• Generalization of Derivative: a Brief Review

In Calculus I, we learned differentiation and integration of real-valued functions in one variable. In Calculus II, we will learn those of vector-valued functions in several variable.

	symbol	norm	variable	function	derivative	chain rule	$\cdots$
scalar	a	$\lvert a\rvert$	single	real-valued	$f'(a)$	$g'(f(a))f'(a)$	$\cdots$
vector	$\mathbf{a}$	$\lVert\mathbf{a}\rVert$	multi	vector-valued	$D\mathbf{f}(\mathbf{a})$	$D\mathbf{g}(\mathbf{f}(\mathbf{a}))D\mathbf{f}(\mathbf{a})$	$\cdots$

Let $\mathbf{f} : X \subseteq \mathbf{R}^n \to \mathbf{R}^m$ be a function defined on an open set $X \subseteq \mathbf{R}^n$. We can write $\mathbf{f} = (f_1, f_2, \cdots, f_m)$ where $f_i = f_i(x_1, x_2, \ldots, x_n)$ is a real-valued function in n variables. If $\mathbf{f}$ is differentiable at $\mathbf{a} \in X$, then the derivative of $\mathbf{f}$ at $\mathbf{a}$ is given by an $m \times n$ matrix

$$D\mathbf{f}(\mathbf{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{bmatrix}.$$

In particular,

- $n = m = 1$. $D\mathbf{f}(\mathbf{a}) = Df(a)$ is a 1×1 matrix, which is a single number $f'(a)$.
- $m = 1$. $D\mathbf{f}(\mathbf{a}) = Df(\mathbf{a})$ is a $1 \times n$ matrix, which is considered as a gradient vector

$$\nabla f(\mathbf{a}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{a}),\, \frac{\partial f}{\partial x_2}(\mathbf{a}),\, \ldots,\, \frac{\partial f}{\partial x_n}(\mathbf{a})\right) = (f_{x_1}(\mathbf{a}),\, f_{x_2}(\mathbf{a}),\, \ldots,\, f_{x_n}(\mathbf{a}))$$

  where $\dfrac{\partial f}{\partial x_i} = f_{x_i}$ is the partial derivative of f with respect to x_i.

  Moreover, the higher-order partial derivatives are defined, e.g. $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are able to be defined. But remember that they are not always equal.
- $n = 1$. $\mathbf{f}(\mathbf{x}) = \mathbf{f}(x) = (f_1(x),\, f_2(x),\, \ldots,\, f_m(x))$ is called a path(in the Textbook, a curve means the image of a path) and the $m \times 1$ matrix $D\mathbf{f}(\mathbf{a}) = D\mathbf{f}(a)$ corresponds to a tangent vector (velocity vector)

$$\mathbf{f}'(a) = (f_1'(a),\, f_2'(a),\, \ldots,\, f_m'(a))$$

  of the path $\mathbf{f}$.
- $n = m \geq 2$. $\mathbf{f}$ is said to be a vector field. It assigns a vector to each point of $\mathbf{R}^n$.

Question. $D\mathbf{f}(\mathbf{a})$ is a matrix (provided it exists). If $\mathbf{v}$ is a vector in $\mathbf{R}^n$, what is $D\mathbf{f}(\mathbf{a})(\mathbf{v})$? What if $m = 1$? Is $Df(\mathbf{a})(\mathbf{v}) = \nabla f \cdot \mathbf{v}$?

- **Continuity and Differentiability of functions in several variable**

Example. $f(\mathbf{x})$ is conti. along every line passing through $\mathbf{x} = \mathbf{a} \;\not\Longrightarrow\; f(\mathbf{x})$ is conti. at $\mathbf{x} = \mathbf{a}$

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

A line passing through $\mathbf{a} = (0,0)$ is written by $\mathbf{x}(t) = (at, bt)$ for some $(a,b) \neq (0,0)$. Then $\mathbf{x}(t) \to \mathbf{a}$ as $t \to 0$. Note that

$$\lim_{t\to 0} f(\mathbf{x}(t)) = \lim_{t\to 0} \frac{(at)^2(bt)}{(at)^4 + (bt)^2} = \lim_{t\to 0} \frac{a^2 bt}{a^4 t + b^2} = 0.$$

That is, $f(\mathbf{x}(t)) \to f(\mathbf{a})$ as $\mathbf{x}(t) \to \mathbf{a}$. Since (a,b) was arbitrary, it means that $f(\mathbf{x})$ is continuous along every line passing through $\mathbf{a}$. However, $f(\mathbf{x})$ is not continuous at $\mathbf{x} = \mathbf{a}$. Because, along the curve $\mathbf{y}(t) = (t, t^2)$,

$$\lim_{t\to 0} f(\mathbf{y}(t)) = \lim_{t\to 0} \frac{t^2(t^2)}{(t)^4 + (t^2)^2} = \lim_{t\to 0} \frac{t^4}{2t^4} = \frac{1}{2}.$$

Thus $f(\mathbf{y}(t)) \not\to f(\mathbf{a})$ as $\mathbf{y}(t) \to \mathbf{a}$. Notice that this function has the directional derivatives at $\mathbf{a}$ in all the directions (identical to Example 3 of Textbook p.161). However, since f is not continuous at $\mathbf{a}$, it is not differentiable at $\mathbf{a}$ (Theorem 2.3.6, 2.3.9: differentiability implies continuity).

In general, it not easy to show the continuity of a given function. It might be required to use an ε-δ argument.

Example. $f(\mathbf{x})$ is continuous and has the directional derivatives at $\mathbf{x} = \mathbf{a}$ in all the directions $\not\Longrightarrow\; f(\mathbf{x})$ is diff. at $\mathbf{x} = \mathbf{a}$

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function defined by

$$f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

Then $f(\mathbf{x})$ is continuous at $(0,0)$ as solved in Homework #2.2.52 and it is easy to see that all the directional derivatives of $f(\mathbf{x})$ at $(0,0)$ exist. But $f(\mathbf{x})$ is not differentiable at $(0,0)$.

Question. How to show that a real-valued function is not differentiable?

It is not a good idea to use a naive definition of differentiability to disprove it. Instead, one can verify a property which must be satisfied by any differentiable function. That is, if a function does not satisfy the property, then it tells that the function is not differentiable. Here are some suggestions: if $f(\mathbf{x})$ is differentiable at $\mathbf{x} = \mathbf{a}$, then $Df(\mathbf{a})$ exists (Definition 2.3.4, 2.3.7, 2.3.8), $\mathbf{f}(\mathbf{x})$ is continuous at $\mathbf{x} = \mathbf{a}$ (Theorem 2.3.6 again), and the directional derivatives exist and satisfy

$$D_{\mathbf{v}} f(\mathbf{a}) = Df(\mathbf{a})(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{R}^n. \quad \text{(Theorem 2.6.2)}$$

Then one can disprove the differentiability of $f(x, y)$ by checking

$$D_{\mathbf{v}_0} f(0,0) \neq Df(0,0)(\mathbf{v}_0) \quad \text{for some } \mathbf{v}_0 \in \mathbf{R}^n.$$

Question. How to show that a vector-valued function $\mathbf{f} = (f_1, \ldots, f_m)$ is not differentiable?
(A) Theorem 2.3.11: $\mathbf{f}$ is diff. $\Leftrightarrow$ f_i is diff. for all i.

- **Chain Diagram**

The following is a chain diagram for functions on $X \subseteq \mathbf{R}^3$.

$$\boxed{\text{scalar}} \xrightarrow[(\nabla)]{\text{grad}} \boxed{\text{vector}} \xrightarrow[(\nabla\times)]{\text{curl}} \boxed{\text{vector}} \xrightarrow[(\nabla\bullet)]{\text{div}} \boxed{\text{scalar}}$$

The diagram can be read as follows. If f has its values in $\mathbf{R}$, then $\mathbf{F} = \nabla f$ has its values in $\mathbf{R}^3$. If $\mathbf{F}$ is a vector field, then $\mathbf{G} = \nabla \times \mathbf{F}$ is also a vector field. For vector-valued $\mathbf{G}$, $\nabla \bullet \mathbf{G}$ becomes real-valued. The diagram says something more. If f is real-valued and of class C^2, then $\nabla \times (\nabla f) = \mathbf{0}$. Also, if $\mathbf{F}$ is a vector field of class C^2, then $\nabla \bullet (\nabla \times \mathbf{F}) = 0$. That is, if a function goes through the arrows twice, then the outcome becomes trivial. One can use this diagram to memorize the rules of differential operations(Theorem 3.4.3, 3.4.4).

Then it is natural to ask that if $\nabla \times \mathbf{F} = \mathbf{0}$($\mathbf{F}$ is irrotational), then $\mathbf{F} = \nabla f$ for some f? Also, if $\nabla \bullet \mathbf{G} = 0$($\mathbf{G}$ is incompressible), then $\mathbf{G} = \nabla \times \mathbf{F}$ for some $\mathbf{F}$? Both answers are negative in general, but positive under a special condition(this is a topic of Section 6.3).

Question. There is a similar diagram over the real line.

$$\boxed{\text{scalar}} \xrightarrow{\frac{d}{dx}} \boxed{\text{scalar}}$$

Is there a similar diagram over the real plane? (A) We will treat this after midterm.

• Line Integrals of Scalar Field and Vector Field

Question. Let f be a scalar field and $\mathbf{F}$ be a vector field which are continuous on $\mathbf{R}^3$. Let $\mathbf{x} : [a, b] \to \mathbf{R}^3$ be a smooth path. Which notations are valid?

$$(1) \int_{\mathbf{x}} f\ ds \qquad (2) \int_{\mathbf{x}} f\ d\mathbf{s} \qquad (3) \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} \qquad (4) \int_{\mathbf{x}} \mathbf{F}\ ds$$

(A) Only (1) and (3) are covered in our course. In this course, keep in mind that the value of integral is always scalar. That is, integrals like

$$\int_{\mathbf{x}} M\mathbf{i} + N\mathbf{j} + P\mathbf{k}\ ds = \left(\int_{\mathbf{x}} M\ ds\right)\mathbf{i} + \left(\int_{\mathbf{x}} N\ ds\right)\mathbf{j} + \left(\int_{\mathbf{x}} P\ ds\right)\mathbf{k} \qquad \text{(type (4))}$$

are not dealt in our course. Instead, we are interested in integrals like

$$\begin{aligned}\int_{\mathbf{x}} (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) \cdot d\mathbf{s} &= \int_{\mathbf{x}} (M\mathbf{i} + N\mathbf{j} + P\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \int_{\mathbf{x}} M\, dx + N\, dy + P\, dz \\ &= \int_a^b M \cdot x'(t) + N \cdot y'(t) + P \cdot z'(t)\ dt \qquad \text{(type (3))}\end{aligned}$$

for $\mathbf{s}(t) = \mathbf{x}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \Rightarrow d\mathbf{s} = \mathbf{x}'(t)\, dt$. When the integrand is a scalar field, we are interested in integrals like

$$\int_{\mathbf{x}} f\ ds = \int_a^b f\, \|\mathbf{x}'(t)\|\, dt = \int_a^b f\, \sqrt{\{x'(t)\}^2 + \{y'(t)\}^2 + \{z'(t)\}^2}\, dt \qquad \text{(type (1))}$$

for $s(t) = \int_a^t \|\mathbf{x}'(\tau)\|\, d\tau \Rightarrow ds = \|\mathbf{x}'(t)\|\, dt$.

Let $\mathbf{T}(t)$ be the unit tangent vector to $\mathbf{x}(t)$. Then type (3) integral is also written by

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F} \cdot \mathbf{x}'(t)\, dt = \int_{\mathbf{x}} \mathbf{F} \cdot \mathbf{T}\, \|\mathbf{x}'(t)\|\, dt = \int_{\mathbf{x}} \mathbf{F} \cdot \mathbf{T}\ ds.$$

If $\mathbf{x}$ is a closed path in $\mathbf{R}^2$, there is a natural choice of a vector associated to $\mathbf{x}$ besides $\mathbf{T}$. It is the (outward) unit normal vector $\mathbf{n}$ to the path $\mathbf{x}$ (refer to Section 6.2). Subtitute $\mathbf{T}$ by $\mathbf{n}$ in the above integral, we obtain the integral

$$\int_{\mathbf{x}} \mathbf{F} \cdot \mathbf{n}\ ds.$$

This is the only phenomenon for line integrals in $\mathbf{R}^2$. It does not happen for those in $\mathbf{R}^3$ because there is no unique choice of a normal vector to a path (up to opposite direction, and the principal normal vector $\mathbf{N}$ has nothing to do with a surface enclosed by the path). However, it is able to choose a normal vector when the integral is taken over a surface in $\mathbf{R}^3$ (codimension 1). Notice that the integration with normal vectors is said to be *flux* of $\mathbf{F}$ across $\mathbf{x}$ whereas that with tangent vectors is called *circulation* of $\mathbf{F}$ along $\mathbf{x}$.

- **Basic Philosophy of Integration Theorems: Decrease/Increase Dimension**

Recall (The Fundamental Theorem of Calculus). Let f be a real-valued differentiable function in one variable. Then the Fundamental Theorem of Calculus states that

$$\int_a^b \frac{d}{dx} f(x)\ dx = f(b) - f(a).$$

The left-hand side is a 1-dimensional integration and the right-hand side can be regarded as a 0-dimensional integration, which is nothing but just evaluation at points.

The fundamental theorem can be generalized to functions in two variable as follows. Let f be a real-valued differentiable function in two variables and let C be a path in $\mathbf{R}^2$ with initial point A and terminal point B. Then Theorem 6.3.3 (p.441) says that

$$\int_C \operatorname{grad} f \cdot d\mathbf{s} = f(B) - f(A).$$

The dimension of integration is lowered as taking the boundary of integration domain like the Fundamental Theorem of Calculus.

Remind that the differential operation 'curl' is defined for vector fields on $\mathbf{R}^3$. But, if we consider a vector field $\mathbf{F} = (M, N)$ on $\mathbf{R}^2$ as a vector field on $\mathbf{R}^3$ after adjoining the third coordinate, i.e. $(M, N, 0)$, the 'curl' is able to be defined for vector fields on $\mathbf{R}^2$ by taking the third coordinate of the usual curl as

$$\operatorname{curl} \mathbf{F} = \frac{\partial}{\partial x} N - \frac{\partial}{\partial y} M \quad \text{(CAUTION: this notation is not standard)}.$$

However, the divergence operation 'div' is naturally defined for vector field on $\mathbf{R}^2$ as

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x} M + \frac{\partial}{\partial y} N.$$

Then the Green's theorem (p.429) and divergence theorem in the plane (p.432) are

$$\iint_D \operatorname{curl} \mathbf{F}\ dA = \oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds \qquad \text{and} \qquad \iint_D \operatorname{div} \mathbf{F}\ dA = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds.$$

The right-hand side of the first equation is said to be a *circulation* and the one in the latter is called a *flux*. Be careful not to confuse a pair with the other:

$$\text{curl} \leftrightarrow \text{circulation} \qquad \text{and} \qquad \text{divergence} \leftrightarrow \text{flux}.$$

As before, one can observe from the theorems that the dimension of integration is reduced by one as going through the equation from left to right.

- **When is a vector field conservative?**

Question. What are the definitions of conservative vector fields, vector fields having path-independent line integrals, simply-connected domains, simple paths, and closed paths?

If $\mathbf{F}$ is conservative, i.e. $\mathbf{F} = \nabla f$ for some f, then line integrals over $\mathbf{F}$ become easier; it does not depend on the shape of a path as

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \operatorname{grad} f \cdot d\mathbf{s} = f(B) - f(A)$$

where A and B are the endpoints of the path C. In particular, if C is a closed path, that is, if $B = A$, then $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$. In other words, $\mathbf{F}$ has path-independent line integrals. The converse also holds (Theorem 6.3.3).

Question. Let $\mathbf{F}$ be a differentiable vector field on $D \subseteq \mathbf{R}^2$. If $\mathbf{F} = \operatorname{grad} f$ for some f, then $\operatorname{curl} \mathbf{F} = 0$. Conversely, if $\operatorname{curl} \mathbf{F} = 0$, then does there exist some f such that $\mathbf{F} = \operatorname{grad} f$? (A) Yes when D is simply connected. Please give to students some examples and non-examples of simply-connected domains.

Example. Let $\mathbf{F} = (-y, x)$ and $\mathbf{x} : [0, 2\pi] \to \mathbf{R}^2;\ t \mapsto (\cos t, \sin t)$ be a circle in $\mathbf{R}^2$. Then

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = \int_0^{2\pi} 1\ dt = 2\pi.$$

So $\mathbf{F}$ is not conservative of course and $\operatorname{curl} \mathbf{F} = \dfrac{\partial}{\partial x} x - \dfrac{\partial}{\partial y}(-y) = 2 \neq 0$.

Example. Take $\mathbf{G} = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right)$ and $\mathbf{x}$ to be as before. Then $\int_{\mathbf{x}} \mathbf{G} \cdot d\mathbf{s} = 2\pi$ so that $\mathbf{G}$ is not conservative, but

$$\operatorname{curl} \mathbf{G} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2) - x(2x) + (x^2+y^2) - y(2y)}{(x^2+y^2)^2} = 0.$$

Thus $\mathbf{G}$ provides an example that $\operatorname{curl} \mathbf{G} = 0$ but $\mathbf{G}$ is not conservative. This is because the domain of $\mathbf{G}$ is not simply-connected as $\mathbf{R}^2 \backslash \{(0,0)\}$.

Let C_1 and C_2 be two paths in $\mathbf{R}^2$ who share the endpoints A and B. Though $\mathbf{G}$ is not conservative, if there is a way to deform C_1 into C_2 while fixing the endpoints, and the deforming region R is contained in the domain of $\mathbf{G}$, we have

$$\int_{C_1} \mathbf{G} \cdot d\mathbf{s} - \int_{C_2} \mathbf{G} \cdot d\mathbf{s} = \oint_{\partial R} \mathbf{G} \cdot d\mathbf{s} = \iint_R \operatorname{curl} \mathbf{G}\ dA = \iint_R 0\ dA = 0$$

by the Green's theorem and hence $\int_{C_1} \mathbf{G} \cdot d\mathbf{s} = \int_{C_2} \mathbf{G} \cdot d\mathbf{s}$.

- **Surface Integrals**

Let S be a surface in $\mathbf{R}^3$ and S be parametrized by a smooth function

$$\mathbf{X}(s,t):\ D \subseteq \mathbf{R}^2 \to \mathbf{R}^3$$

that is one-to-one on D. Let f be a scalar field and $\mathbf{F}$ be a vector field which are continuous on $\mathbf{R}^3$. The valid notations for surface integrals are

$$\iint_S f\ dS \qquad \text{and} \qquad \iint_S \mathbf{F} \cdot d\mathbf{S}.$$

To compute the integrals, one needs to know that

$$dS = \|\mathbf{X}_s(s,t) \times \mathbf{X}_t(s,t)\|\, ds\, dt \qquad \text{and} \qquad d\mathbf{S} = \mathbf{X}_s(s,t) \times \mathbf{X}_t(s,t)\, ds\, dt.$$

If we denote by $\mathbf{N} = \mathbf{X}_s \times \mathbf{X}_t$ and by $\mathbf{n}$ the unit normal vector determined by the surface (unique up to opposite direction as mentioned in Recitation 8), then $\mathbf{N} = \|\mathbf{X}_s \times \mathbf{X}_t\|\, \mathbf{n}$ and

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot (\mathbf{X}_s \times \mathbf{X}_t)\ ds\, dt = \iint_S \mathbf{F} \cdot \mathbf{N}\ ds\, dt \\ &= \iint_S \mathbf{F} \cdot \mathbf{n}\, \|\mathbf{X}_s \times \mathbf{X}_t\|\, ds\, dt = \iint_S \mathbf{F} \cdot \mathbf{n}\ dS.\end{aligned}$$

Comparing with line integrals, for a curve parametrization $\mathbf{x}(t) :\ I \subseteq \mathbf{R}^2 \to \mathbf{R}^3$, we had

$$\int_C f\ ds \quad \text{with} \quad ds = \|\mathbf{x}'(t)\|\, dt \qquad \text{and} \qquad \int_C \mathbf{F} \cdot d\mathbf{s} \quad \text{with} \quad d\mathbf{s} = \mathbf{x}'(t)\, dt.$$

Moreover, $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ for the unit tangent vector $\mathbf{T}$ to the path $\mathbf{x}$.

Example. We will find the surface area of the paraboloid $z = 1 - x^2 - y^2$ bounded by $z \geq 0$.

Solution. The surface has a parametrization

$$\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, 1 - r^2), \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$

From $\mathbf{X}_r(r,\theta) = (\cos\theta, \sin\theta, -2r)$, $\mathbf{X}_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 0)$, we obtain normal vectors $\mathbf{N}(r,\theta) = (2r^2\cos\theta, 2r^2\sin\theta, r)$ pointing outward and $\|\mathbf{N}(r,\theta)\| = \sqrt{4r^4 + r^2}$. Thus the surface area is

$$\begin{aligned}\iint_S dS &= \iint_{[0,1]\times[0,2\pi]} \|\mathbf{N}(s,t)\|\, dr\, d\theta = \int_0^{2\pi}\int_0^1 r\sqrt{1+4r^2}\ dr\, d\theta \\ &= \int_0^{2\pi}\int_1^5 \frac{1}{8}\sqrt{s}\ ds\, d\theta = \int_0^{2\pi} \frac{5\sqrt{5}-1}{12}\ d\theta = \frac{5\sqrt{5}-1}{6}\pi.\end{aligned}$$

• Chain Diagram Revisited: Way to Memorize Integration Theorems

Recall (The Fundamental Theorem of Calculus). There was a so-called 'chain diagram' for dimension one.

$$\boxed{\text{scalar}} \xrightarrow{\frac{d}{dx}} \boxed{\text{scalar}}$$

$$f(b) - f(a) \quad = \quad \int_a^b \frac{d}{dx} f \, dx$$

The 0-dimensional integral of f is changed into 1-dimensional integral after taking the differential operation $\frac{d}{dx}$ to f. Also, note that the domain of 1-dimensional integral is changed into that of 0-dimensional by taking its boundary. The points a and b are indeed the boundary of the interval $[a, b]$. Thus it can be considered that the left box represents dimension 0 and the right box represents dimension 1.

The following is a chain diagram for dimension three introduced in Recitation 2. Each arrow of the diagram corresponds to a theorem we covered so far.

$$\begin{array}{lcll}
\text{(0-dim)} & \boxed{\text{scalar}} & & \\
 & \big\downarrow \nabla & \displaystyle\int_C \nabla f \cdot d\mathbf{s} = f(B) - f(A) & \text{(Theorem 6.3.3)} \\
\text{(1-dim)} & \boxed{\text{vector}} & & \\
 & \big\downarrow \nabla\times & \displaystyle\iint_S \nabla\times\mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} & \text{(Stokes's Theorem)} \\
\text{(2-dim)} & \boxed{\text{vector}} & & \\
 & \big\downarrow \nabla\cdot & \displaystyle\iiint_D \nabla\cdot\mathbf{F} \, dV = \oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} & \text{(Gauss's Theorem)} \\
\text{(3-dim)} & \boxed{\text{scalar}} & &
\end{array}$$

Question. How about the diagram for dimension two?
(A) One possible answer is as follows.

$$\begin{tikzcd}
\boxed{\text{scalar}} \arrow[r, "\text{grad}"] & \boxed{\text{vector}} \arrow[r, "\text{curl}"] \arrow[dr, "\text{div}"'] & \boxed{\text{scalar}} \\
 & & \boxed{\text{scalar}}
\end{tikzcd}$$

Then the above arrow in the second column corresponds to the Green's theorem and the below one corresponds to the divergence theorem in the plane. If one asks that which arrow is more natural, then the answer would be the above one because curl $\circ$ grad $= 0$ whereas div $\circ$ grad is not always zero.