Fractional Matchings and Matchings of Graphs

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Joint work with Jaehoon Kim and Suil O

April 25, 2015

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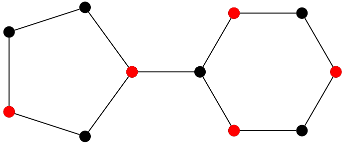

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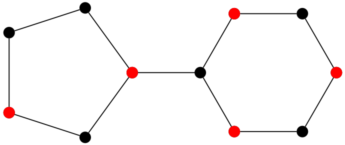

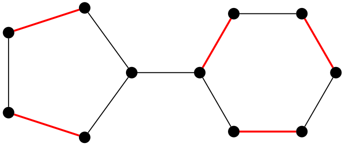

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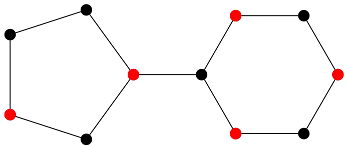

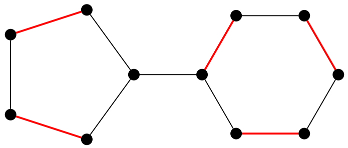

In other words......

A matching is a function ϕ such that

– for each edge e: $\phi(e) \in \{0, 1\}$

– for each vertex v: $\sum_{u \sim v} \phi(uv) \leq 1$.

The size of ϕ is $\sum_{e \in E(G)} \phi(e)$.

The matching number $\alpha'(G)$ of G is the max size of a matching.

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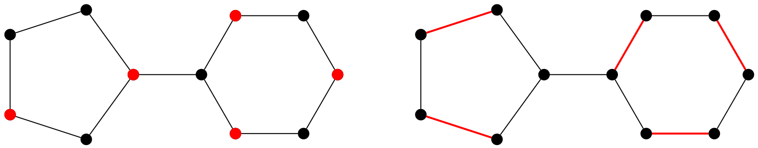

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What if we relax the range of ϕ.............?

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Fractional Ramsey theory.. Fractionally Hamiltonian graphs..

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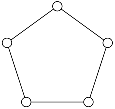

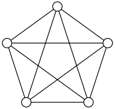

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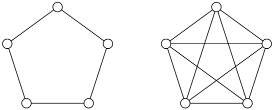

For a constant c, there are infinitely many G where $\alpha'_f(G) - \alpha'(G) > c$ and $\alpha'_f(G)/\alpha'(G) > c$

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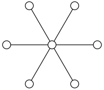

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Corollary (C.–Kim–O 2015+)

For any n-vertex graph G, we have $\alpha'_f(G) - \alpha'(G) \le \frac{n}{6}$, and equality holds if any only if G is the disjoint union of copies of K_3.

Corollary (C.–Kim–O 2015+)

For any n-vertex graph G with at least one edge, we have $\frac{\alpha'_f(G)}{\alpha'(G)} \le \frac{3}{2}$, and equality holds if any only if G is the disjoint union of copies of K_3.

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Theorem (C.–Kim–O 2015+)

For $n \geq 5$, if G is an n-vertex connected graph, then

$$\alpha'_f(G) - \alpha'(G) \leq \frac{n-2}{6} \quad \text{and} \quad \frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{3n}{2n+2}$$

Equality holds if and only if either
(i) $n = 5$ and either C_5 is a subgraph of G or $K_2 + K_3$ is a subgraph of G
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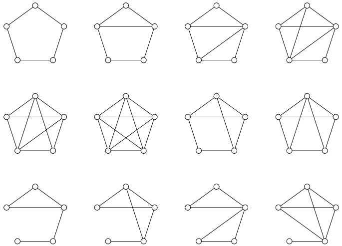

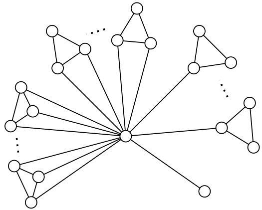

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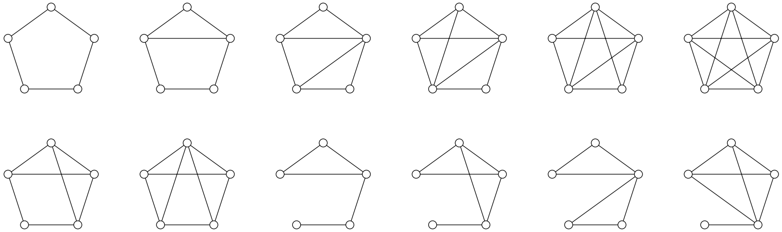

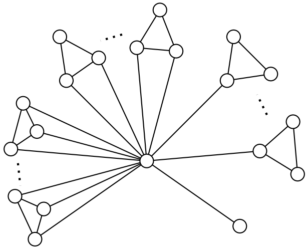

For a graph H, let $o(H)$ denote the number of odd components of H.

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Proposition

For any graph G, $2\alpha'_f(G)$ is an integer.
Moreover, there is a fractional matching ϕ where $\sum_{e \in E(G)} \phi(e) = \alpha'_f(G)$ and $\phi(e) \in \{0, \frac{1}{2}, 1\}$ for each edge e.

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Pf: Let G maximize $\alpha'_f(G) - \alpha'(G)$; S max set with max $o(G-S) - |S|$.

$x = i(G-S)$	:	number of vertex components of $G-S$
$y = o(G-S) - x$	:	number of non-vertex components of $G-S$
$n \geq \|S\| + x + 3y$	:	$G-S$ has no even components

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$\alpha'_f(G) - \alpha'(G)$

$$= \frac{1}{2}\left[\left(n - \max_{S_f}\{i(G - S_f) - |S_f|\}\right) - \left(n - \max_{S}\{o(G - S) - |S|\}\right)\right]$$

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$$\alpha'_f(G) - \alpha'(G)$$

$$\begin{aligned}
&= \frac{1}{2}\left[\left(n - \max_{S_f}\{i(G-S_f) - |S_f|\}\right) - \left(n - \max_{S}\{o(G-S) - |S|\}\right)\right] \\
&= \frac{1}{2}\left[\max_{S}\{o(G-S) - |S|\} - \max_{S_f}\{i(G-S_f) - |S_f|\}\right] \\
&\leq \frac{1}{2}\Big[x + y - |S| - (x - |S|)\Big] = \frac{y}{2} \leq \frac{n - x - |S|}{6} \leq \frac{n-2}{6}
\end{aligned}$$

What about for k-uniform hypergraphs?!

What about the difference and the ratio of other graph parameters and their fractional versions?

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Thank you for your attention!