Choosability of Toroidal Graphs with Forbidden Structures

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August 6, 2014

A list assignment L assigns each vertex v a list $L(v)$ of available colors.

An L-coloring is a function f on $V(G)$ where

- for each vertex v: $f(v) \in L(v)$
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A graph G is k-choosable if there is an L-coloring for each L where
- for each vertex v: $|L(v)| \geq k$.

The list chromatic number or choosability $\chi_l(G)$ is the minimum such k.

A graph G is k-colorable if there is an L-coloring where
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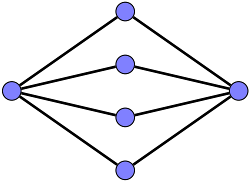

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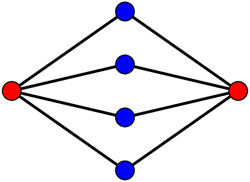

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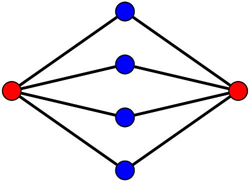

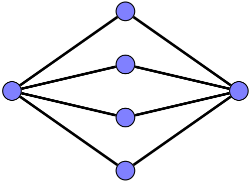

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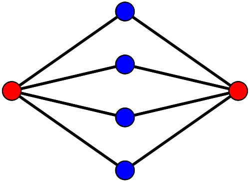

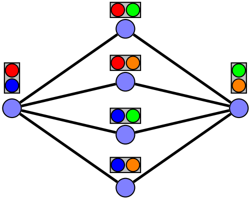

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Planar graphs are 5-choosable, and not all planar graphs are 4-choosable.

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Planar graphs with no k-cycle for some $k \in \{3, 4, 5, 6, 7\}$ are 4-choosable.

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Guaranteeing 3-choosability

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Question (Borodin 1996)

Are planar graphs without cycles of length 4 to 8 indeed 3-choosable?

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For a graph G on the Klein Bottle, $\chi(G) \leq H(2) - 1 = 6$.

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Theorem (Dirac 1956, 1957, Albertson–Hutchinson 1979)

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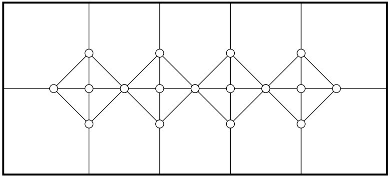

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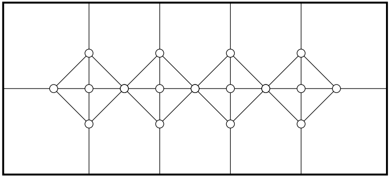

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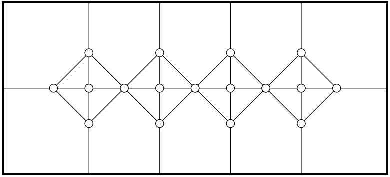

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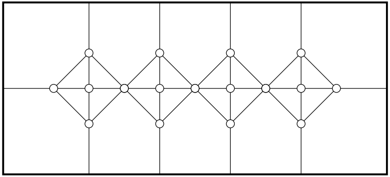

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For a projective plane graph G: if $C_6 \not\subseteq G$ and $K_5 \not\subseteq G$, then $\chi_\ell(G) \leq 4$?
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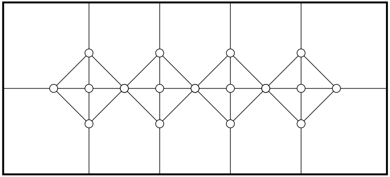

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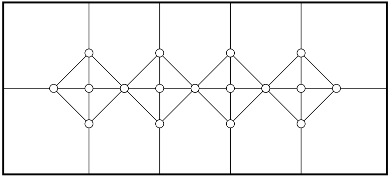

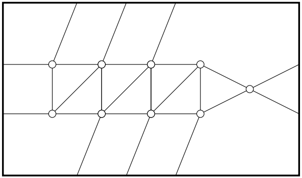

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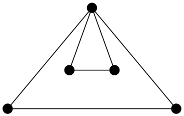

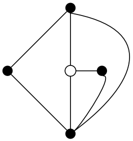

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Thank you for your attention!