Avoiding Large Squares in Partial Words

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Outline

Words and Partial Words

Alphabet: Σ – a non-empty finite set of symbols, or *letters*
Hole: $\diamond$ – a character outside the alphabet that is viewed as representing any letter of the alphabet.

A *word* is a concatenation of symbols from Σ
A *partial word* is a concatenation of symbols from $\Sigma_\diamond = \Sigma \cup \{\diamond\}$.
Σ^* denotes the set of words over Σ

Length of w: $|w|$ – the number of symbols in w

Example

For example, in the binary alphabet $\{a, b\}$,

$$\begin{aligned} \diamond &: \quad a, b \\ \diamond\diamond &: \quad aa, ab, ba, bb \\ bb\diamond a\diamond &: \quad bbaaa, bbaab, bbbaa, bbbab \\ abaa &: \quad abaa \end{aligned}$$

Since each $\diamond$ may represent any letter in the alphabet, a partial word with h holes *represents* $|\Sigma|^h$ words.

Note that a (full) word is a concatenation of letters from $\Sigma_\diamond$ with no holes.

Compatibility

For two partial words u, v, where $|u| = |v|$, u is *compatible* with v ($u \uparrow v$), if u and v agree at each position where neither has a hole.

In other words, u and v could potentially represent the same word.

For example,

$ab\diamond\diamond$	$a\diamond b\diamond$	$ab\diamond\diamond$
$a\diamond b\diamond$	$aab\diamond$	$aab\diamond$

$ab\diamond\diamond \uparrow a\diamond b\diamond$ $\quad$ $a\diamond b\diamond \uparrow aab\diamond$ $\quad$ $ab\diamond\diamond \not\uparrow aab\diamond$

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$a\diamond b\diamond$	$aab\diamond$	$aab\diamond$
$ab\diamond\diamond \uparrow a\diamond b\diamond$	$a\diamond b\diamond \uparrow aab\diamond$	$ab\diamond\diamond \not\uparrow aab\diamond$

Square Word

A word of the form $uu = u^2$ is a *square (word)* if it is nonempty.
A word of the form $vvv = v^3$ is a *cube (word)* if it is nonempty.

A partial word W *has* a square S if S is compatible with some consecutive subword of W.

For example, over $\{a, b, n, s, t\}$,

$bananas$ $\qquad$ $aaaaa$ $\qquad$ $banan\diamond s$

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For example, over $\{a, b, n, s, t\}$,

$bananas$	$aaaaa$	$banan\diamond s$
$anan = (an)^2$ $nana = (na)^2$	$aa = a^2$ $aaaa = (aa)^2$	$anan = (an)^2$ $nana = (na)^2$ $nn = n^2$ (from $n\diamond$)

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$anan = (an)^2$	$aa = a^2$	$anan = (an)^2$
$nana = (na)^2$	$aaaa = (aa)^2$	$nana = (na)^2$
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		$ss = s^2$ (from $\diamond s$)

Morphism

A *morphism*, $\varphi : A^* \to \Sigma^*$, is a function such that if $u = a_0 a_1 a_2 \cdots$ (where $a_i \in A$ for all i), $\varphi(u) = \varphi(a_0)\varphi(a_1)\varphi(a_2)\cdots$.

For example, let

$$\begin{array}{llll} \varphi : & \{a,b,c\}^* & \to & \{d,e\}^* \\ & a & \mapsto & d \\ & b & \mapsto & ee \\ & c & \mapsto & de \end{array}$$

Then $\varphi(abc) = \varphi(a)\varphi(b)\varphi(c) = deede$.

Note that the domain of the morphism is typically a (full) word.

A morphism, $\varphi : A^* \to \Sigma^*$, is *$n$-uniform* if $|\varphi(a)| = n$ for all $a \in A$.
A morphism, $\varphi : A^* \to \Sigma^*$, is *iterative* if $A = \Sigma$.

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ab

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aba

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$abab$

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In 1995, Fraenkel and Simpson give an infinite binary word that has only three squares a^2, b^2, $(ab)^2$.

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In 2007, Manea and Mercaş introduce the concept of squares in partial words.

In 2009, we investigate the avoidance of squares in both partial words and cube-free partial words.

Goal

Let $g(h, s)$ denote the maximum length of a binary partial word with h holes that has at most s distinct squares.

The goal is to fill up the table below.

$s \backslash h$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							

Simple Results for Words (Franekel and Simpson)

$g(0,1) = 7$ (*aaabaaa*, *abaaaba*, *abbbabb*, their complements, reverses).

$g(0,2) = 18$ (*abaabbaaabbbaabbab*, *babbaabbbaaabbaaba*).

$g(0,s) = \infty$ for all integers $s \geq 3$.

Simple Results for Partial Words

one hole :

$g(1,0)=1$ ($\diamond$), $g(1,1)=5$ ($\diamond aaba$), $g(1,2)=16$ ($\diamond abbaaabbbaabbab$)

two holes:

$g(2,0)=0$, $g(2,1)=3$ ($\diamond a\diamond$), $g(2,2)=14$ ($\diamond abbaaabbbaab\diamond$)

three holes:

$g(3,0)=0$, $g(3,1)=0$, $g(3,2)=9$ ($\diamond abb\diamond aab\diamond$)

The *only* words valid for $g(3,2)$ are
$\diamond\diamond\diamond$, $\diamond a\diamond b\diamond$, $\diamond a\diamond bba\diamond$, $\diamond abb\diamond aab\diamond$, their reverses and complements.

Moreover, for four holes there exist no partial word that has fewer than three squares.

Table so far

$s \backslash h$	0	1	2	3	4	5	6
0	3	1	0	0	0		
1	7	5	3	0	0		
2	18	16	14	9	0		
3	∞						
4	...						
5	...						

Rampersad, Shallit, and Wang's Morphisms, 1

5-letter 24-uniform morphism α:

$$\begin{array}{lcl} a & \mapsto & abcdcbabcdeabcbabcdcbcde \\ b & \mapsto & abcbabcdedcdeabcdedcbcde \\ c & \mapsto & abcbabcdcbcdeabcdcbabcde \\ d & \mapsto & abcdcbcdedcdeabcdcbabcde \\ e & \mapsto & abcdcbcdeabcbabcdedcbcde \end{array}$$

6-uniform morphism β:

$$\begin{array}{lcl} a & \mapsto & abbbaa \\ b & \mapsto & babbaa \\ c & \mapsto & bbbaaa \\ d & \mapsto & bbaaba \\ e & \mapsto & bbaaab \end{array}$$

Rampersad, Shallit, and Wang's Morphisms, 1

5-letter 24-uniform morphism α:

$$\begin{array}{lcl}
a & \mapsto & \color{red}{abc}dcbabcdeabcbabcdc\color{red}{bcde} \\
b & \mapsto & \color{red}{abc}babcdedcdeabcdedc\color{red}{bcde} \\
c & \mapsto & \color{red}{abc}babcdcbcdeabcdcba\color{red}{bcde} \\
d & \mapsto & \color{red}{abc}dcbcdedcdeabcdcba\color{red}{bcde} \\
e & \mapsto & \color{red}{abc}dcbcdeabcbabcdedc\color{red}{bcde}
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\end{array}$$

No Holes

Lemma (Rampersad, Shallit, Wang)

If $u \in \{a, b, c, d, e\}^*$ is square-free and avoids the patterns ac, ad, ae, be, ca, da, eb, then $\alpha(u)$ is square-free and avoids the patterns ac, ad, ae, bd, be, ca, ce, da, db, eb, ec, aba, ede.

Lemma (Rampersad, Shallit, Wang)

If $u \in \{a, b, c, d, e\}^*$ is square-free and avoids the patterns ac, ad, ae, bd, be, ca, ce, da, db, eb, ec, $edeaba$, then the only squares in $\beta(u)$ are aa, bb, $abab$.

Theorem (Rampersad, Shallit, Wang)

There exists an infinite binary word, $w_0 = \beta(\alpha^\omega(a))$, that has only the three squares: a^2, b^2, $(ab)^2$.

One and Two Holes

Theorem (B.-S., C., M. (2011))

There exists an infinite binary partial word with one hole, $w_1 = \diamond\beta(\alpha^\omega(a))$, that has only the three squares: a^2, b^2, $(ab)^2$.

Theorem (B.-S., C., M. (2011))

There exists an infinite binary partial word with two holes, $w_2{=}\diamond\diamond\beta(\alpha^\omega(a))$, that has only the three squares: a^2, b^2, $(ab)^2$.

Proof of One Hole

Suppose not. Then, the 4th square must be compatible with $\diamond u a_0 u'$, where $\diamond u a_0 u' a_1$ is a prefix of $w_1 {=} \diamond\beta(\alpha^\omega(a))$ where $u{=}u' {\in} \{a, b\}^*$ and $a_0 {\neq} a_1$ are letters. Note that $u = u'$ has prefix $\beta(ab) = abbbaababbaa$. Let $|u| {=} 6q + r$.

6-uniform morphism β:

$$\begin{array}{lll} a & \mapsto & abbbaa \\ b & \mapsto & babbaa \\ c & \mapsto & bbbaaa \\ d & \mapsto & bbaaba \\ e & \mapsto & bbaaab \end{array}$$

If $r{=}0$, then a_0abbb is a prefix of $\beta(f)$ for some f
If $r{=}1$, then a_0abbb is a suffix of $\beta(f)$ for some f
If $r{=}2$, then a_0abb is a suffix of $\beta(f)$ for some f
If $r{=}4$, then $bbbaab$ is $\beta(f)$ for some f

If $r{=}5$, since all images of β have unique 5-letter prefixes, $a_0 = a_1$.

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If $r = 3$, then there are letters f, g, h where

$$\begin{array}{lcr} \beta(f) \text{ has suffix } a_0 ab & \Rightarrow & f = e \\ \beta(g) = bbaaba & \Rightarrow & g = d \\ \beta(h) \text{ has prefix } bbaa & \Rightarrow & h \in \{d, e\} \end{array}$$

If $h = d$, $\alpha^\omega(a)$ has square dd
If $h = e$, then $\alpha^\omega(a)$ has ede

Three and Four Holes

Proposition

For large m, $w_3 = \beta(\alpha^m(a))\diamond$ *has only the three squares:* a^2, b^2, $(ab)^2$.

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For large m, $w_4 = \beta(\alpha^m(a))\diamond\diamond$ *has only three squares:* a^2, b^2, $(ab)^2$.

Theorem (B.-S., C., M. (2011))

There exist an arbitrarily long finite partial word with four holes, $\diamond\diamond\beta(\alpha^m(a))\diamond\diamond$*, that has only the three squares:* a^2, b^2, $(ab)^2$.

Table so far

$s \backslash h$	0	1	2	3	4	5	6
0	3	1	0	0	0		
1	7	5	3	0	0		
2	18	16	14	9	0		
3	∞	∞	∞	$<\infty$?	$<\infty$?		
4	$\cdots$	$\cdots$	$\cdots$				
5	$\cdots$	$\cdots$	$\cdots$				

Results

We found that all binary partial words of the form $u_0 \diamond u_1$ with $|u_0| = |u_1| = 8$ have at least four distinct squares.

Thus, all partial words with at most three distinct squares and at least three holes must have the holes placed within the first, or last, eight positions.

Also, we found that all partial words with at least three holes within the first eight symbols that have no more than three distinct squares have maximum length 13.

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Also, we found that all partial words with at least three holes within the first eight symbols that have no more than three distinct squares have maximum length 13.

Theorem (B.-S., C., M., (2011))

All infinite partial words with at least three holes have at least four distinct squares.

Theorem (B.-S., C., M., (2011))

All partial words with more than four holes have at least four distinct squares.

Table so far

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0	3	1	0	0	0	0	$\cdots$
1	7	5	3	0	0	0	$\cdots$
2	18	16	14	9	0	0	$\cdots$
3	∞	∞	∞	$<\infty$	$<\infty$	0	$\cdots$
4	$\cdots$	$\cdots$	$\cdots$				
5	$\cdots$	$\cdots$	$\cdots$				

Modification of the Morphism

5-letter 24-uniform morphism α:

$$\begin{array}{lcl} a & \mapsto & abcdcbabcdeabcbabcdcbcde \\ b & \mapsto & abcbabcdedcdeabcdedcbcde \\ c & \mapsto & abcbabcdcbcdeabcdcbabcde \\ d & \mapsto & abcdcbcdedcdeabcdcbabcde \\ e & \mapsto & abcdcbcdeabcbabcdedcbcde \end{array}$$

6-uniform morphism β:

$$\begin{array}{lcl} a & \mapsto & abbbaa \\ b & \mapsto & babbaa \\ c & \mapsto & bbbaaa \\ d & \mapsto & bbaaba \\ e & \mapsto & bbaaab \end{array}$$

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6-uniform morphism β:

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6-uniform morphism $\hat{\beta}$:

$$\begin{array}{lcl} a & \mapsto & \diamond bbbaa \\ b & \mapsto & babbaa \\ c & \mapsto & bbbaaa \\ d & \mapsto & bbaaba \\ e & \mapsto & bbaaab \end{array}$$

Infinitely Many Holes

Theorem (B.-S., C., M. (2011))

There exists an infinite binary partial word with infinitely many holes, $w_5 = \hat{\beta}(\alpha^\omega(a))$, that has only the four squares: a^2, b^2, $(ab)^2$, $(bb)^2$.

Complete Table

Note that $s \geq 4$, $g(h, s) = \infty$ for all non-negative integers h. Also, note that when $h \geq 5$, $g(h, s) = 0$ for all non-negative integers $s \leq 3$.

$s \backslash h$	0	1	2	3	4	5	6
0	3	1	0	0	0	0	$\cdots$
1	7	5	3	0	0	0	$\cdots$
2	18	16	14	9	0	0	$\cdots$
3	∞	∞	∞	$<\infty$	$<\infty$	0	$\cdots$
4	$\cdots$	$\cdots$	$\cdots$	∞	∞	∞	$\cdots$
5	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Ternary Alphabet

What if we increase the size of the alphabet? Is it possible to get fewer squares?

Theorem (Blanchet-Sadri, Mercaş, Scott (2009))

There exist infinitely many infinite partial words with infinitely many holes over a three-letter alphabet that have at most two squares.

Is it possible to construct a partial word with one hole that has only one square?

One Hole

The Hall morphism

$$\begin{array}{rcl} \phi : \{a, b, c\}^* & \to & \{a, b, c\}^* \\ a & \mapsto & abc \\ b & \mapsto & ac \\ c & \mapsto & b \end{array}$$

Theorem

There exists an infinite square-free ternary word, $\phi^\omega(a)$.

Proposition (B.-S., C., M. (2011))

There exists an infinite partial word with one hole, $\diamond\phi^\omega(a)$, that has only one square.

History

In 1974, Entringer, Jackson, and Schatz conjecture that an infinite cube-free binary word must have an arbitrarily long square.

In 1976, Dekking shows that there exists an infinite cube-free binary word that avoids all squares xx such that $|x| \geq 4$. Dekking also proves that the bound 4 is best possible.

Note that a word that has $(aa)^2$, $(bb)^2$, $(aaa)^2$, or $(bbb)^2$ is not cubefree.

Proposition

All cubefree binary words of length greater than 113 *avoiding squares* xx *such that* $|x| \geq 4$ *must have all of the following ten squares:* a^2, b^2, $(ab)^2$, $(ba)^2$, $(aab)^2$, $(aba)^2$, $(abb)^2$, $(baa)^2$, $(bab)^2$, *and* $(bba)^2$.

Rampersad, Shallit, and Wang's Morphisms, 2

The 10-uniform morphism γ:

$$\begin{array}{rcl} a & \mapsto & adbacabacd \\ b & \mapsto & adbacdabac \\ c & \mapsto & acabadbacd \\ d & \mapsto & acadabacab \end{array}$$

The 6-uniform morphism δ:

$$\begin{array}{rcl} a & \mapsto & abaabb \\ b & \mapsto & ababba \\ c & \mapsto & abbaab \\ d & \mapsto & abbaba \end{array}$$

Rampersad, Shallit, and Wang's Morphisms, 2

The 10-uniform morphism γ:

$$\begin{array}{lcl} a & \mapsto & \color{red}{a}dbacabacd \\ b & \mapsto & \color{red}{a}dbacdabac \\ c & \mapsto & \color{red}{a}cabadbacd \\ d & \mapsto & \color{red}{a}cadabacab \end{array}$$

The 6-uniform morphism δ:

$$\begin{array}{lcl} a & \mapsto & abaabb \\ b & \mapsto & ababba \\ c & \mapsto & abbaab \\ d & \mapsto & abbaba \end{array}$$

One and Two Holes

Theorem (B.-S., C., M. (2011))

There exists an infinite binary cube-free partial word with one hole, $v_1 = \diamond(\delta(a))^{-1}v_0$, that has only ten distinct squares.

$$abaabbabbabaababbaabaabbabbaababaabbababbaabaabbabbaababb\cdots$$

Theorem (B.-S., C., M. (2011))

There exists an infinite binary cube-free partial word with two holes, $v_2 = \diamond baabv_1$, that has only ten distinct squares.

One and Two Holes

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$abaabbabbabaababbaabaabbabbaababaabbababbaabaabbabbaababb\cdots$

Theorem (B.-S., C., M. (2011))

There exists an infinite binary cube-free partial word with two holes, $v_2 = \diamond baabv_1$, that has only ten distinct squares.

One and Two Holes

Theorem (B.-S., C., M. (2011))

There exists an infinite binary cube-free partial word with one hole, $v_1 = \diamond(\delta(a))^{-1}v_0$, that has only ten distinct squares.

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Results

We found that all cube-free binary partial words of the form $u_0 \diamond u_1$ with $|u_0| = |u_1| = 9$ have at least eleven distinct squares. This implies that all cube-free binary partial words that have at most ten distinct squares must have all the holes in the first or last nine positions.

It is easy to check that it is impossible to avoid a cube in a partial word of length 11 with three holes in the first nine positions.

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Proposition (B.-S., C., M., (2011))

All infinite cube-free binary partial words with at least three holes have at least eleven distinct squares.

Proposition (B.-S., C., M., (2011))

All cube-free binary partial words with more than four holes have at least eleven distinct squares.

Four Holes

Proposition

For large m, $v_3 = \delta(\gamma^{3m+1}(a)(acd)^{-1})\diamond baab\diamond$*, an arbitrarily long finite cube-free binary partial word with two holes, has only ten squares.*

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For large m, $v_4 = \diamond baab\diamond\delta(a^{-1}\gamma^{3m+1}(a)(acd)^{-1})\diamond baab\diamond$*, an arbitraily long finite cube-free binary partial word with four holes, has only ten squares.*

Results

What happens when we have more than four holes?

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In 1976, in the context of repetition threshold, Dejean gives an iterative morphism over a three letter alphabet that preserves $\frac{7}{4}^+$ freeness.

In 2006, Ochem gives a non-iterative morphism τ that, for any $\frac{7}{4}^+$-free word $w \in \{a, b, c\}^*$, $\tau(w) \in \{a, b\}^*$ is $\frac{5}{2}^+$-free, $\frac{7}{3}^+$-free for words of length 3 and $\frac{823}{412}^+$-free for words of length 4.

Modification of Ochem's Morphism

We replace an occurrence of an a with $\diamond$ in $\tau(b)$ where τ is the original morphism of Ochem, and get the 103-uniform morphism $\hat{\tau}$:

$$\begin{array}{lcl}
a & \mapsto & aabaabbabaababbaabaabbabbaababaabbababbaabaabbabbaababba \\
 & & \qquad baabbabbaabaabbababbaababaabbabbaababbabaabbabb \\
b & \mapsto & aabaabbabaababbaabaabbabbaababaabbababbaabaabbabbaababba \\
 & & \qquad baabbabbaabaabbabaababbaabaabbababbaababaabbabb \\
c & \mapsto & aabaabbabaababbaabaabbabbaababaabbababbaabaabbabaababbaa \\
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\end{aligned}
$$

Infinitely Many Holes

Theorem (B.-S., C., M. (2011))

There exists an infinite cube-free binary partial word with infinitely many holes that has only eleven distinct squares of the form xx, where $|x| \leq 4$.

Thank you!

F. Blanchet-Sadri, I. Choi, R. Mercaș, "Avoiding Large Squares in Partial Words", *Theoretical Computer Science*, 412 (2011) 3752–3758