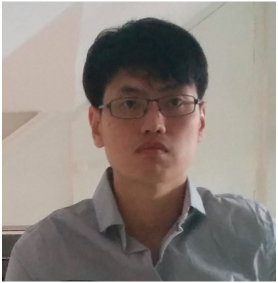
Strong edge-colorings of sparse graphs with large maximum degree

ILKYOO CHOI

KAIST, Korea

Joint work with



Jaehoon Kim

Alexandr Kostochka

André Raspaud

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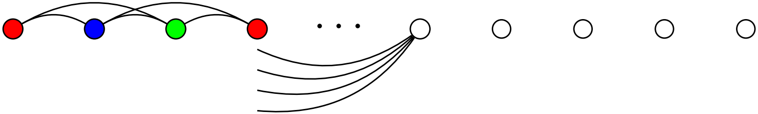

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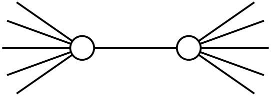

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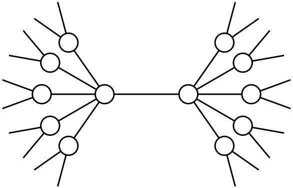

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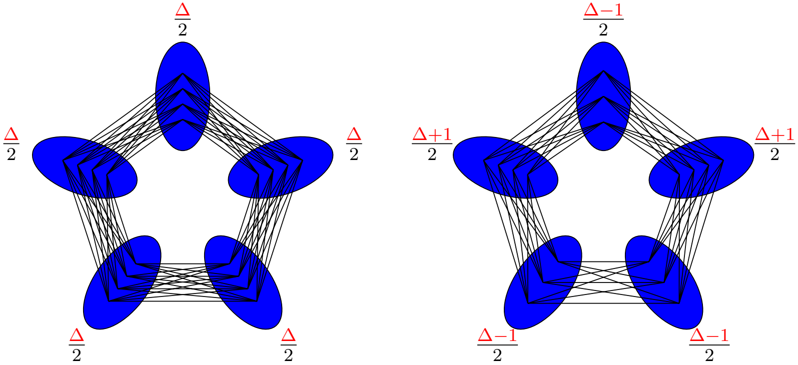

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If $\Delta(G) = 3$, then $\chi'_s(G) \leq 10$. *Conjecture is true for $\Delta(G) = 3$!*
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Investigated on many other graph classes.......

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For a graph G with $\Delta(G) \leq 3$,
(1): $\chi'_s(G) \leq 10$
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(3): $\chi'_s(G) \leq 9$ *if* G *is planar*
(4): $\chi'_s(G) \leq 6$ *if* G *is bipartite and* $d(x) + d(y) \leq 5$ *for each edge* xy*.*
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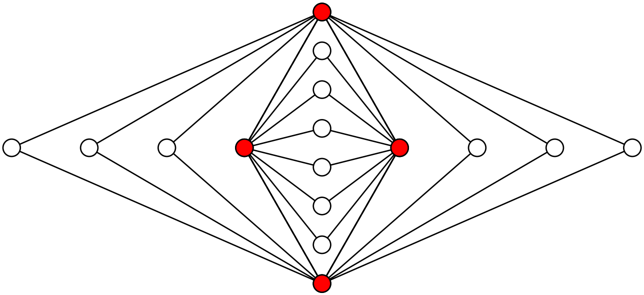

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Theorem (2013 Borodin, Ivanova)

If G is planar with $\Delta(G) \geq 3$ and girth $\geq 40\lfloor \frac{\Delta(G)}{2} \rfloor + 1$ then $\chi'_s(G) \leq 2\Delta(G) - 1$.

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Fix one color class and contract each edge; the resulting graph is planar.
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Example: Glue $K_{2,\Delta-2}$ and $K_{2,\Delta}$ in a smart way.

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If G is planar with $\Delta(G) \geq 3$ and girth $\geq 40\lfloor\frac{\Delta(G)}{2}\rfloor + 1$ then $\chi'_s(G) \leq 2\Delta(G) - 1$.

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If G is planar with $\Delta(G) \geq 4$ and girth $\geq 10\Delta(G) + 46$ then $\chi'_s(G) \leq 2\Delta(G) - 1$.

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There exists a planar graph G with $\chi'_s(G) = 4\Delta(G) - 4$.

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$\mathrm{Mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$. If G is planar with girth g, then $\mathrm{Mad}(G) < \frac{2g}{g-2}$.

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Assume $\Delta(G) \leq 3$.
If $Mad(G) < \frac{15}{7}$, *then* $\chi'_s(G) \leq 6$
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Thank you for your attention!