Equicovering Subgraphs of Graphs and Hypergraphs

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Outline

t-Equal Union Property

A hypergraph H has the *t-Equal Union Property* (t-EUP) if there are t distinct subhypergraphs $H_1, \ldots, H_t$ of H such that

1. $E(H_i) \cap E(H_j) = \emptyset$ for $1 \leq i < j \leq t$
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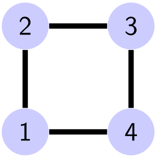

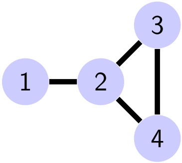

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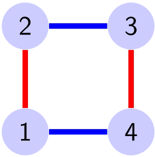

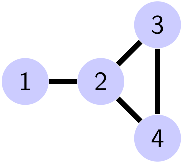

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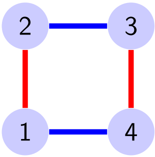
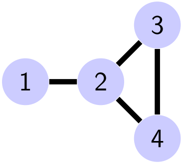


Has 2-EUP



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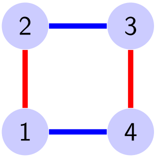

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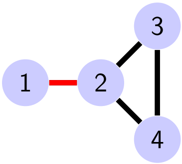

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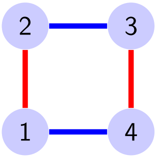

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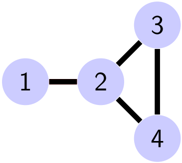

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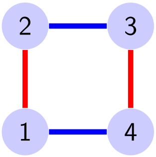

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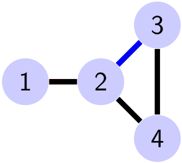

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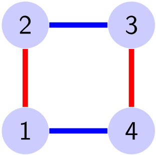

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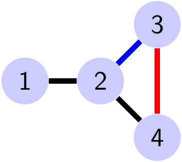

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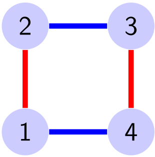

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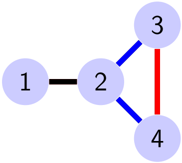

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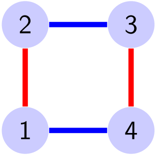

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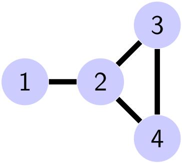

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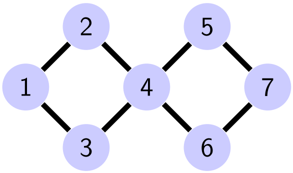

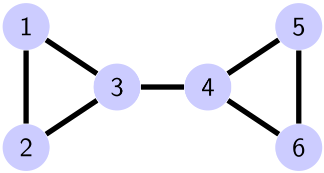

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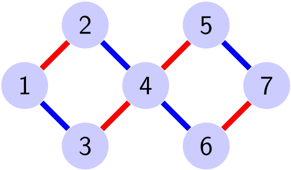

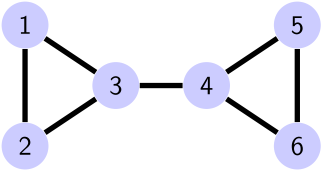

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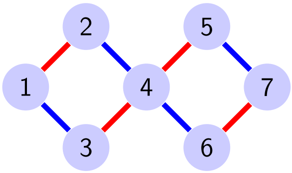

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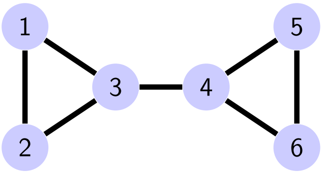

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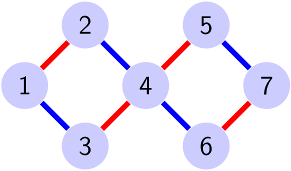

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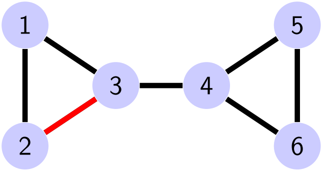

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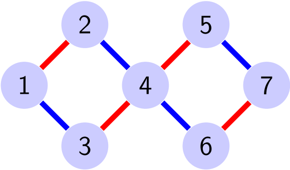

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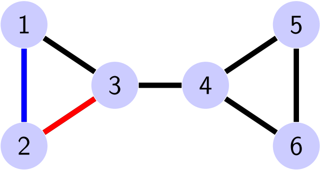

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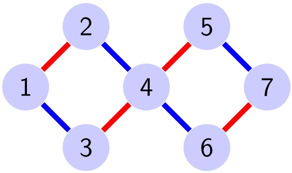

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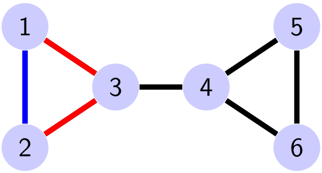

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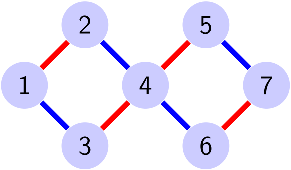

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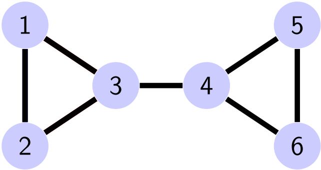

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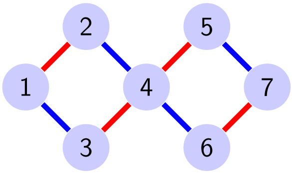

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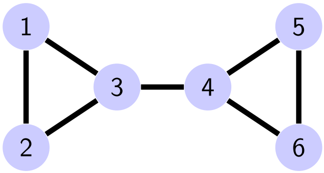

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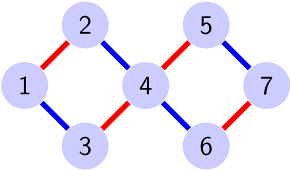

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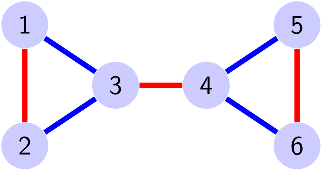

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Notation

Theorem (Lindström (1972))

If a hypergraph H has more than $(t-1)n$ edges, then H has the t-EUP.

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If a hypergraph H has more than $(t-1)n$ edges, then H has the t-EUP.

Maximum number of edges $\mathbb{U}(n, t)$ in a hypergraph without the t-EUP.
Maximum number of edges $\mathbb{V}(n, t)$ in a hypergraph without the t-EVP.

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Theorem (Lindström (1972))

$$\mathbb{U}(n, t) \leq (t-1)n$$

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Maximum number of edges $\mathbb{V}(n, t)$ in a hypergraph without the t-EVP.

n-vertex	t-EUP	t-EVP
graphs	$\mathbb{U}_2(n, t)$	$\mathbb{V}_2(n, t)$
k-uniform hypergraphs	$\mathbb{U}_k(n, t)$	$\mathbb{V}_k(n, t)$
hypergraphs	$\mathbb{U}(n, t)$	$\mathbb{V}(n, t)$

Note that $A(n, t) \geq A_k(n, t)$ for $A \in \{\mathbb{U}, \mathbb{V}\}$.

Characterization of Graphs with the 2-EUP

Theorem

A graph G has the 2-EUP if and only if G has an even cycle or has two odd cycles in the same component.

Corollary

$$\mathbb{U}_2(n, 2) = n$$

Equality holds only for either connected graphs with only one odd cycle or the disjoint union of odd cycles.

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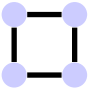

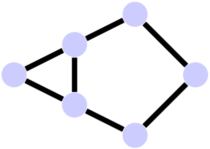

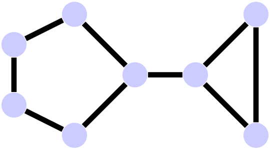

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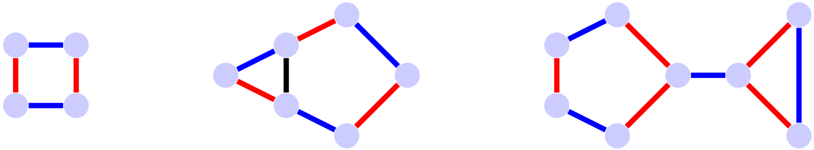

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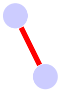

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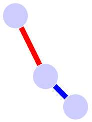

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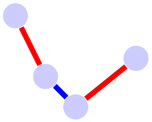

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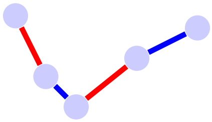

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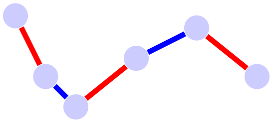

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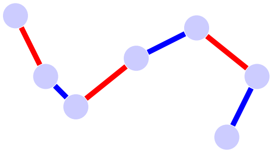

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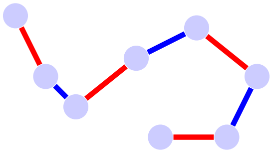

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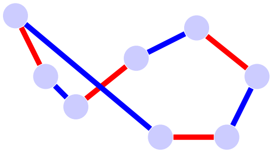

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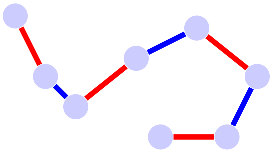

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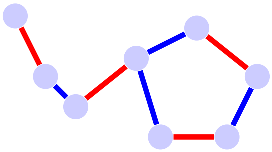

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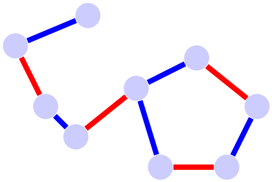

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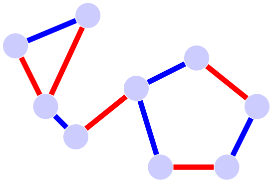

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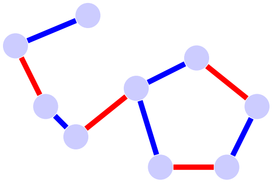

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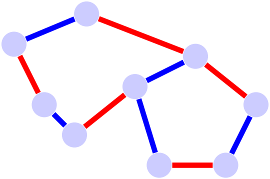

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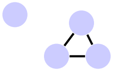

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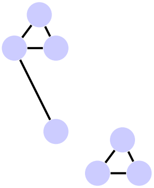

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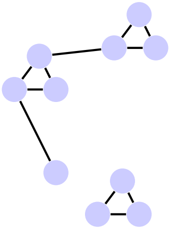

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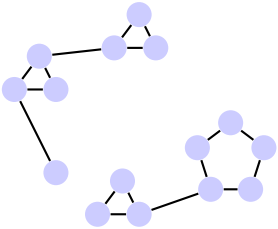

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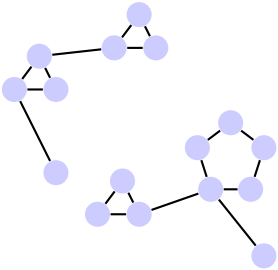

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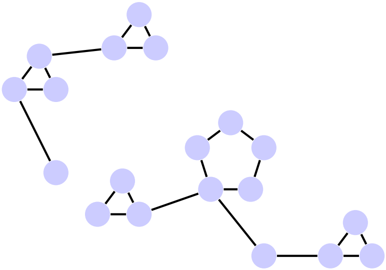

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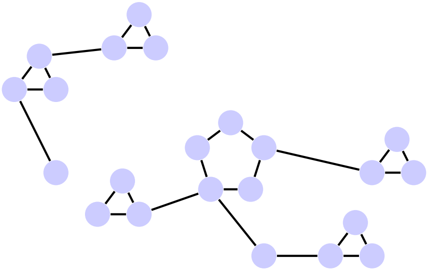

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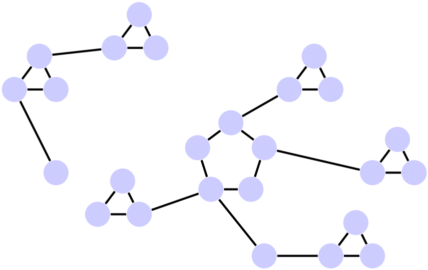

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Theorem

A graph G does not have the 2-EVP if any only if G is an odd-cycle-forest.

Corollary

$$\mathbb{V}_2(n, 2) = \left\lfloor \frac{4}{3} n \right\rfloor - 1$$

Equality holds only for odd-cycle-trees obtained by replacing all vertices in a tree by triangles.

The t-EVP of Graphs: Upper Bound

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A graph Q is a *q-divisible graph* if the degree of each vertex in Q is a multiple of an integer q.

Lemma

If Q is a q-divisible bipartite graph, then Q has the q-EVP.

The t-EVP of Graphs: Lower Bound

Every $(t-1)$-degenerate graph does not have the t-EVP. This gives

$$\mathbb{V}_2(n,t) \geq (t-1)n$$

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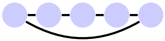

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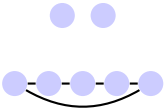

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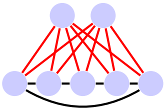

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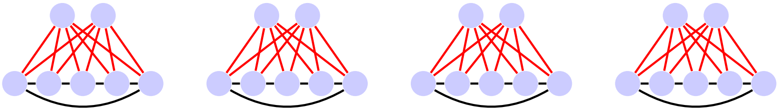

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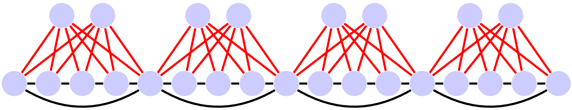

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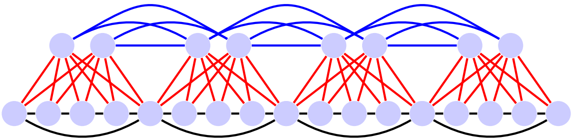

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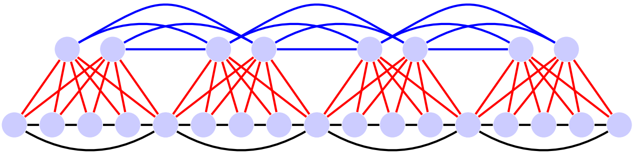

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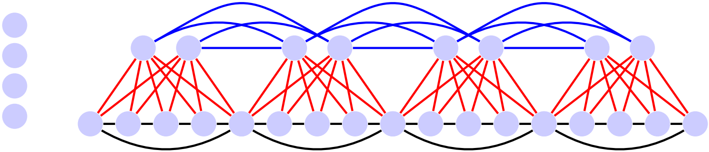

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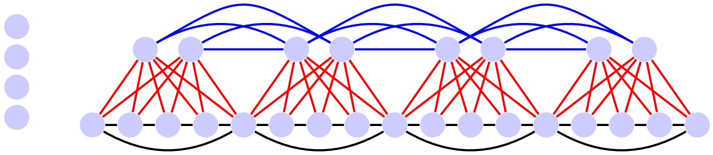

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Theorem

For some polynomial $f_k(t)$ with degree at most 2,

$$\mathbb{V}_k(n, t) \geq \left(t - 1 + \frac{1}{2(t-1)}\right) n - f_k(t)$$

The 2-EVP of Hypergraphs: Proof

Theorem

For an n-vertex k-uniform hypergraph with $k \geq 3$,

$$\mathbb{V}_k(n, 2) < (\log_2 k + (1 + \varepsilon_k) \log_2 \log_2 k)n$$

for some $\varepsilon_k > 0$, where $\varepsilon_k \to 0$ as $k \to \infty$.

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Proof. H with $m = (\log k + (1 + \varepsilon_k) \log \log k)n$ edges has 2^m subgraphs.

Let $(d_1, \ldots, d_n)$ be the degree list of H. Since $\sum_{i=1}^n d_i = km$, there exists at most $\prod_{i=1}^n (d_i + 1) \leq \left(\frac{km}{n} + 1\right)^n$ degree lists for subhypergraphs.

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If H does not have 2-EVP $\Rightarrow$ subhypergraphs must have different lists.

$$2^m \leq \left(\frac{km}{n}+1\right)^n \Rightarrow (k(\log_2 k)^{1+\varepsilon_k})^n \leq (k\log_2 k + (1+\varepsilon_k)k\log_2\log_2 k + 1)^n$$

With appropriate ε_k, contradiction. Hence, H has the 2-EVP. □

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ε_k must be greater than a root of

$$(\log_2 k)^{1+\varepsilon_k} - \log_2 k - (1+\varepsilon_k)\log_2\log_2 k - \frac{1}{k} = 0$$

As k increases, $\varepsilon_k \to 0$, but at a very slow rate.
When $k = 10^{42}$ the constant ε_k still needs to be larger than 0.01.

Corollary

$$\mathbb{V}_3(n,2) < 3.5377n$$

The t-EVP of Hypergraphs

By refining the previous argument using known results about r-Δ-systems, we prove the following theorems.

Theorem

Let $t \in \mathbb{N}_{\geq 3}$ and $\varepsilon > 0$. There exists $N = N(t, \varepsilon)$ such that for $n \geq N$,

$$\mathbb{V}(n, t) < (4 + \varepsilon)n^2 \left(\frac{\log n}{\log \log \log n} \right)^2$$

Theorem

Let $t \in \mathbb{N}_{\geq 3}, k \in \mathbb{N}_{\geq 2}$, and $\varepsilon > 0$. There exists $N = N(t, k, \varepsilon)$ such that for $n \geq N$,

$$\mathbb{V}_k(n, t) < (1 + \varepsilon)n^2 \left(\frac{\log n}{\log \log \log n} \right)^2$$

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2. We know $\left(t - 1 + \frac{1}{2(t-1)}\right) n - f_k(t) \le \mathbb{V}_2(n,t) \le 4(t-1)n$ for some function $f_k(t)$ of degree at most two. Close the gap?

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3. Better lower bounds?

Thank you!