Conjectures regarding chi-bounded classes of graphs

ILKYOO CHOI[1], O-joung Kwon[2], and Sang-il Oum[1]

Korea Advanced Institute of Science and Technology (KAIST), Daejeon, Korea

Hungarian Academy of Sciences, Budapest, Hungary

December 7, 2015

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

(1) subgraph

– deleting vertices/edges

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

(1) subgraph
  - deleting vertices/edges

(2) topological minor
  - deleting vertices/edges
  - smoothing vertices

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

(1) subgraph
- deleting vertices/edges

(2) topological minor
- deleting vertices/edges
- smoothing vertices

(3) minor
- deleting vertices/edges
- contracting edges

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

(1) subgraph
- deleting vertices/edges

(2) topological minor
- deleting vertices/edges
- smoothing vertices

(3) minor
- deleting vertices/edges
- contracting edges

[1] induced subgraph
- deleting vertices

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

(1) subgraph
- deleting vertices/edges

(2) topological minor
- deleting vertices/edges
- smoothing vertices

(3) minor
- deleting vertices/edges
- contracting edges

[1] induced subgraph
- deleting vertices

[2] pivot-minor
- deleting vertices
- pivoting edges

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

(1) subgraph
- deleting vertices/edges

(2) topological minor
- deleting vertices/edges
- smoothing vertices

(3) minor
- deleting vertices/edges
- contracting edges

[1] induced subgraph
- deleting vertices

[2] pivot-minor
- deleting vertices
- pivoting edges

[3] vertex-minor
- deleting vertices
- local complementations at vertices

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

(1) subgraph
- deleting vertices/edges

(2) topological minor
- deleting vertices/edges
- smoothing vertices

(3) minor
- deleting vertices/edges
- contracting edges

[1] induced subgraph
- deleting vertices

[2] pivot-minor
- deleting vertices
- pivoting edges

[3] vertex-minor
- deleting vertices
- local complementations at vertices

More operations imply easier to get the structure!
No H-vertex-minor implies no H-pivot-minor implies H-free.

A clique is a set of pairwise adjacent vertices in a graph.
The clique number $\omega(G)$ is the size of a largest clique in a graph G.

A clique is a set of pairwise adjacent vertices in a graph.
The clique number $\omega(G)$ is the size of a largest clique in a graph G.

A graph G is k-colorable if the following is possible:

– each vertex receives a color from $\{1, \ldots, k\}$
– adjacent vertices receive different colors

The chromatic number $\chi(G)$ is the minimum such k.

A clique is a set of pairwise adjacent vertices in a graph.
The clique number $\omega(G)$ is the size of a largest clique in a graph G.

A graph G is k-colorable if the following is possible:

- each vertex receives a color from $\{1, \ldots, k\}$
- adjacent vertices receive different colors

The chromatic number $\chi(G)$ is the minimum such k.

$$\omega(G) \leq \chi(G)$$

A clique is a set of pairwise adjacent vertices in a graph.
The clique number $\omega(G)$ is the size of a largest clique in a graph G.

A graph G is k-colorable if the following is possible:

- each vertex receives a color from $\{1, \ldots, k\}$
- adjacent vertices receive different colors

The chromatic number $\chi(G)$ is the minimum such k.

$$\omega(G) \leq \chi(G)$$

Strong Perfect Graph Conjecture (1961 Berge)

Given a graph G, every induced subgraph H satisfies $\omega(H) = \chi(H)$ iff G contains no C_k and no $\overline{C_k}$ as induced subgraphs for any odd $k \geq 5$.

A clique is a set of pairwise adjacent vertices in a graph.
The clique number $\omega(G)$ is the size of a largest clique in a graph G.

A graph G is k-colorable if the following is possible:
- each vertex receives a color from $\{1, \ldots, k\}$
- adjacent vertices receive different colors

The chromatic number $\chi(G)$ is the minimum such k.

$$\omega(G) \leq \chi(G)$$

Strong Perfect Graph Conjecture (1961 Berge)

Given a graph G, every induced subgraph H satisfies $\omega(H) = \chi(H)$ iff G contains no C_k and no $\overline{C_k}$ as induced subgraphs for any odd $k \geq 5$.

Theorem (2006 Chudnovsky–Robertson–Seymour–Thomas)

The Strong Perfect Graph Conjecture is true.

Is there a function f such that $\chi(G) \leq f(\omega(G))$ for all graphs G?

Is there a function f such that $\chi(G) \leq f(\omega(G))$ for all graphs G?

Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)

For any k, there exists a graph G with no triangle and $\chi(G) \geq k$.
For any k, there exists a graph G with girth at least 6 and $\chi(G) \geq k$.
For any k, g, there exists a graph G with girth at least g and $\chi(G) \geq k$.

Is there a function f such that $\chi(G) \le f(\omega(G))$ for all graphs G? *NO!*

Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)

For any k, there exists a graph G with no triangle and $\chi(G) \ge k$.
For any k, there exists a graph G with girth at least 6 and $\chi(G) \ge k$.
For any k, g, there exists a graph G with girth at least g and $\chi(G) \ge k$.

Is there a function f such that $\chi(G) \leq f(\omega(G))$ for all graphs G? *NO!*

Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)

For any k, there exists a graph G with no triangle and $\chi(G) \geq k$.
For any k, there exists a graph G with girth at least 6 and $\chi(G) \geq k$.
For any k, g, there exists a graph G with girth at least g and $\chi(G) \geq k$.

Strong Perfect Graph Thm (06 Chudnovsky-Robertson-Seymour-Thomas)

Given a graph G, every induced subgraph H satisfies $\omega(H) = \chi(H)$ iff G contains no C_k and no $\overline{C_k}$ as induced subgraphs for any odd $k \geq 5$.

Is there a function f such that $\chi(G) \leq f(\omega(G))$ for all graphs G? *NO!*

Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)

For any k, there exists a graph G with no triangle and $\chi(G) \geq k$.
For any k, there exists a graph G with girth at least 6 and $\chi(G) \geq k$.
For any k, g, there exists a graph G with girth at least g and $\chi(G) \geq k$.

Strong Perfect Graph Thm (06 Chudnovsky-Robertson-Seymour-Thomas)

Given a graph G, every induced subgraph H satisfies $\omega(H) = \chi(H)$ iff G contains no C_k and no $\overline{C_k}$ as induced subgraphs for any odd $k \geq 5$.

Forbidding (infinitely many) induced subgraphs makes a graph perfect.

Is there a function f such that $\chi(G) \leq f(\omega(G))$ for all graphs G? *NO!*

Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)

For any k, there exists a graph G with no triangle and $\chi(G) \geq k$.
For any k, there exists a graph G with girth at least 6 and $\chi(G) \geq k$.
For any k, g, there exists a graph G with girth at least g and $\chi(G) \geq k$.

Strong Perfect Graph Thm (06 Chudnovsky-Robertson-Seymour-Thomas)

Given a graph G, every induced subgraph H satisfies $\omega(H) = \chi(H)$ iff G contains no C_k and no $\overline{C_k}$ as induced subgraphs for any odd $k \geq 5$.

Forbidding (infinitely many) induced subgraphs makes a graph perfect.

Definition

A class $\mathcal{C}$ of graphs is χ-*bounded* if there is a function f where $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{C}$.

Is there a function f such that $\chi(G) \leq f(\omega(G))$ for all graphs G? *NO!*

Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)

For any k, there exists a graph G with no triangle and $\chi(G) \geq k$.
For any k, there exists a graph G with girth at least 6 and $\chi(G) \geq k$.
For any k, g, there exists a graph G with girth at least g and $\chi(G) \geq k$.

Strong Perfect Graph Thm (06 Chudnovsky-Robertson-Seymour-Thomas)

Given a graph G, every induced subgraph H satisfies $\omega(H) = \chi(H)$ iff G contains no C_k and no $\overline{C_k}$ as induced subgraphs for any odd $k \geq 5$.

Forbidding (infinitely many) induced subgraphs makes a graph perfect.

Definition

A class $\mathcal{C}$ of graphs is χ-*bounded* if there is a function f where $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{C}$.

What happens when we forbid one induced subgraph?

Is there a function f such that $\chi(G) \leq f(\omega(G))$ for all graphs G? *NO!*

Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)

For any k, there exists a graph G with no triangle and $\chi(G) \geq k$.
For any k, there exists a graph G with girth at least 6 and $\chi(G) \geq k$.
For any k, g, there exists a graph G with girth at least g and $\chi(G) \geq k$.

Strong Perfect Graph Thm (06 Chudnovsky-Robertson-Seymour-Thomas)

Given a graph G, every induced subgraph H satisfies $\omega(H) = \chi(H)$ iff G contains no C_k and no $\overline{C_k}$ as induced subgraphs for any odd $k \geq 5$.

Forbidding (infinitely many) induced subgraphs makes a graph perfect.

Definition

A class $\mathcal{C}$ of graphs is χ-*bounded* if there is a function f where $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{C}$.

What happens when we forbid one induced subgraph?

For which H is the class of H-(induced subgraph)-free graphs χ-bounded?

Is there a function f such that $\chi(G) \le f(\omega(G))$ for all graphs G? *NO!*

Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)

For any k, there exists a graph G with no triangle and $\chi(G) \ge k$.
For any k, there exists a graph G with girth at least 6 and $\chi(G) \ge k$.
For any k, g, there exists a graph G with girth at least g and $\chi(G) \ge k$.

Strong Perfect Graph Thm (06 Chudnovsky-Robertson-Seymour-Thomas)

Given a graph G, every induced subgraph H satisfies $\omega(H) = \chi(H)$ iff G contains no C_k and no $\overline{C_k}$ as induced subgraphs for any odd $k \ge 5$.

Forbidding (infinitely many) induced subgraphs makes a graph perfect.

Definition

A class $\mathcal{C}$ of graphs is χ-*bounded* if there is a function f where $\chi(G) \le f(\omega(G))$ for all $G \in \mathcal{C}$.

What happens when we forbid one induced subgraph?

For which H is the class of H-(induced subgraph)-free graphs χ-bounded?

H cannot contain a cycle!

Conjecture (1975 Gyárfás, 1981 Sumner)

The class of H-free graphs is χ-bounded if and only if H is a forest.

Conjecture (1975 Gyárfás, 1981 Sumner)

The class of H-free graphs is χ-bounded if and only if H is a forest.

1980 Gyárfás–Szemerédi–Tuza: for triangle-free classes and H is a broom
1987 Gyárfás: H is a broom
1990 Kierstead–Penrise and 1993 Sauer: may assume H is a tree
1993 Kierstead–Rődl: for $K_{n,n}$-free classes
1994 Kierstead–Penrise: H is any tree of radius at most 2
2004 Kierstead–Zhu: H is a special tree of radius at most 3

Conjecture (1975 Gyárfás, 1981 Sumner)

The class of H-free graphs is χ-bounded if and only if H is a forest.

1980 Gyárfás–Szemerédi–Tuza: for triangle-free classes and H is a broom
1987 Gyárfás: H is a broom
1990 Kierstead–Penrise and 1993 Sauer: may assume H is a tree
1993 Kierstead–Rődl: for $K_{n,n}$-free classes
1994 Kierstead–Penrise: H is any tree of radius at most 2
2004 Kierstead–Zhu: H is a special tree of radius at most 3

Theorem (1997 Scott)

For any tree T, the class of graphs with no T-subdivision is χ-bounded.

Conjecture (1975 Gyárfás, 1981 Sumner)

The class of H-free graphs is χ-bounded if and only if H is a forest.

1980 Gyárfás–Szemerédi–Tuza: for triangle-free classes and H is a broom
1987 Gyárfás: H is a broom
1990 Kierstead–Penrise and 1993 Sauer: may assume H is a tree
1993 Kierstead–Rődl: for $K_{n,n}$-free classes
1994 Kierstead–Penrise: H is any tree of radius at most 2
2004 Kierstead–Zhu: H is a special tree of radius at most 3

Theorem (1997 Scott)

For any tree T, the class of graphs with no T-subdivision is χ-bounded.

Conjecture (1997 Scott)

For any graph H, the class of graphs with no H-subdivision is χ-bounded.

Conjecture (1975 Gyárfás, 1981 Sumner)

The class of H-free graphs is χ-bounded if and only if H is a forest.

1980 Gyárfás–Szemerédi–Tuza: for triangle-free classes and H is a broom
1987 Gyárfás: H is a broom
1990 Kierstead–Penrise and 1993 Sauer: may assume H is a tree
1993 Kierstead–Rődl: for $K_{n,n}$-free classes
1994 Kierstead–Penrise: H is any tree of radius at most 2
2004 Kierstead–Zhu: H is a special tree of radius at most 3

Theorem (1997 Scott)

For any tree T, the class of graphs with no T-subdivision is χ-bounded.

Conjecture (1997 Scott *DISPROVED!*)

For any graph H, the class of graphs with no H-subdivision is χ-bounded.

Conjecture (1975 Gyárfás, 1981 Sumner)

The class of H-free graphs is χ-bounded if and only if H is a forest.

1980 Gyárfás–Szemerédi–Tuza: for triangle-free classes and H is a broom
1987 Gyárfás: H is a broom
1990 Kierstead–Penrise and 1993 Sauer: may assume H is a tree
1993 Kierstead–Rődl: for $K_{n,n}$-free classes
1994 Kierstead–Penrise: H is any tree of radius at most 2
2004 Kierstead–Zhu: H is a special tree of radius at most 3

Theorem (1997 Scott)

For any tree T, the class of graphs with no T-subdivision is χ-bounded.

Conjecture (1997 Scott *DISPROVED!*)

For any graph H, the class of graphs with no H-subdivision is χ-bounded.

2013 Pawlik–Kozik–Krawczyk–Lasoń–Micek–Trotter–Walczak:
A family of triangle-free intersection graphs of segments in the plane
– with unbounded chromatic number
– NOT containing a subdivision of any 1-planar graph.

Conjecture (1975 Gyárfás, 1981 Sumner)

The class of H-free graphs is χ-bounded if and only if H is a forest.

1980 Gyárfás–Szemerédi–Tuza: for triangle-free classes and H is a broom
1987 Gyárfás: H is a broom
1990 Kierstead–Penrise and 1993 Sauer: may assume H is a tree
1993 Kierstead–Rődl: for $K_{n,n}$-free classes
1994 Kierstead–Penrise: H is any tree of radius at most 2
2004 Kierstead–Zhu: H is a special tree of radius at most 3

Theorem (1997 Scott)

For any tree T, the class of graphs with no T-subdivision is χ-bounded.

Conjecture (1997 Scott *DISPROVED!*)

For any graph H, the class of graphs with no H-subdivision is χ-bounded.

Conjecture (1975 Gyárfás, 1981 Sumner)

The class of H-free graphs is χ-bounded if and only if H is a forest.

1980 Gyárfás–Szemerédi–Tuza: for triangle-free classes and H is a broom
1987 Gyárfás: H is a broom
1990 Kierstead–Penrise and 1993 Sauer: may assume H is a tree
1993 Kierstead–Rődl: for $K_{n,n}$-free classes
1994 Kierstead–Penrise: H is any tree of radius at most 2
2004 Kierstead–Zhu: H is a special tree of radius at most 3

Theorem (1997 Scott)

For any tree T, the class of graphs with no T-subdivision is χ-bounded.

Conjecture (1997 Scott *DISPROVED!*)

For any graph H, the class of graphs with no H-subdivision is χ-bounded.

If no forest is forbidden, then infinitely many graphs must be forbidden.

Conjecture (1975 Gyárfás, 1981 Sumner)

The class of H-free graphs is χ-bounded if and only if H is a forest.

1980 Gyárfás–Szemerédi–Tuza: for triangle-free classes and H is a broom
1987 Gyárfás: H is a broom
1990 Kierstead–Penrise and 1993 Sauer: may assume H is a tree
1993 Kierstead–Rődl: for $K_{n,n}$-free classes
1994 Kierstead–Penrise: H is any tree of radius at most 2
2004 Kierstead–Zhu: H is a special tree of radius at most 3

Theorem (1997 Scott)

For any tree T, the class of graphs with no T-subdivision is χ-bounded.

Conjecture (1997 Scott *DISPROVED!*)

For any graph H, the class of graphs with no H-subdivision is χ-bounded.

If no forest is forbidden, then infinitely many graphs must be forbidden.
Natural to forbid infinitely many cycles!

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *Given k, the class of graphs with no induced cycles of length $\geq k$*
3. *Given k, the class of graphs with no induced cycles of odd length $\geq k$*

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *Given k, the class of graphs with no induced cycles of length $\geq k$*
3. *Given k, the class of graphs with no induced cycles of odd length $\geq k$*

3. implies both 1. and 2.!

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. The class of graphs with no induced cycles of odd length ≥ 5

2. Given k, the class of graphs with no induced cycles of length $\geq k$

3. Given k, the class of graphs with no induced cycles of odd length $\geq k$

3. implies both 1. and 2.!

Partial cases............. no induced cycles of.......

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *Given k, the class of graphs with no induced cycles of length $\geq k$*
3. *Given k, the class of graphs with no induced cycles of odd length $\geq k$*

3. implies both 1. and 2.!

Partial cases............ no induced cycles of.......

2008 Addario-Berry–Chudnovsky–Havet–Reed–Seymour: even length
2013 Bonamy–Charbit–Thomassé: length divisible by 3
2015+ Lagoutte: length 3 and even length ≥ 6
2015+ Chudnovsky–Scott–Seymour: length 3 and odd length ≥ 7
length 5 and length 3 and odd length $\geq k$
length 5 and length $\geq k$

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *Given k, the class of graphs with no induced cycles of length $\geq k$*
3. *Given k, the class of graphs with no induced cycles of odd length $\geq k$*

3. implies both 1. and 2.!

Partial cases............ no induced cycles of.......

2008 Addario-Berry–Chudnovsky–Havet–Reed–Seymour: even length
2013 Bonamy–Charbit–Thomassé: length divisible by 3
2015+ Lagoutte: length 3 and even length ≥ 6
2015+ Chudnovsky–Scott–Seymour: length 3 and odd length ≥ 7
length 5 and length 3 and odd length $\geq k$
length 5 and length $\geq k$
2015+ Scott–Seymour: length 3 and odd length $\geq k$

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. The class of graphs with no induced cycles of odd length ≥ 5

2. Given k, the class of graphs with no induced cycles of length $\geq k$

3. Given k, the class of graphs with no induced cycles of odd length $\geq k$

3. implies both 1. and 2.!

Partial cases............ no induced cycles of......

2008 Addario-Berry–Chudnovsky–Havet–Reed–Seymour: even length
2013 Bonamy–Charbit–Thomassé: length divisible by 3
2015+ Lagoutte: length 3 and even length ≥ 6
2015+ Chudnovsky–Scott–Seymour: length 3 and odd length ≥ 7
length 5 and length 3 and odd length $\geq k$
length 5 and length $\geq k$
2015+ Scott–Seymour: length 3 and odd length $\geq k$

2015+ Scott–Seymour: *CLAIM TO SOLVE 1.!*
2015+ Chudnovsky–Scott–Seymour: *CLAIM TO SOLVE 2.!*

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

(1) subgraph
- deleting vertices/edges

(2) topological minor
- deleting vertices/edges
- smoothing vertices

(3) minor
- deleting vertices/edges
- contracting edges

[1] induced subgraph
- deleting vertices

[2] pivot-minor
- deleting vertices
- pivoting edges

[3] vertex-minor
- deleting vertices
- local complementations at vertices

More operations imply easier to get the structure!
No H-vertex-minor implies no H-pivot-minor implies H-free.

Bipartite graph

Distance-hereditary

Parity graph

Circle graph

Bipartite graph: vertex set has a partition into two independent sets

Distance-hereditary: distances are preserved in every induced subgraph

Parity graph: shortest paths joining a pair of vertices have the same parity

Circle graph: intersection graph of chords on a circle

Bipartite graph: vertex set has a partition into two independent sets

no C_3-pivot-minor

Distance-hereditary: distances are preserved in every induced subgraph

no C_5-vertex-minor (1987, 1988 Bouchet)

no C_5, C_6-pivot-minors (1986 Bandelt–Mulder)

Parity graph: shortest paths joining a pair of vertices have the same parity

no C_5-pivot-minor (1984 Burlet–Uhry)

Circle graph: intersection graph of chords on a circle

three forbidden vertex-minors (1994 Bouchet)

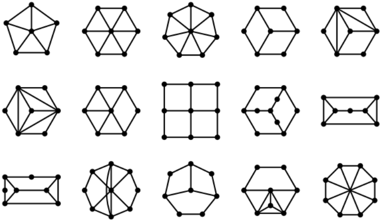
fifteen forbidden pivot-minors (2009 Geelen–Oum)

Bipartite graph: vertex set has a partition into two independent sets
no C_3-pivot-minor

Distance-hereditary: distances are preserved in every induced subgraph
no C_5-vertex-minor (1987, 1988 Bouchet)
no C_5, C_6-pivot-minors (1986 Bandelt–Mulder)

Parity graph: shortest paths joining a pair of vertices have the same parity
no C_5-pivot-minor (1984 Burlet–Uhry)

Circle graph: intersection graph of chords on a circle
three forbidden vertex-minors (1994 Bouchet)
fifteen forbidden pivot-minors (2009 Geelen–Oum)



Bipartite graph: vertex set has a partition into two independent sets

no C_3-pivot-minor

Distance-hereditary: distances are preserved in every induced subgraph

no C_5-vertex-minor (1987, 1988 Bouchet)

no C_5, C_6-pivot-minors (1986 Bandelt–Mulder)

Parity graph: shortest paths joining a pair of vertices have the same parity

no C_5-pivot-minor (1984 Burlet–Uhry)

Circle graph: intersection graph of chords on a circle

three forbidden vertex-minors (1994 Bouchet)

fifteen forbidden pivot-minors (2009 Geelen–Oum)

Bipartite graph: vertex set has a partition into two independent sets

no C_3-pivot-minor

Distance-hereditary: distances are preserved in every induced subgraph

no C_5-vertex-minor (1987, 1988 Bouchet)

no C_5, C_6-pivot-minors (1986 Bandelt–Mulder)

Parity graph: shortest paths joining a pair of vertices have the same parity

no C_5-pivot-minor (1984 Burlet–Uhry)

Circle graph: intersection graph of chords on a circle

three forbidden vertex-minors (1994 Bouchet)

fifteen forbidden pivot-minors (2009 Geelen–Oum)

Bipartite, distance-hereditary, parity graphs are perfect, thus χ-bounded.

Circle graphs are χ-bounded. (1997 Kostochka–Kratochvíl)

Bipartite graph: vertex set has a partition into two independent sets
no C_3-pivot-minor

Distance-hereditary: distances are preserved in every induced subgraph
no C_5-vertex-minor (1987, 1988 Bouchet)
no C_5, C_6-pivot-minors (1986 Bandelt–Mulder)

Parity graph: shortest paths joining a pair of vertices have the same parity
no C_5-pivot-minor (1984 Burlet–Uhry)

Circle graph: intersection graph of chords on a circle
three forbidden vertex-minors (1994 Bouchet)
fifteen forbidden pivot-minors (2009 Geelen–Oum)

Bipartite, distance-hereditary, parity graphs are perfect, thus χ-bounded.
Circle graphs are χ-bounded. (1997 Kostochka–Kratochvíl)

Conjecture (Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

Conjecture (Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

Conjecture (Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

2012 Dvořák–Král': H is W_5

Conjecture (Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

2012 Dvořák–Král': H is W_5

2015+ Chudnovsky–Scott–Seymour: H is a cycle

Conjecture (Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

2012 Dvořák–Král': H is W_5
2015+ Chudnovsky–Scott–Seymour: H is a cycle
(The class of graphs with no induced cycles of long length is χ-bounded)

Conjecture (Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

2012 Dvořák–Král': H is W_5
2015+ Chudnovsky–Scott–Seymour: H is a cycle
(The class of graphs with no induced cycles of long length is χ-bounded)

Theorem (2015+ C.–Kwon–Oum)

For any fan F, the class of graphs with no F-vertex-minor is χ-bounded.

Conjecture (Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

2012 Dvořák–Kráľ: H is W_5
2015+ Chudnovsky–Scott–Seymour: H is a cycle
(The class of graphs with no induced cycles of long length is χ-bounded)

Theorem (2015+ C.–Kwon–Oum)

For any fan F, the class of graphs with no F-vertex-minor is χ-bounded.

Theorem (2015+ C.–Kwon–Oum)

For any cycle C, the class of graphs with no C-pivot-minor is χ-bounded.

Conjecture (Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

2012 Dvořák–Král': H is W_5
2015+ Chudnovsky–Scott–Seymour: H is a cycle
(The class of graphs with no induced cycles of long length is χ-bounded)

Theorem (2015+ C.–Kwon–Oum)

For any fan F, the class of graphs with no F-vertex-minor is χ-bounded.

Theorem (2015+ C.–Kwon–Oum)

For any cycle C, the class of graphs with no C-pivot-minor is χ-bounded.

Question (2015+ C.–Kwon–Oum)

For any H, is the class of graphs with no H-pivot-minor χ-bounded?

No H-vertex-minor implies no H-pivot-minor implies H-free.

No H-vertex-minor implies no H-pivot-minor implies H-free.

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *The class of graphs with no induced cycles of length $\geq k$*
3. *The class of graphs with no induced cycles of odd length $\geq k$*

Conjecture (1995 Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

No H-vertex-minor implies no H-pivot-minor implies H-free.

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *The class of graphs with no induced cycles of length $\geq k$*
3. *The class of graphs with no induced cycles of odd length $\geq k$*

Question (2015+ C.–Kwon–Oum)

For any H, is the class of graphs with no H-pivot-minor χ-bounded?

Conjecture (1995 Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

Is there time to do sketch of proof.................?

Is there time to do sketch of proof.................?

Theorem (2015+ C.–Kwon–Oum)

For any fan F, class of graphs with no F-vertex-minor is χ-bounded.

Is there time to do sketch of proof.................?

Theorem (2015+ C.–Kwon–Oum)

For any fan F, class of graphs with no F-vertex-minor is χ-bounded.
For any k, every graph G with no F_k-vertex-minor satisfies

$$\chi(G) \leq 2(\omega(G)-1)^{g_3[k,2g_2(k)\{g_5(2g_2(k)-1,g_4(k)-2)-1\}+1]-1}$$

Is there time to do sketch of proof.................?

Theorem (2015+ C.–Kwon–Oum)

For any fan F, class of graphs with no F-vertex-minor is χ-bounded.
For any k, every graph G with no F_k-vertex-minor satisfies

$$\chi(G) \leq 2(\omega(G)-1)^{g_3[k,2g_2(k)\{g_5(2g_2(k)-1,g_4(k)-2)-1\}+1]-1}$$

For positive integers k and ℓ:

- $g_1(k,\ell) := R\left(3(R(k,k)+k-1)((2^{k-1}-1)^2k^2+1), 2\ell+1\right)$
- $h(1) := 2$ and $h(i+1) := g_1(k, h(i))$ for all $i \geq 1$
- $g_2(k) := h(2^{k-1}+1)$
- $g_3(k,\ell) := k^{k^{2\ell+1}-1} - 1$
- $g_4(k) := (2^k-1)^2+1$
- $g_5(k,\ell) := \frac{k}{k-2}(k-1)^\ell$

Is there time to do sketch of proof.................?

Theorem (2015+ C.–Kwon–Oum)

For any fan F, class of graphs with no F-vertex-minor is χ-bounded.
For any k, every graph G with no F_k-vertex-minor satisfies

$$\chi(G) \leq 2(\omega(G)-1)^{g_3[k,2g_2(k)\{g_5(2g_2(k)-1,g_4(k)-2)-1\}+1]-1}$$

For positive integers k and ℓ:

- $g_1(k,\ell) := R\left(3(R(k,k)+k-1)((2^{k-1}-1)^2k^2+1), 2\ell+1\right)$
- $h(1) := 2$ and $h(i+1) := g_1(k, h(i))$ for all $i \geq 1$
- $g_2(k) := h(2^{k-1}+1)$
- $g_3(k,\ell) := k^{k^{2\ell+1}-1} - 1$
- $g_4(k) := (2^k-1)^2+1$
- $g_5(k,\ell) := \frac{k}{k-2}(k-1)^\ell$

Proof idea:
(1) if $\chi(G)$ is large, then G contains a ladder graph as a vertex-minor
(2) ladder graph contains F_k-vertex-minor

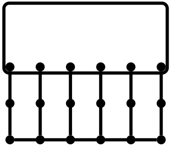
Theorem (2015+ C.–Kwon–Oum)

For any fan F, class of graphs with no F-vertex-minor is χ-bounded.
For any k, every graph G with no F_k-vertex-minor satisfies

$$\chi(G) \leq 2(\omega(G)-1)^{g_3[k,2g_2(k)\{g_5(2g_2(k)-1,g_4(k)-2)-1\}+1]-1}$$

Proposition

Let k be a positive integer and let $\ell \geq R(k,k)^{4k^2-1}+1$. Let H be a connected graph with at least ℓ vertices. If $v_1,\dots,v_\ell$ are pairwise distinct vertices of H and $w_1,\dots,w_\ell$ are the ℓ leaves of E_ℓ, then the graph obtained by identifying v_i and w_i for each $i \in \{1,\dots,\ell\}$ results in a graph that contains a vertex-minor isomorphic to F_k.



Theorem (2015+ C.–Kwon–Oum)

For any cycle C, class of graphs with no C-pivot-minor is χ-bounded.

Theorem (2015+ C.–Kwon–Oum)

For any cycle C, class of graphs with no C-pivot-minor is χ-bounded.
For any k, every graph G with no C_k-pivot-minor satisfies

$$\chi(G) \leq 2(\omega(G)-1)^{2(k-2)(12k^2+k-3)+1}$$

Theorem (2015+ C.–Kwon–Oum)

For any cycle C, class of graphs with no C-pivot-minor is χ-bounded.
For any k, every graph G with no C_k-pivot-minor satisfies

$$\chi(G) \leq 2(\omega(G) - 1)^{2(k-2)(12k^2+k-3)+1}$$

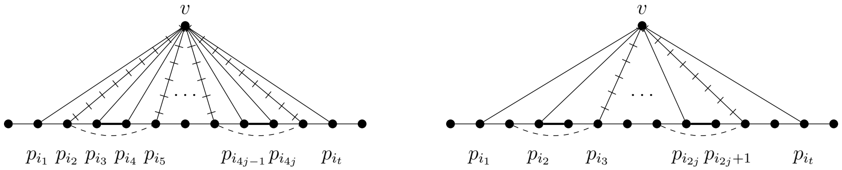
Proof idea:
(1) if $\chi(G)$ is large, then G contains an incomplete fan as a pivot-minor
(2) an incomplete fan contains C_k-pivot-minor

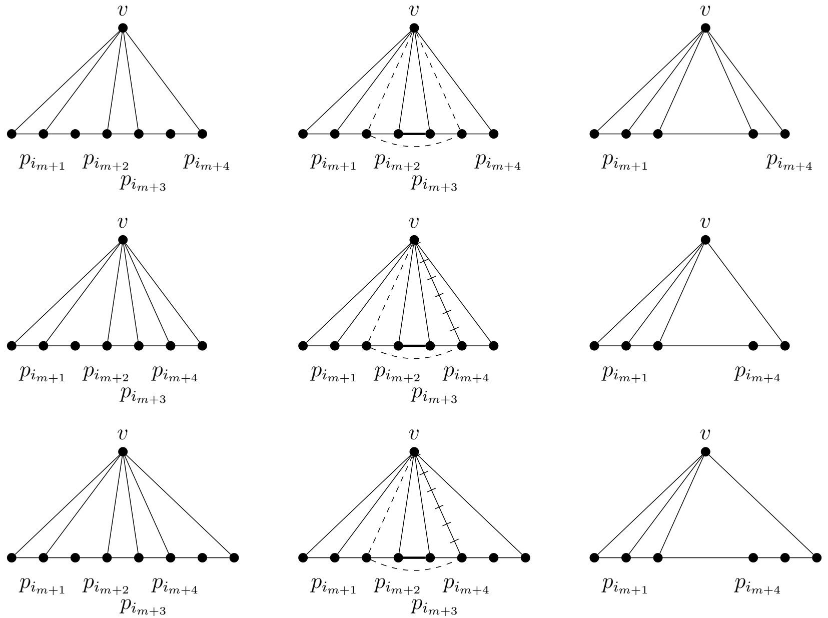
Theorem (2015+ C.–Kwon–Oum)

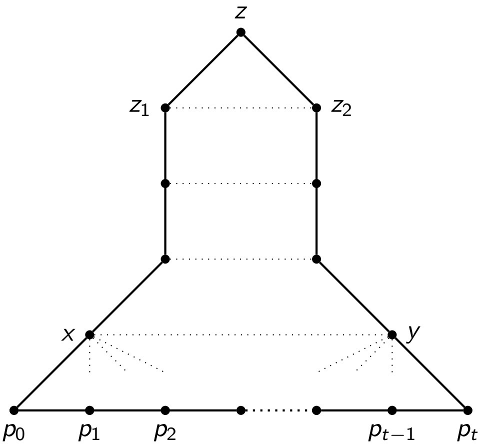
For any cycle C, class of graphs with no C-pivot-minor is χ-bounded.
For any k, every graph G with no C_k-pivot-minor satisfies

$$\chi(G) \leq 2(\omega(G)-1)^{2(k-2)(12k^2+k-3)+1}$$

Proof idea:
(1) if $\chi(G)$ is large, then G contains an incomplete fan as a pivot-minor
(2) an incomplete fan contains C_k-pivot-minor




v
$p_{i_{m+1}}$
$p_{i_{m+2}}$
$p_{i_{m+3}}$
$p_{i_{m+4}}$


z
z_1
z_2
x
y
p_0
p_1
p_2
p_{t-1}
p_t

No H-vertex-minor implies no H-pivot-minor implies H-free.

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *The class of graphs with no induced cycles of length $\geq k$*
3. *The class of graphs with no induced cycles of odd length $\geq k$*

Question (2015+ C.–Kwon–Oum)

For any H, is the class of graphs with no H-pivot-minor χ-bounded?

Conjecture (1995 Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

Thank you for your attention!