kD$$

$$\exists\ k\text{-section } B \text{ on } X \text{ with } B^2 = 0 \quad \Leftrightarrow \quad S^k E \text{ is destabilized by ``rk 1''}$$

In particular for $k = 2$, if E is not stable, then $S^2 E$ is not stable.

Question

$\exists$ E such that

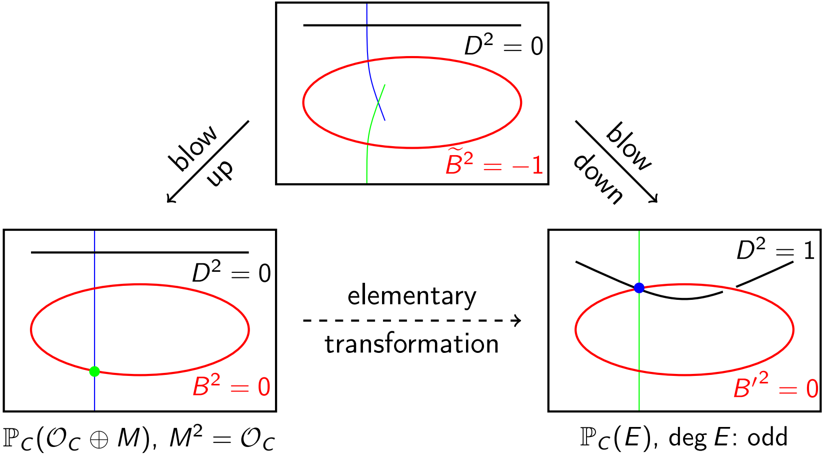
- E is stable but $S^2 E$ is destabilized by "rk 1" ($\Leftrightarrow$ not stable)?

That is, $\exists$ X which admits

- no section D with $D^2 = 0$ but a 2-section B with $B^2 = 0$?

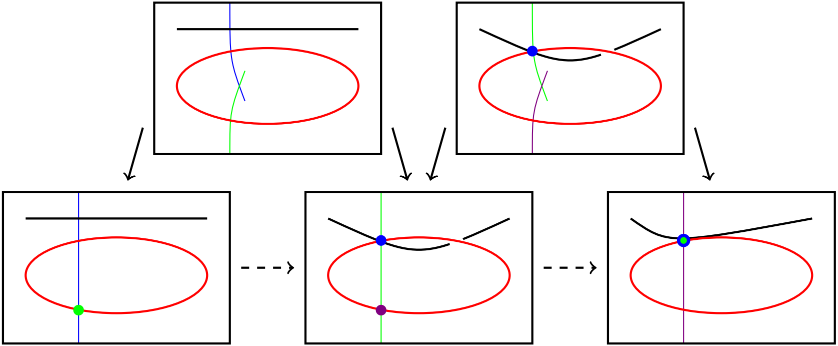
It is answered in [Choi-Park(최영욱-박의성) '15]. They construct such examples using elementary transformations.

$k = 2$



[Choi-Park '15] starts with the ruled surface associated to $\mathcal{O}_C \oplus M$ which admits an irreducible 2-section B with $B^2 = 0$. They show that elementary transformations at general points on B yield stable E.

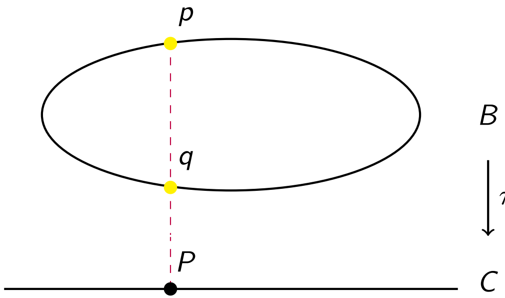
$k = 2$



$\mathbb{P}_C(\mathcal{O}_C \oplus M)$, $M^2 = \mathcal{O}_C$ $\quad$ $\mathbb{P}_C(F)$, $\deg F$: odd $\quad$ $\mathbb{P}_C(E)$, $\deg E$: even

When $\deg E$ is even and S^2E is not stable, the construction generates *orthogonal bundles*. According to the classification of [Mumford '71], an orthogonal bundle is given by $\pi_* R$ for some $R \in \mathrm{Pr}(B/C)$ and unramified 2-covering $\pi : B \to C$.

$k = 2$



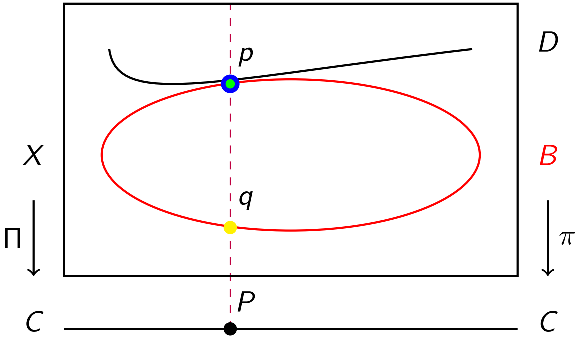
Let $\pi : B \to C$ be a nontrivial unramified 2-covering. Then the Prym of B over C is defined by

$$\Pr(B/C) = \{R \in J^0(B) \,|\, \mathrm{Nm}_{B/C}(R) = \mathcal{O}_C\}.$$

Here, $\mathrm{Nm}_{B/C}$ is for instance defined by

$$\mathrm{Nm}_{B/C}(\mathcal{O}_B(p - q)) = \mathcal{O}_C(\pi(p) - \pi(q)).$$

$k = 2$



Note that

- $E \approx \Pi_* \mathcal{O}_X(D - Pf)$ up to *normalization* (twist by line bundles),
- $\mathcal{O}_B(D - Pf) = \mathcal{O}_B(p - q) \in \mathrm{Pr}(B/C)$.

We have $\Pi_* \mathcal{O}_X(D - Pf) \cong \pi_* \mathcal{O}_B(D - Pf)$ by pushing forward

$$0 \to \mathcal{O}_X(-C_1 + \mathfrak{b} f) \to \mathcal{O}_X(D - Pf) \to \mathcal{O}_B(D - Pf) \to 0.$$

$k = 3$

Question

$\exists$ E such that

- E is stable but S^3E is destabilized by "rk 1"?

That is, $\exists$ X which admits

- no section of zero self-intersection but a 3-section D with $D^2 = 0$?

Because if we take an elementary transformation at a point on

- a 2-section B of $B^2 = 0 \ \rightarrow \ B'^2 = 0$,
- a 3-section D of $D^2 = 0 \ \rightarrow \ D'^2 \neq 0$,

it seems no longer effective to apply elementary transformations.

$k = 3$

Theorem [K. '20]

Let E be a stable bundle on C of rank 2 with even degree. Then

$$S^3E \text{ is destabilized by "rk 1"} \;\Rightarrow\; S^2E \text{ is strictly semi-stable},$$

and $\Leftarrow$ **holds whenever** $E \approx \pi_* R$ **for some** $R \in J_6(B) \backslash \pi^* J^0(C)$ with $R \in \mathrm{Pr}(B/C)$ associated to a nontrivial unramified 2-covering $\pi : B \to C$ up to normalization.

Remark

There are only finitely many

- unramified 2-coverings $\pi : B \to C$,
- 6-torsion line bundles on B.

Hence the number of such E's is finite if $\det E$ is fixed.

$k = 4$

Question

$\exists$ E such that

- S^2E is stable but S^4E is destabilized by "rk 1"?

That is, $\exists$ X which admits

- no 2-section B with $B^2 = 0$ but a 4-section of zero self-intersection?

(Answer) Yes.

Let

- $\pi : D \to C$ be an unramified cyclic 3-covering,
- $M \in J_2(D)$.

Then $\pi_* M = S^2E \otimes A$ for some 2-bundle E and line bundle A.
We can check that

- π is nontrivial $\Rightarrow$ S^4E is destabilized by "rk 1",
- $M \in \pi^* J^0(C)$ $\Leftrightarrow$ S^2E is destabilized by "rk 1".

So a desired example comes from nontrivial π and $M \in J_2(D) \backslash \pi^* J^0(C)$.

$k = 4$

Theorem [K. '20]

Let E be a stable bundle on C of rank 2 with even degree. Then

S^4E is destabilized by a "rk 1" $\Leftrightarrow$ S^2E is strictly semi-stable,

except the previous example up to normalization.

Remark

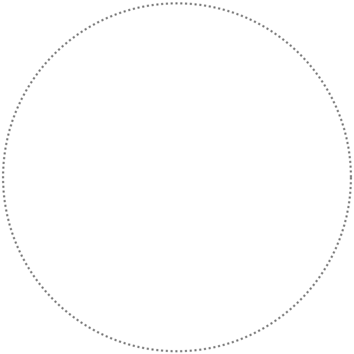
There are only finitely many

- unramified 3-coverings $\pi : D \to C$,
- 2-torsion line bundles on D,
- F's satisfying $S^2E \cong S^2F$.

Hence the number of such E's is finite if $\det E$ is fixed.

Summary

(k) $X = \mathbb{P}_C(E)$ has a k-section of zero self-intersection.

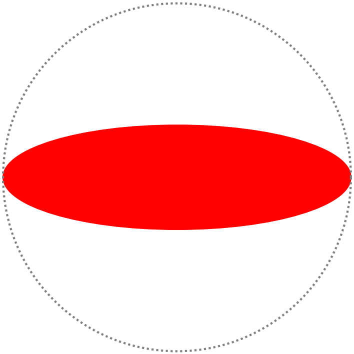


The collection of X's
for $X = \mathbb{P}_C(E)$ with semi-stable E of even degree

Summary

(k) $X = \mathbb{P}_C(E)$ has a k-section of zero self-intersection.

(1)

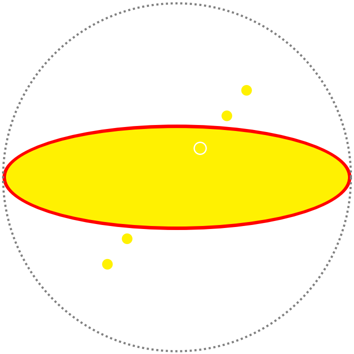


The collection of X's
for $X = \mathbb{P}_C(E)$ with semi-stable E of even degree

Summary

(k) $X = \mathbb{P}_C(E)$ has a k-section of zero self-intersection.

$$\mathbf{(1)} \subsetneq \mathbf{(3)}$$



The collection of X's
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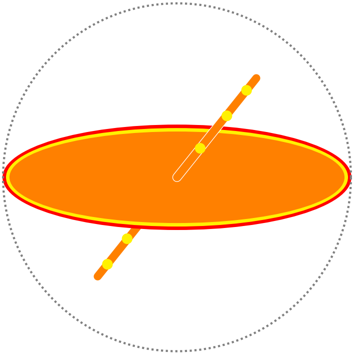
Summary

(k) $X = \mathbb{P}_C(E)$ has a k-section of zero self-intersection.

$$(1) \subsetneq (3) \subsetneq (2)$$

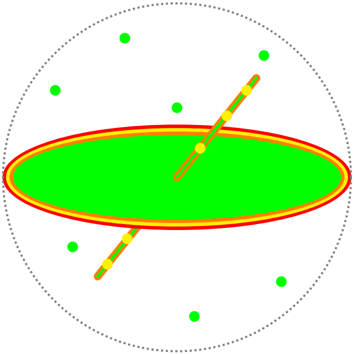


The collection of X's
for $X = \mathbb{P}_C(E)$ with semi-stable E of even degree

Summary

(k) $X = \mathbb{P}_C(E)$ has a k-section of zero self-intersection.

$$\textbf{(1)} \subsetneq \textbf{(3)} \subsetneq \textbf{(2)} \subsetneq \textbf{(4)}$$



The collection of X's
for $X = \mathbb{P}_C(E)$ with semi-stable E of even degree

References

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감사합니다.