The List Version of the Borodin–Kostochka Conjecture for Graphs with Large Max Degree

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A list assignment L assigns each vertex v a list $L(v)$ of available colors.

An L-coloring is a function f on $V(G)$ where

- for each vertex v: $f(v) \in L(v)$
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A graph G is k-choosable if there is an L-coloring for each L where

- for each vertex v: $|L(v)| \geq k$.

The list chromatic number or choosability $\chi_l(G)$ is the minimum such k.

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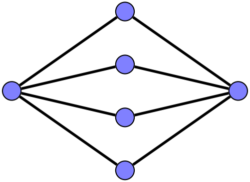

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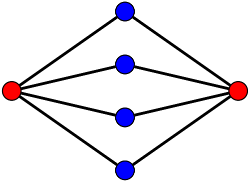

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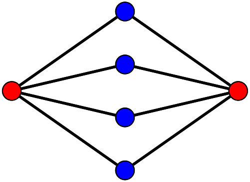

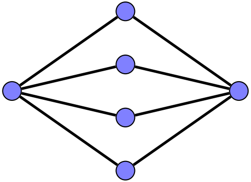

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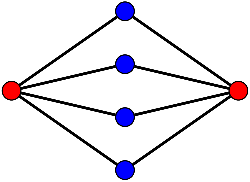

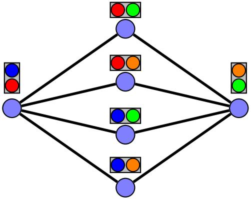

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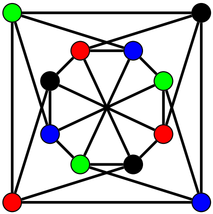
Conjecture (Reed 1998)

For any graph G,

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil$$



$$\frac{2+4+1}{2} < 4 \leq \left\lceil \frac{2+4+1}{2} \right\rceil$$

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Can this be extended?

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Given a graph G with $\Delta(G) \ge 9$,

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If true, then sharp. Blow each vertex of a 5-cycle into a 3-cycle.

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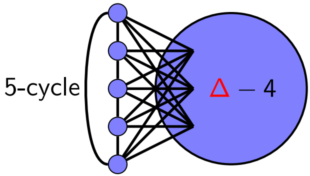

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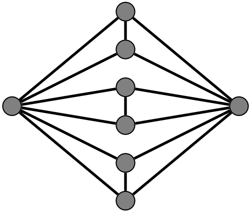

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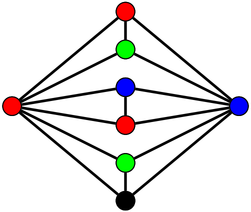

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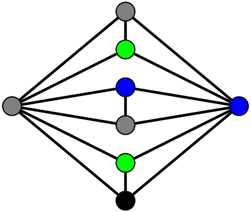

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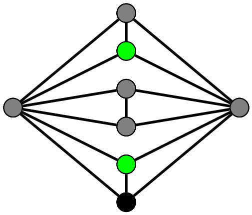

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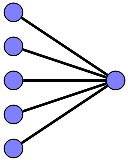

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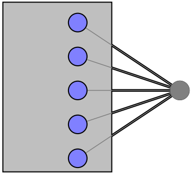

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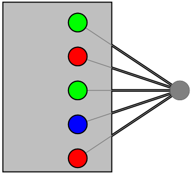

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Given a graph G with $\Delta(G) \geq 10^{20}$,

$$\text{if } \omega(G) \leq \Delta(G) - 1, \text{ then } \chi_l(G) \leq \Delta(G) - 1$$

Proof Sketch: Take a counterexample G with the minimum number of vertices and a list assignment L where $|L(v)| = \Delta(G) - 1$ for each vertex v.

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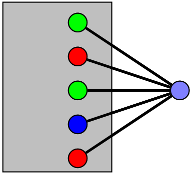

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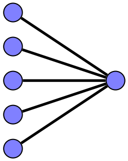

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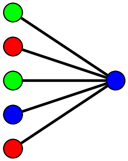

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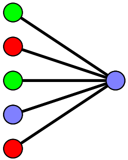

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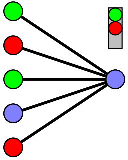
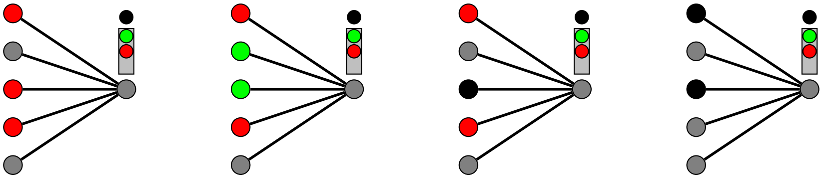


Safe Vertex

Given a partial coloring of G, an uncolored vertex v is *safe* if one of the following occurs:

- a color is repeated three times in $N(v)$;
- two colors are repeated twice in $N(v)$;
- a color is repeated twice in $N(v)$ and a color not in $L(v)$ is in $N(v)$;
- two colors not in $L(v)$ appear in $N(v)$.



Note that a vertex with two uncolored neighbors can always be colored.

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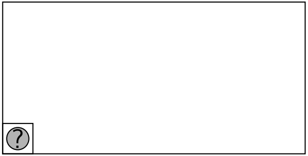

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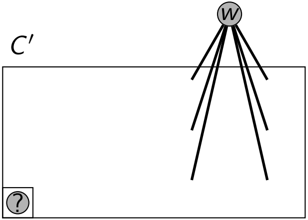

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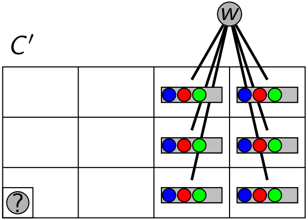

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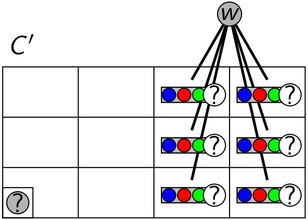

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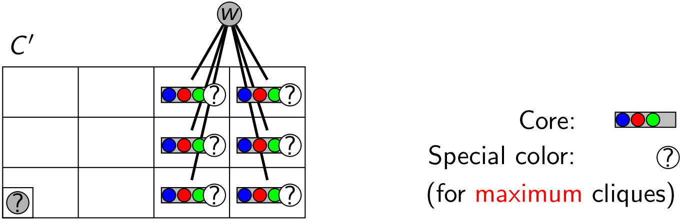

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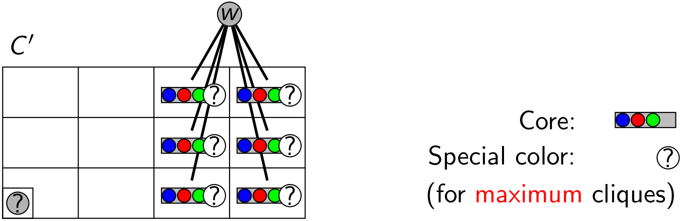

- Each vertex in a big clique C has at most one neighbor outside of C with more than 4 neighbors in C.
- There is at most one vertex outside of a $(\Delta - 1)$-clique with more than 4 neighbors in the clique.
- Each special color appears in at most 5 lists.

1. Decompose G "nicely" so we can analyze in smaller pieces.

Lovász Local Lemma

Consider a set of (bad) events $\mathcal{E}$ where for each $E \in \mathcal{E}$:

- *$Pr(E) \leq p$*
- *E is mutually independent to a set of all but at most d other events.*

If $ep(d+1) \leq 1$, then with positive probability, no event in $\mathcal{E}$ occurs.

2. Color randomly to obtain a good (enough) partial coloring.

Azuma's Inequality

Let X be a random variable determined by n trials $T_1, \ldots, T_n$. For each i and any two possible outcomes, if the following holds:
$|E(X|T_1 = t_1, \ldots, T_i = t_i) - E(X|T_1 = t_1, \ldots, T_i = t_i')| \leq c_i$,
then $Pr(|X - E(X)| > t) \leq 2e^{-t^2/(2\sum c_i^2)}$.

3. For the remaining graph, color greedily to show that G cannot exist.

Decomposition

We can partition $V(G)$ into a sparse set S and dense sets $D_1, \ldots, D_l$ so:

- *every vertex v in S is has $||N(v)|| < \binom{\Delta}{2} - o(\Delta^2)$;*
- *$\exists w_i \in D_i$ such that $D_i - w_i$ is a clique of size $\Delta - o(\Delta)$;*
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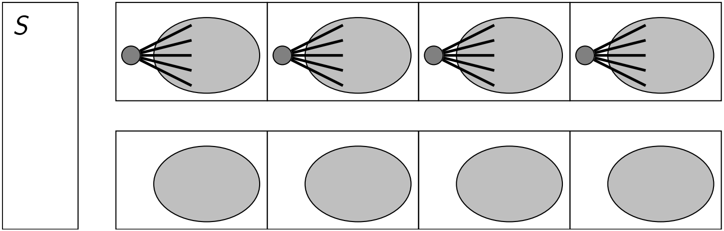

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Partition the cliques further.

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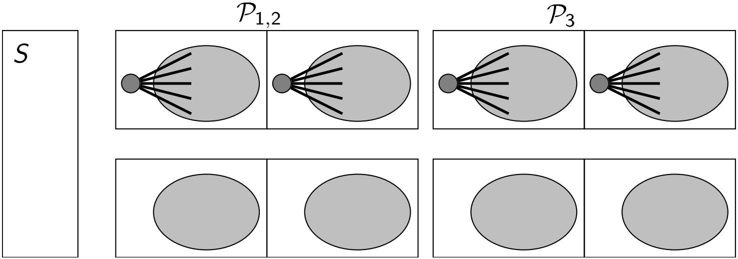

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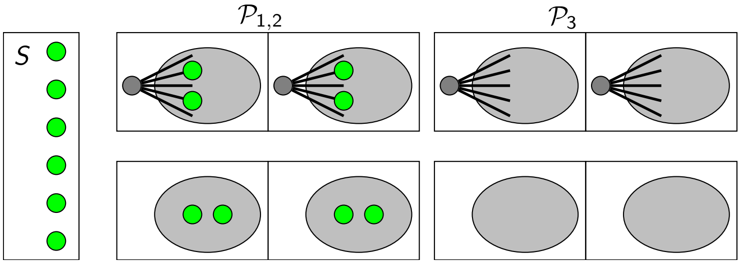

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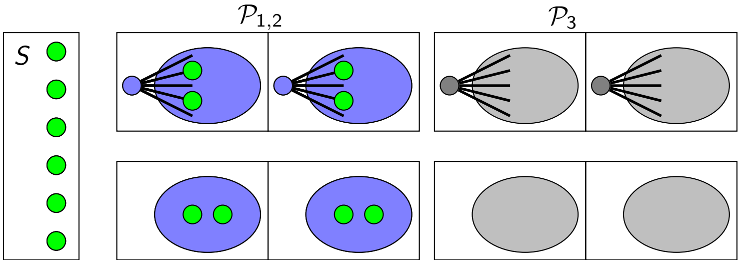

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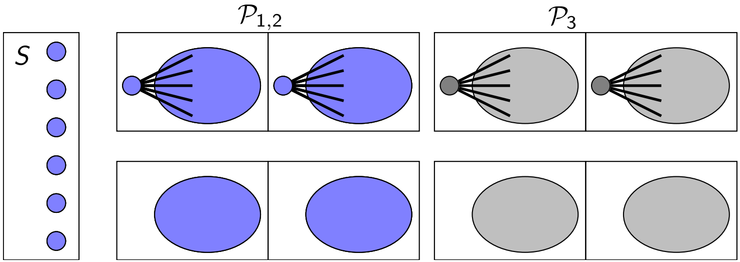

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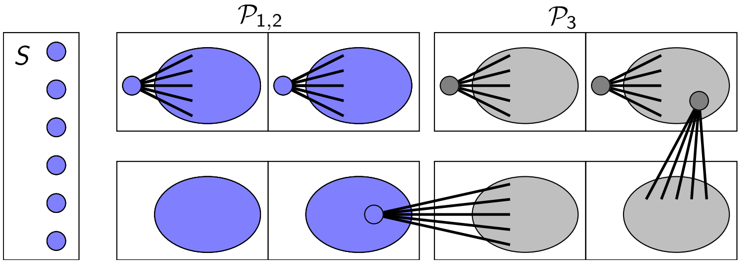

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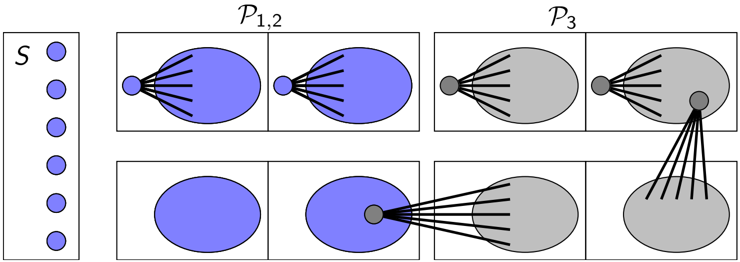

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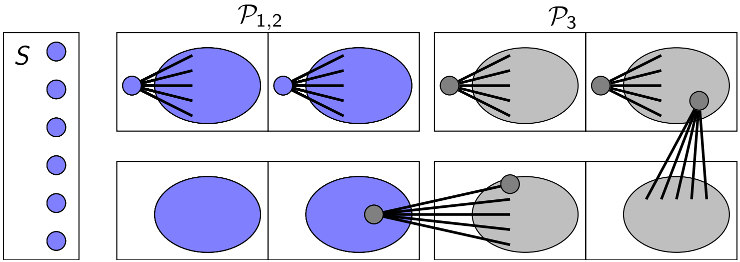

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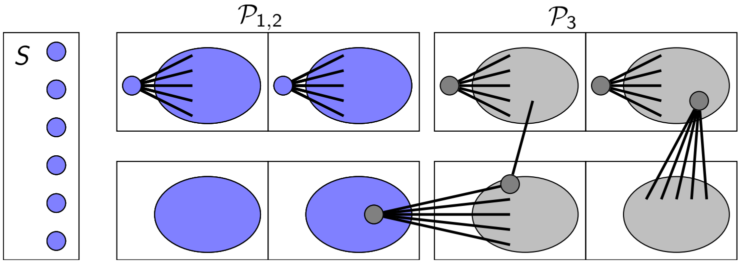

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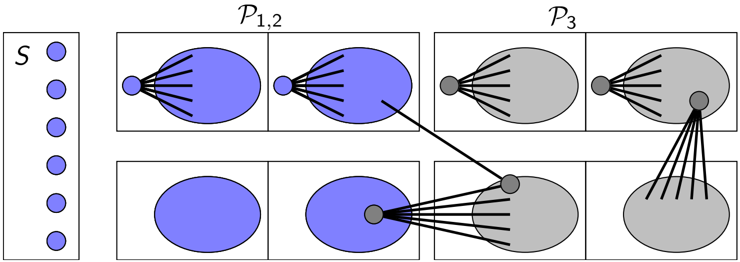

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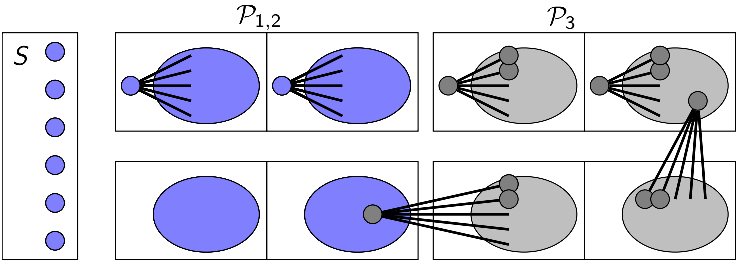

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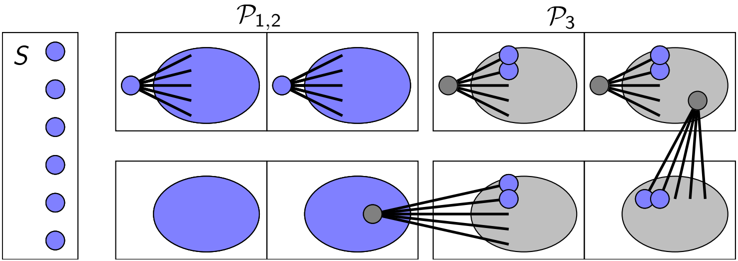

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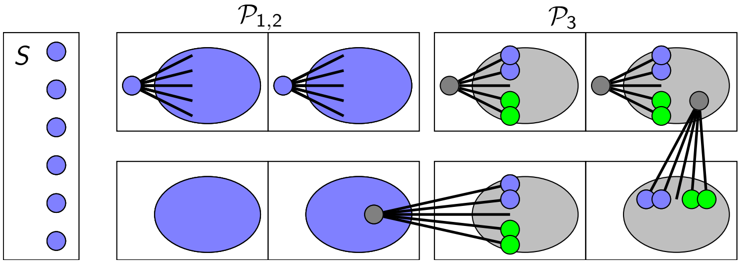

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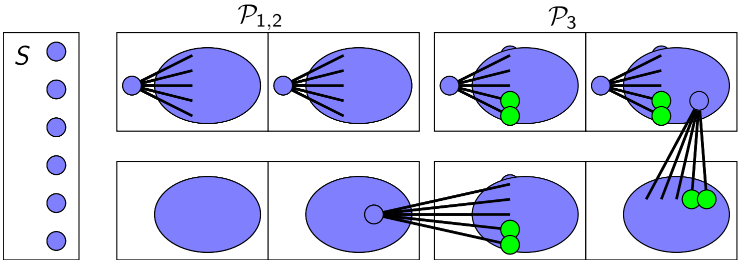
– $\mathcal{P}_3$: maximum cliques C where some $v \notin C$ has "many" neighbors in C.

– $\mathcal{P}_{1,2}$: other cliques.



Decomposition

We can partition $V(G)$ into a sparse set S and dense sets $D_1, \ldots, D_l$ so:

- *every vertex v in S is has $||N(v)|| < \binom{\Delta}{2} - o(\Delta^2)$;*
- *$\exists w_i \in D_i$ such that $D_i - w_i$ is a clique of size $\Delta - o(\Delta)$;*
- *no vertex outside of D_i has more than $\frac{3\Delta}{4}$ neighbors in D_i and w_i has at least $\frac{3\Delta}{4}$ neighbors in D_i.*

Partition the cliques further.

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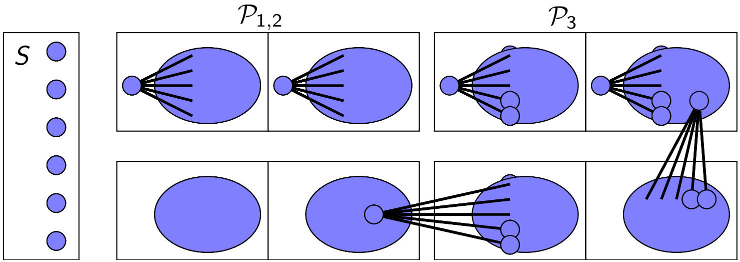

1. Decompose G "nicely" so we can analyze in smaller pieces.

Lovász Local Lemma

Consider a set of (bad) events $\mathcal{E}$ where for each $E \in \mathcal{E}$:

- $Pr(E) \leq p$
- *E is mutually independent to a set of all but at most d other events.*

If $ep(d+1) \leq 1$, then with positive probability, no event in $\mathcal{E}$ occurs.

2. Color randomly to obtain a good (enough) partial coloring.

Azuma's Inequality

Let X be a random variable determined by n trials $T_1, \ldots, T_n$. For each i and any two possible outcomes, if the following holds:
$|E(X|T_1 = t_1, \ldots, T_i = t_i) - E(X|T_1 = t_1, \ldots, T_i = t_i')| \leq c_i$,
then $Pr(|X - E(X)| > t) \leq 2e^{-t^2/(2\sum c_i^2)}$.

3. For the remaining graph, color greedily to show that G cannot exist.

Thank you!