# Characterization of Cycle Obstructions for Improper Coloring Planar Graphs

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Joint work with

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What about 3-colorable?!

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**Theorem (2011 Borodin–Glebov, 2014 Dvořák)**

*Planar graphs with $d^3 \geq 2$ and no $C_5$ are 3-colorable.*
*Planar graphs with $d^{4^-} \geq 26$ are 3-colorable.*

Guaranteeing 3-colorings of planar graphs via forbidding cycle lengths

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**Question (Erdös)**

*Find the min $k$ where planar graphs with no $C_4, \ldots, C_k$ are 3-colorable.*

There is a planar graph with no $C_4$ that is NOT 3-colorable.

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**Question**

*Are planar graphs with no $C_4$, $C_5$, $C_6$ indeed 3-colorable?*

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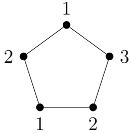

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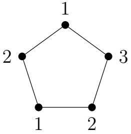


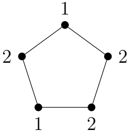

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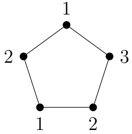

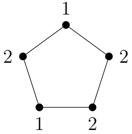


A class $\mathcal{C}$ of graphs is balanced $k$-partitionable,
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A class $\mathcal{C}$ of graphs is unbalanced $k$-partitionable,
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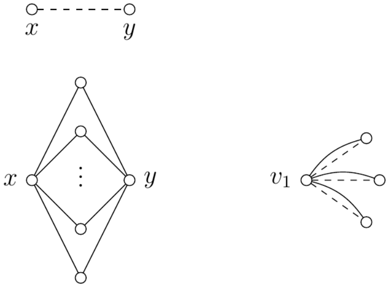

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**Theorem (2000 Škrekovski)**

*Planar graphs with girth at least 5 are balanced 2-partitionable.*

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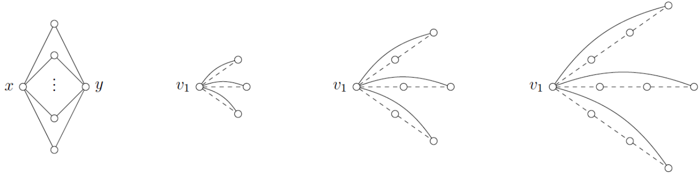

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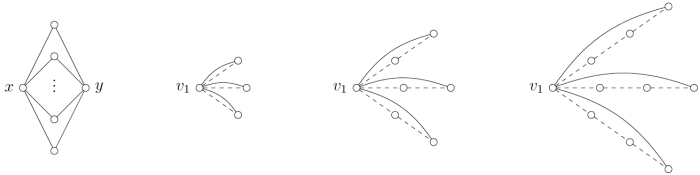


4 | 4 and 3 | 4 and 5 | 4 and 7

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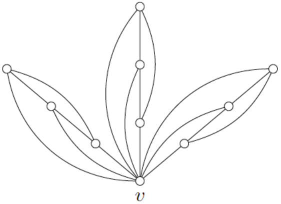


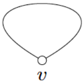


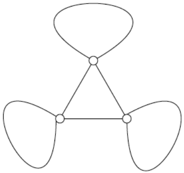

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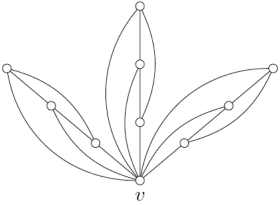


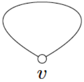


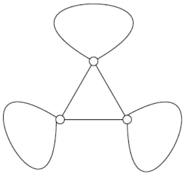

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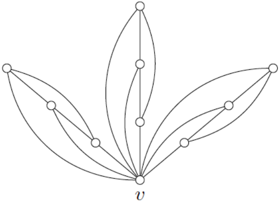


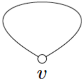


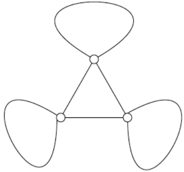

3 and 4

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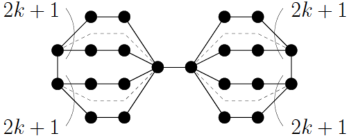

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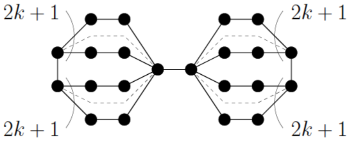


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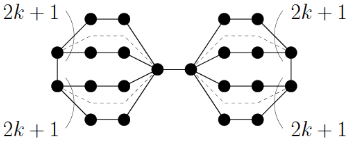


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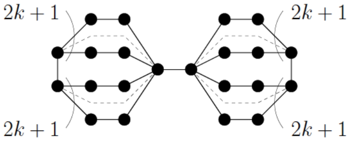


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# Thank you for your attention!