Improper Coloring of Planar Graphs

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Joint work with Hojin Choi, Jisu Jeong, and Geewon Suh

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- partition $V(G)$ into k parts
- each part has maximum degree at most 0

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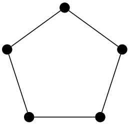

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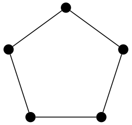

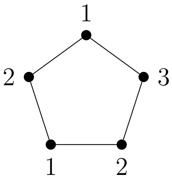

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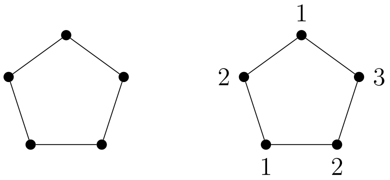

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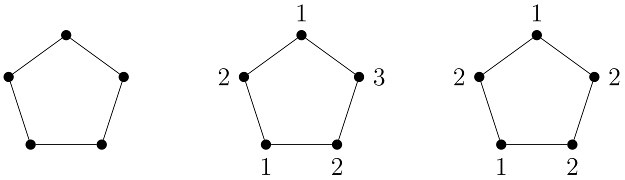

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Theorem (Appel–Haken 1977)

Every planar graph is $(0,0,0,0)$-colorable.

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Theorem (Cowen–Cowen–Woodall 1986)

Every planar graph is $(2, 2, 2)$-colorable.

Theorem (Eaton–Hull 1999, Škrekovski 1999)

For each k, there exists a non-$(1, k, k)$-colorable planar graph.

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Theorem (Cowen–Goddard–Jesurum 1997)

Every toroidal graph is $(1,1,1,1,1)$-colorable and $(2,2,2)$-colorable.

Question (Cowen–Goddard–Jesurum 1997)

Is every toroidal graph $(1,1,1,1)$-colorable?

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Problem

Given (d_1, d_2), determine the supremum x such that every graph with $Mad(G) \leq x$ is (d_1, d_2)-colorable.

$\text{Mad}(G) = \max_{H \subseteq G} \frac{2|E(H)|}{|V(H)|}$. If G is planar with girth g, then $\text{Mad}(G) < \frac{2g}{g-2}$.

Problem

Given (d_1, d_2), determine the min $g = g(d_1, d_2)$ such that every planar graph with girth g is (d_1, d_2)-colorable.

Problem

Given $(g; d_1)$, determine the min $d_2 = d_2(g; d_1)$ such that every planar graph with girth g is (d_1, d_2)-colorable.

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Theorem (Havet–Sereni 2006)

For $l \geq 2$, every G with $Mad(G) < \frac{l(l+2k)}{l+k}$ is $(k, \dots, k)$-choosable.

Theorem (Borodin–Ivanova–Montassier–Ochem–Raspaud 2009)

For $k \geq 0$, every G with $Mad(G) < \frac{3k+4}{k+2}$ is $(0, k)$-colorable.

Theorem (Borodin–Ivanova–Montassier–Raspaud 2010)

For $k \geq 2$, every G with $Mad(G) < \frac{10k+22}{3k+9}$ is $(1, k)$-colorable.

Theorem (Borodin–Kostochka 2011, Borodin–Kostochka–Yancey 2011)

Given $d_1 \geq 0$ and $d_2 \geq 2d_1 + 2$, every G with $Mad(G) < 2\left(2 - \frac{d_2+2}{(d_1+2)(d_2+1)}\right)$ is (d_1, d_2)-colorable, and this is sharp.
Every G with $Mad(G) \leq 14/5$ is $(1, 1)$-colorable, and this is sharp.

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Given (d_1, d_2), *determine the min* $g = g(d_1, d_2)$ *such that every planar graph with girth* g *is* (d_1, d_2)*-colorable.*

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Theorem (Cowen–Cowen–Woodall 1986)

For each d, there are non-(d, d)-colorable planar graphs. $g(d, d) \geq 4$.

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Škrekovski 1999 2000 Havet–Sereni 2006 Borodin–Kostochka–Yancey 13

Theorem

$g(1, 1) \in \{6, 7\}$
$g(2, 2) \in \{5, 6\}$
$g(3, 3) \in \{5, 6\}$
$g(d, d) = 5$ *for* $d \geq 4$.

$g(d_1, d_2) = 5$ for $d_1, d_2 \geq 4$ since $g(d_1, d_2 + 1) \leq g(d_1, d_2)$.

Theorem (Montassier–Ochem 2014+, Borodin–Kostochka 2011, 2014)

$g(0,k) = 7$ *for* $k \geq 4$
$g(0,3) \in \{7,8\}$
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$g(0,1) \in \{10,11\}$

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Theorem

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Theorem (Škrekovski 2000, C.–Raspaud 2014+)

$$g(3,5) = 5$$

$d_2 \setminus d_1$	0	1	2	3	4	5
0	$\times$					
1	10 or 11	6 or 7				
2	8	6 or 7	5 or 6			
3	7 or 8	6 or 7	5 or 6	5 or 6		
4	7	5 or 6	5 or 6	5 or 6	5	
5	7	5 or 6	5 or 6	5	5	5
6	7	5 or 6	5	5	5	5

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Given $(g; d_1)$*, determine the min* $d_2 = d_2(g; d_1)$ *such that every planar graph with girth* g *is* (d_1, d_2)*-colorable.*

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All values of $d_2(g; d_1)$ are finite, except maybe $d_2(5; 1)$.

girth	$(0, k)$	$(1, k)$	$(2, k)$	$(3, k)$	$(4, k)$
3	$\times$	$\times$	$\times$	$\times$	$\times$
4	$\times$	$\times$	$\times$	$\times$	$\times$
5	$\times$		$(2, 6)$	$(3, 5)$	$(4, 4)$
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5	$\times$	?	$(2, 6)$	$(3, 5)$	$(4, 4)$
6	$\times$	$(1, 4)$	$(2, 2)$		
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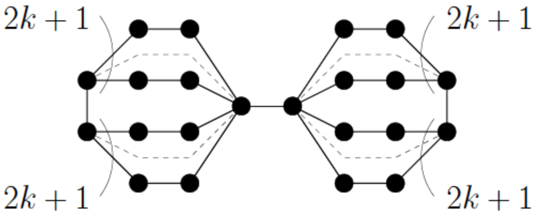
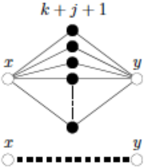
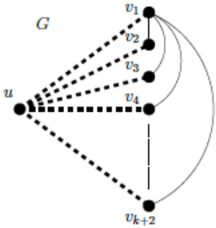
Question (Montassier–Ochem 2014+)

Is $d_2(5; 1)$ finite or not?



Non-$(0, k)$-colorable planar graph with girth 6.





There are also non-(d_1, d_2)-colorable planar graphs with girth 4.

girth	$(0,k)$	$(1,k)$	$(2,k)$	$(3,k)$	$(4,k)$
3	$\times$	$\times$	$\times$	$\times$	$\times$
4	$\times$	$\times$	$\times$	$\times$	$\times$
5	$\times$	$(1, ?)$	$(2,6)$	$(3,5)$	$(4,4)$
6	$\times$	$(1,4)$	$(2,2)$		
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Question (Raspaud 2013)

Is every planar graph with girth 5 indeed (j,k)-colorable for all $j+k\geq 8$?

girth	$(0,k)$	$(1,k)$	$(2,k)$	$(3,k)$	$(4,k)$
3	$\times$	$\times$	$\times$	$\times$	$\times$
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Question (Raspaud 2013, Montassier–Ochem 2014+)

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– Is there some k where planar graphs with girth 5 are $(1,k)$-colorable?

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Theorem (Havet–Sereni 06, C.–Raspaud 14+, Borodin–Kostochka 14)

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Theorem (C.)

There exists a planar graph with girth 5 that is not $(1,1)$-colorable.

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Theorem (Montassier–Ochem 2014+)

There exists a planar graph with girth 5 that is not $(1,3)$-colorable.

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3	$\times$	$\times$	$\times$	$\times$	$\times$
4	$\times$	$\times$	$\times$	$\times$	$\times$
5	$\times$	$(1, 4 \leq ? \leq 10)$	$(2,6)$	$(3,5)$	$(4,4)$
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Theorem (Choi–C.–Jeong–Suh 2014+)

Every planar graph with girth 5 is $(1,10)$-colorable.

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Theorem (Choi–C.–Jeong–Suh 2014+)

Every planar graph with girth 5 is $(1, 10)$*-colorable.*

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Proof: Discharging!

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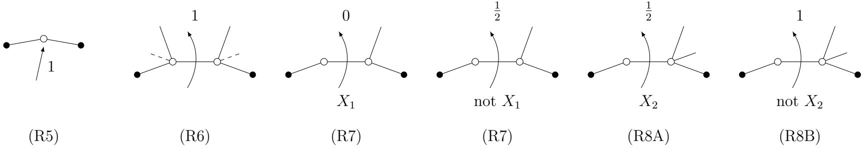
Assign initial charge... $\mu(v) = d(v) - 6$ and $\mu(f) = 2l(f) - 6$
Initial charge sum...

$$\sum_v (d(v) - 6) + \sum_f (2 \cdot l(f) - 6) = -6v + 6e - 6f < 0$$

Discharging rules...



Show each element has nonnegative final charge...

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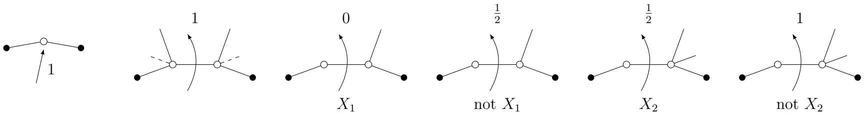

(R5) (R6) (R7) (R7) (R8A) (R8B)

Show each element has nonnegative final charge...

CONTRADICTION!

Thank you for your attention!