EXTREMAL PROBLEMS ON VARIATIONS OF GRAPH COLORINGS

BY

ILKYOO CHOI

DISSERTATION

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Doctoral Committee:

Professor Bruce Reznick, Chair
Professor Alexandr V. Kostochka, Director of Research
Professor Emeritus Douglas B. West
J.L. Doob Research Assistant Professor Bernard Lidický

Abstract

This thesis investigates various coloring problems in graph theory. Graph coloring is an essential part of combinatorics and discrete mathematics, as it deals with the fundamental problem of partitioning objects so that each part satisfies a certain condition. In particular, we study how forbidding certain structures (subgraphs) affects a given coloring parameter.

Open since 1977, the Borodin–Kostochka Conjecture states that given a graph G with maximum degree $\Delta(G)$ at least 9, if G has no clique of size $\Delta(G)$, then G is $(\Delta(G)-1)$-colorable. The current best result by Reed shows that the statement of the Borodin–Kostochka Conjecture is true for graphs with maximum degree at least 10^{14}. We produce a result of this type for the list chromatic number; namely, we prove that given a graph G with maximum degree at least 10^{20}, if G has no clique of size $\Delta(G)$, then G is $(\Delta(G)-1)$-choosable.

Cai, Wang, and Zhu proved that a toroidal graph with no 6-cycles is 5-choosable, and they conjectured that the only case when it is not 4-choosable is when the graph contains K_5. We disprove this conjecture by constructing a family of graphs containing neither 6-cycles nor K_5 that are not even 4-colorable. This family is embeddable not only on a torus, but also on any surface except the plane and the projective plane. We prove a slightly weaker statement suggested by Zhu that toroidal graphs containing neither 6-cycles nor K_5^- are 4-choosable.

We also study questions regarding variants of coloring. We provide additional positive support for a question by Škrekovski regarding choosability with separation for planar graphs, and we completely answer a question by Raspaud and Wang regarding vertex ar-

boricity for toroidal graphs. We also improve results regarding improper coloring of planar graphs, responding to a question of Montassier and Ochem.

Dedicated to my family,

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Chapter 1

Introduction

1.1 The List Borodin–Kostochka Conjecture

Brooks' Theorem [10] states that for a graph G with maximum degree $\Delta(G)$ at least 3, the chromatic number is at most $\Delta(G)$ when the clique number is at most $\Delta(G)$. Vizing [48] proved that the list chromatic number is also at most $\Delta(G)$ under the same conditions. In 1977, Borodin and Kostochka [8] conjectured that a graph G with maximum degree at least 9 must be $(\Delta(G)-1)$-colorable when the clique number is at most $\Delta(G)-1$; this was proven for graphs with maximum degree at least 10^{14} by Reed [41] in 1999. We prove the following analogous result for the list chromatic number:

Theorem 1.1.1 ([16]). *For a graph G with $\Delta(G) \geq 10^{20}$, if $\omega(G) \leq \Delta(G) - 1$, then $\chi_\ell(G) \leq \Delta(G) - 1$.*

This is joint work with H. A. Kierstead, L. Rabern, and B. Reed.

1.2 Choosability of Toroidal Graphs

The choosability $\chi_\ell(G)$ of a graph G is the minimum k such that having k colors available at each vertex guarantees a proper coloring. Given a toroidal graph G, it is known that $\chi_\ell(G) \leq 7$, and that $\chi_\ell(G) = 7$ if and only if G contains K_7. Cai, Wang, and Zhu [11] proved that a toroidal graph G without 7-cycles is 6-choosable, and that $\chi_\ell(G) = 6$ if and only if G contains K_6. They also proved that a toroidal graph G without 6-cycles is 5-choosable, and they conjectured that $\chi_\ell(G) = 5$ if and only if G contains K_5. We disprove this conjecture by constructing an infinite family of non-4-colorable toroidal graphs containing neither K_5 nor cycles of length at least 6; moreover, this family of graphs is embeddable on every surface except the plane and the projective plane. We prove the following slightly weaker statement suggested by Zhu:

Theorem 1.2.1 ([15]). *A toroidal graph containing neither K_5^- nor 6-cycles is 4-choosable.*

This is sharp in the sense that forbidding only one of the two structures does not ensure that the graph is 4-choosable.

1.3 Choosability with Separation for Planar Graphs

We study choosability with separation, which is a constrained version of list coloring of graphs. A (k,d)-*list assignment* L of a graph G is a function that assigns to each vertex v a list $L(v)$ of at least k colors such that for any edge xy, the lists $L(x)$ and $L(y)$ share at most d colors. A graph G is (k,d)-choosable if there exists an L-coloring of G for every (k,d)-list assignment L. This concept is also known as "choosability with separation". Škrekovski [43] asked whether planar graphs are $(3,1)$-choosable. We prove the following two theorems that supports the aforementioned question in the affirmative.

Theorem 1.3.1 ([17]). *Every planar graph without 4-cycles is $(3,1)$-choosable.*

Theorem 1.3.2 ([17]). *Every planar graph with no 5-cycle and no 6-cycle is $(3,1)$-choosable.*

In addition, we give an alternative and slightly stronger proof that triangle-free planar graphs are $(3,1)$-choosable, which is a result by Kratochvíl, Tuza, and Voigt [33]. This is joint work with D. Stolee and B. Lidický.

1.4 Vertex Arboricity of Toroidal Graphs

The vertex arboricity $a(G)$ of a graph G is the minimum k such that $V(G)$ can be partitioned into k sets such that each set induces a forest. For a planar graph G, it is known that

$a(G) \leq 3$. In two recent papers, it was proved that planar graphs without k-cycles for any one $k \in \{3, 4, 5, 6, 7\}$ have vertex arboricity at most 2. For a toroidal graph G, it is known that $a(G) \leq 4$. Let us consider the following question: do toroidal graphs without k-cycles have vertex arboricity at most 2? It was known that the answer is yes for $k = 3$. Recently, Zhang [55] proved that the answer is yes for $k = 5$. Since a complete graph on 5 vertices is a toroidal graph having no cycles of length at least 6 and has vertex arboricity at least 3, the only unknown case was $k = 4$. We solve this case in the affirmative:

Theorem 1.4.1 ([19]). *If G is a toroidal graph with no 4-cycles, then $a(G) \leq 2$.*

This is joint work with H. Zhang.

1.5 Improper Coloring of Planar Graphs

A graph is $(d_1, \ldots, d_r)$-colorable if its vertex set can be partitioned into r sets $V_1, \ldots, V_r$ such that the maximum degree of the graph induced by V_i is at most d_i for each $i \in \{1, \ldots r\}$. Let $\mathcal{G}_g$ denote the class of planar graphs with girth at least g. We focus on graphs in $\mathcal{G}_5$, since for any d_1 and d_2, Montassier and Ochem [38] constructed graphs in $\mathcal{G}_4$ that are not (d_1, d_2)-colorable. It is known that graphs in $\mathcal{G}_5$ are $(2, 6)$-colorable and $(4, 4)$-colorable, but they are not all $(3, 1)$-colorable. We prove the following theorem:

Theorem 1.5.1 ([18]). *Planar graphs with girth at least 5 are $(3, 5)$-colorable.*

We leave two interesting questions open: (1) are graphs in $\mathcal{G}_5$ also $(3, d_2)$-colorable for some $d_2 \in \{2, 3, 4\}$? (2) are graphs in $\mathcal{G}_5$ indeed (d_1, d_2)-colorable whenever $d_1 + d_2 = 8$ with $d_2 \geq d_1 \geq 1$? This is joint work with A. Raspaud.

1.6 Definitions and Notation

The definitions and notation in this thesis mostly follow [54]. We repeat important notions here. We would also like to point out that the scope of the definitions made in a specific chapter is that one chapter.

Given a positive integer n, let $[n]$ denote the set $\{1, \dots, n\}$. Only finite, simple graphs are considered in this thesis, unless specified otherwise.

For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively. A graph H is a *subgraph* of a graph G if there exists an injection $f : V(H) \to V(G)$ such that $xy \in E(H)$ implies $f(x)f(y) \in E(G)$. We use $H \subseteq G$ to mean "H is a subgraph of G". For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G whose vertex set is S and whose edge set is $\{xy : xy \in E(G) \text{ and } x, y \in S\}$; we say the subgraph $G[S]$ is *induced* by S. A graph H is an *induced subgraph* of G if there is a subset of $V(G)$ that induces H.

The *join* of two graphs G_1 and G_2, denoted $G_1 \vee G_2$, is the disjoint union of G_1 and G_2 plus edges making all of $V(G_1)$ adjacent to all of $V(G_2)$. Given a graph G and $e \in E(G)$, let $G - e$ denote the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$. Given a graph G and $v \in V(G)$, let $G - v$ denote the graph with vertex set $V(G) \setminus \{v\}$ and edge set $E(G) \setminus \{vz : vz \in E(G) \text{ and } z \in V(G)\}$. The *complement* of a given graph G, denoted $\overline{G}$, has vertex set $V(G)$ and edge set $\{uv : uv \notin E(G)\}$.

Let K_n denote the complete graph on n vertices, and let E_n denote the empty graph on n vertices; in other words, E_n and $\overline{K_n}$ denote the same graph. A graph G is *k-partite*, if we can partition $V(G)$ into k sets $V_1, \dots, V_k$ such that V_i is independent for each $i \in [k]$. A 2-partite graph is often called a *bipartite graph.* A *path* is a graph whose vertices can be labelled $x_1, \dots, x_n$ so that $x_i x_{i+1}$ is an edge for every $i \in [n-1]$. A *cycle* is a graph whose vertices can be labelled $x_1, \dots, x_n$ so that its edge set is $\{x_i x_{i+1} : i \in [n-1]\} \cup \{x_n x_1\}$. A path, or cycle, with n vertices is typically denoted by P_n, or C_n, respectively.

A graph is *connected* if for every pair x, y of vertices in the graph, there exists a path starting at x and ending at y. A *component* of a graph is a maximal connected subgraph. The *girth* of a graph is the length of its shortest cycle.

A *neighbor* of a vertex v is a vertex adjacent to v. Let $N(v)$ denote the set of neighbors of v. The *degree* of v, denoted $d(v)$, is $|N(v)|$. For $H \subseteq G$ and a vertex v, let $N_H(v) =$

$N(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$. The *degree* of a face f, denoted $d(f)$, is the minimum length of a boundary walk of f. A *k-vertex*, *k^+-vertex*, or *k^--vertex* is a vertex of degree k, at least k, or at most k, respectively. A *k-face* or *k^+-face* is a face of degree k or at least k, respectively. For a graph G, let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and maximum degree of (a vertex of) G, respectively.

A *k-clique* is the vertex set of a complete graph on k vertices. For a graph G, the *clique number* of G, denoted $\omega(G)$, is the size of the largest clique in G. A set S of vertices is *independent* in G if $G[S]$ is empty. For a graph G, the *independence number* of G, denoted $\alpha(G)$, is the size of the largest independent set in G. A *matching* of a graph is a set of edges where no two edges share an endpoint.

Given a graph G, a *proper coloring* is a function from $V(G)$ to a set of colors such that the two endpoints of each edge receive different colors. A *list assignment* L is a function on $V(G)$ such that $L(v)$ is the set of *available colors* for each $v \in V(G)$. Given a list assignment L, an *L-coloring* (or an *acceptable coloring* when the lists are clear from the context) is a proper coloring f such that $f(v) \in L(v)$ for each vertex v. A graph is *k-choosable* if it has an L-coloring whenever $|L(v)| \geq k$ for each vertex v. The *list chromatic number* of G, denoted $\chi_\ell(G)$, is the minimum k such that G is k-choosable. A graph is *k-colorable* if it has an L-coloring where all the lists have the same k colors. The *chromatic number* of G, denoted $\chi(G)$, is the minimum k such that G is k-colorable. It follows that for every graph G, it must be the case that $\chi(G) \leq \chi_\ell(G)$.

A *planar graph*, *projective planar graph*, or *toroidal graph* is a graph that can be embedded on the plane, in the projective plane, or on the torus, respectively, with no edge crossings. A *plane graph* is a planar graph with a specific embedding on the plane.

Chapter 2

The List Borodin–Kostochka Conjecture

2.1 Introduction

It follows immediately from greedy coloring that a graph G can be properly colored with $\Delta(G) + 1$ colors. Also, $\Delta(G) + 1$ is the least upper bound on $\omega(G)$. In 1941, Brooks [10] proved the following classical result that connects $\Delta(G)$, $\omega(G)$, and $\chi(G)$.

Theorem 2.1.1 ([10]). *For a graph G with $\Delta(G) \geq 3$, if $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.*

The condition on the maximum degree is tight, as the conclusion does not follow for odd cycles. Actually, in 1976, Vizing [48] showed that an analogous result holds for the list chromatic number under the same conditions.

Theorem 2.1.2 ([48]). *For a graph G with $\Delta(G) \geq 3$, if $\omega(G) \leq \Delta(G)$, then $\chi_\ell(G) \leq \Delta(G)$.*

Shortly after, in 1977, Borodin and Kostochka [8] conjectured a similar type of result when the upper bound on the clique number is one less. The condition on the maximum degree is tight, as there exist graphs with maximum degree 9 where the conclusion is not true. We state the contrapositive.

Conjecture 2.1.3 ([8]). *Every graph G satisfying $\chi(G) \geq \Delta(G) \geq 9$ contains $K_{\Delta(G)}$.*

There are various partial results regarding this conjecture. Kostochka [32] proved the following result, which guarantees a clique of size almost the maximum degree.

Theorem 2.1.4 ([32]). *Every graph G satisfying $\chi(G) \geq \Delta(G)$ contains $K_{\Delta(G)-28}$.*

A relaxation of the lower bound on the maximum degree allows a theorem by Mozhan in his thesis, which ensures a clique that is close to big enough.

Theorem 2.1.5 ([39])**.** *Every graph G satisfying $\chi(G) \geq \Delta(G) \geq 31$ contains $K_{\Delta(G)-3}$.*

By drastically increasing the lower bound on the maximum degree, Reed [41] finally shows the existence of a clique of size equal to the maximum degree using probabilistic arguments.

Theorem 2.1.6 ([41])**.** *Every graph G satisfying $\chi(G) \geq \Delta(G) \geq 10^{14}$ contains $K_{\Delta(G)}$.*

In this section, we address Conjecture 2.1.3 for the list chromatic number. We prove that the conjecture is true even for the list chromatic number when the maximum degree is sufficiently large. The main result in this section is the following.

Theorem 2.1.7. *For a graph G with $\Delta(G) \geq 10^{20}$, if $\omega(G) \leq \Delta(G) - 1$, then $\chi_\ell(G) \leq \Delta(G) - 1$.*

Throughout this section, unless specified otherwise, G will be a counterexample to Theorem 2.1.7 with the minimum number of vertices and maximum degree Δ. Let L be a list assignment for G such that each list has size exactly $\Delta(G) - 1$ and G is not L-colorable. By the minimality of G, every proper subgraph of G is L-colorable. Moreover, each vertex of G must have degree either Δ or $\Delta - 1$. If there exists a vertex v of degree less than $\Delta - 1$, then we can obtain an L-coloring of G since an L-coloring on $G - v$ exists by the minimality of G, and this L-coloring extends to G.

We will prove that such a counterexample G cannot not exist by showing that an L-coloring actually exists when $\Delta \geq 10^{20}$. The proof will come in two steps. The first step (Section 4) is to construct a decomposition of G that will facilitate the second step. The second step (Section 5) is to show that G is actually L-colorable via a probabilistic argument involving the Lovász Local Lemma and Azuma's Inequality. We will first discuss some tools we use in the proof in Section 2, and then we will obtain some properties of the list assignment L in Section 3.

2.2 Tools

We will often refer to the tools in this section in the proofs in this chapter.

For a graph H and a function f on $V(H)$, the graph H is *f-choosable* if it has an L-coloring whenever $|L(v)| \geq f(v)$ for each vertex v. For an integer r, a graph is *d_r-choosable* if it is f-choosable where $f(v) = d(v) - r$. Every graph is d_{-1}-choosable.

Cranston and Rabern [22] studied minimal counterexamples to Conjecture 2.1.3; one of their tools was d_1-choosable graphs. The following lemma is used in this section. Recall that the *join* of two graphs G_1 and G_2, denoted $G_1 \vee G_2$, is the disjoint union of G_1 and G_2 plus edges making all of $V(G_1)$ adjacent to all of $V(G_2)$.

Lemma 2.2.1 ([22]). *If B is a graph with $\omega(B) \leq |B| - 2$, then $K_6 \vee B$ is d_1-choosable.*

Note that G cannot have a d_1-choosable graph as an induced subgraph. If G has a d_1-choosable induced subgraph H, then $G - V(H)$ is L-colorable by the minimality of G, and an L-coloring of G extends to H. In particular, $K_6 \vee E_3$ cannot be an induced subgraph of G.

Given a graph H, a *matching* of H is a set of edges where no two edges share an endpoint. A matching M *saturates* a set S if every vertex in S is incident to one edge of M. Recall a classical result by Hall [28] that characterizes when a bipartite graph has a matching that saturates one part.

Theorem 2.2.2 (Hall's Theorem [28]). *Let B be a bipartite graph with parts X and Y. A matching that saturates X exists if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.*

Let B be the bipartite graph with parts $V(G)$ and $L(V(G))$ such that $v \in V(G)$ is adjacent to $c \in L(V(G))$ if and only if $c \in L(v)$. A matching in B corresponds to a partial L-coloring of G. A matching that saturates $V(G)$ corresponds to an L-coloring of G.

Now we introduce some tools from probability theory; definitions and terms not defined regarding probability theory can be found in [37]. The next tool is the Lovász Local Lemma.

This lemma is a very powerful tool since we can prevent every (bad) event in a certain set from happening by bounding the probability and the number of dependent events of each event.

Theorem 2.2.3 (Lovász Local Lemma)**.** *Consider a set $\mathcal{E}$ of (bad) events where each event $E \in \mathcal{E}$ satisfies the following.*

(i) $Pr(E) \leq p$

(ii) *E is mutually independent of a set of all but at most d other events.*

If $p(d+1) \leq 1/e$, then with positive probability, none of the events in $\mathcal{E}$ occur.

The next probabilistic tool is Azuma's Inequality. Azuma's Inequality is a concentration bound; in other words, Azuma's Inequality shows that with high probability, a random variable is close its mean. The inequality actually says that the probability that a random variable is far from its mean is very small.

Theorem 2.2.4 (Azuma's Inequality)**.** *Let X be a random variable determined by n trials $T_1, \ldots, T_n$. If for each i and any two possible sequences of outcomes $t_1, \ldots, t_{i-1}, t_i$ and $t_1, \ldots t_{i-1}, t_i'$ the following holds:*

$$|E(X|T_1 = t_1 \ldots, T_i = t_i) - E(X|T_1 = t_1, \ldots, T_i = t_i')| \leq c_i$$

then $Pr(|X - E(X)| > t) \leq 2e^{-t^2/\left(2\sum c_i^2\right)}$.

Azuma's Inequality often is used after showing that the expectation of a random variable is high. This enables us to show that a certain structure occurs enough times.

2.3 Properties of the list assignment L

Recall that G is a counterexample to Theorem 2.1.7 with the fewest number of vertices. The lemmas in this section will reveal some aspects of the list assignment L of the vertices of cliques of G. In particular, the available colors of vertices in $(\Delta - 1)$-cliques are analyzed. Recall that Lemma 2.2.1 shows that G cannot have $K_6 \vee B$ as an induced subgraph whenever $\omega(B) \leq |B| - 2$.

Definition 2.3.1. Given a partial L-coloring f of G, an uncolored vertex v of degree $\Delta+1-i$ in G is *safe* if there exists a subset Z of $N(v)$ with $3-i$ vertices such that for every vertex z in Z, either $f(z) \in f(N(v) - Z)$ or $f(z) \notin L(v)$.

Given a partial L-coloring f on G, at most $\Delta - 2$ colors can appear in the neighborhood of a safe vertex, by definition. Since each vertex of G has $\Delta - 1$ available colors, there is always a color in $L(v)$ for a safe vertex v that is not used on $N(v)$ and therefore can be used on v. This will be the general idea of the proofs regarding colorings in this section: ensure a partial L-coloring of G that colors all vertices except the safe vertices, and then extend the partial L-coloring to an L-coloring of G.

We will first show that the lists of all but at most one vertex in a clique of G have many colors in common. This lemma will be used in Section 5 when we show that there exists an L-coloring of G. Given a partial L-coloring f on G, let $L_f(v)$ denote the remaining available colors on v; in other words, $L_f(v) = L(v) - \{f(u) : u \in N(v) \text{ and } f(u) \text{ is defined}\}$.

Lemma 2.3.2. *If C is a clique of G, then there exists $C' \subset C$ such that $|C'| = |C| - 1$ and $|L(x) \cap L(y)| \geq |C| - 3$ for all $x, y \in C'$.*

Proof. Let f be an L-coloring of $G - C$, which exists by the minimality of G. For $v \in C$, since v has at most $\Delta - (|C| - 1)$ neighbors outside C, it follows that $|L_f(v)| \geq |C| - 2$. Since G is a smallest counterexample, a system of distinct representatives for $\{L_f(v) : v \in C\}$ does not exist. Thus, by Hall's theorem, there exists a subset F of C such that the union of $\{L_f(v) : v \in F\}$ has size less than $|F|$. Since each list under L_f has size at least $|C| - 2$, we know that $|F|$ has size at least $|C| - 1$. If $|F|$ has size $|C| - 1$, then every vertex in F has the same list under L_f, which is of size $|C| - 2$ so we are done. Otherwise, F is C, and $\bigcup_{v \in C} L_f(C)$ has at most $|C| - 1$ elements. We may assume that some two vertices x and y

have distinct lists, since otherwise we are done. Now the union of these lists under L_f has size exactly $|C|-1$ and every vertex in C has at most one color missing from $L_f(x) \cup L_f(y)$. $\square$

Next we prove two lemmas that analyze the distribution of colors in the lists of available colors on vertices in $(\Delta-1)$-cliques of G. We first prove a lemma that will be used heavily in the second lemma.

Lemma 2.3.3. *For a $(\Delta-1)$-clique C of G and a vertex $w \notin C$ such that $|N(w) \cap C| \geq 5$, if f is a partial L-coloring on $G-C-w$, then $L_f(u) = L_f(v)$ for all $u, v \in N(w) \cap C$.*

Proof. Let $A = N(w) \cap C$. Since w cannot be adjacent to every vertex of C, there must be a vertex x in $C-A$. Let f be an L-coloring of $G-C-w$, which exists by the minimality of G. Since each vertex in A has at most one neighbor not in $C \cup \{w\}$, each list under L_f has size at least $\Delta-2$ for a vertex in A. By similar logic, $|L_f(w)| \geq 4$ and $|L_f(x)| \geq \Delta-3$.

We will show that the list under L_f is the same for all vertices in A. Assume for the sake of contradiction that there exist $u, v \in A$ such that $L_f(u) \neq L_f(v)$. If there exists a color $c \in L_f(w) \cap L_f(x)$, then by using c on both w and x, the L-coloring f on $G-C-w$ can be extended to G by Hall's Theorem, which is a contradiction. Thus, $L_f(w) \cap L_f(x) = \emptyset$, which implies that $|L_f(w) \cup L_f(x)| \geq \Delta+1$. If $c \in (L_f(w) \cup L_f(x)) - L_f(A)$, then by coloring w and x using c and an arbitrary color, f can be extended to G by Hall's Theorem, which is again a contradiction. This implies that $|L_f(A)| \geq \Delta+1$. If we cannot extend f to G by coloring w and x arbitrarily from their respective lists under L_f , then there must exist a nonempty $T \subseteq A$ such that $|L_f(T)| < |T|$. Since $|L_f(A)| \geq \Delta+1$, it must be that $|T| = \Delta-3$ and $|L_f(T)| = \Delta-4$. By coloring w and x using a color in $(L_f(w) \cup L_f(x)) - L_f(T)$ and another color, f can be extended to G by Hall's Theorem, which is a contradiction. $\square$

Lemma 2.3.4. *For a $(\Delta-1)$-clique C of G and a vertex $w \notin C$ such that $|N(w) \cap C| \geq 5$, the following holds:*

(i) each vertex in $N(w) \cap C$ has degree Δ;

(ii) there exists a set S of $\Delta - 2$ colors that are in $L(v)$ for every $v \in N(w) \cap C$;

(iii) each vertex $y \notin C \cup \{w\}$ has at most four neighbors in $N(w) \cap C$;

(iv) for each $v \in N(w) \cap C$, the color in $L(v) - S$ appears in the lists of at most 5 vertices in $N(w) \cap C$.

Proof. of (i) and (ii). Let f be an L-coloring of $G - C - w$, which exists by the minimality of G. Assume for the sake of contradiction that there exists a vertex $v \in N(w) \cap C$ with degree $\Delta - 1$, which implies $|L(v)| = |L_f(v)| = \Delta - 1$. By Lemma 2.3.3, the L_f lists are all the same for vertices in $N(w) \cap C$. Therefore, the size of the L_f lists must all be $\Delta - 1$ for vertices in $N(w) \cap C$. By reasoning as in the proof of Lemma 2.3.3, we can extend f to G, which is a contradiction. This proves both (i) and (ii). □

Proof. of (iii). If the claim fails, then there exists a vertex $y \notin C \cup \{w\}$ that has at least five neighbors in $N(w) \cap C$. Let $x \in C - N(w)$ and let $z \in C - N(w) - x$ be a vertex not adjacent to y. Such a z must exist since if not, then y and w are adjacent to every vertex in C except x. Now, x, w, y form an independent set since adding any edge would create a Δ-clique. This implies that G has $E_3 \vee K_6$ as an induced subgraph; this is a contradiction to Lemma 2.2.1, since $E_3 \vee K_6$ is d_1-choosable.

Let f be an L-coloring of $G - C - w - y$, which exists by the minimality of G. Since y and w each have at least five neighbors in C, it follows that $|L_f(y)| \geq 4$ and $|L_f(w)| \geq 4$. By similar reasoning, $|L_f(x)| \geq \Delta - 3$ and $|L_f(z)| \geq \Delta - 3$. Let $v, u \in N(w) \cap N(y) \cap C$ so that $|L_f(v)| = |L(v)| = \Delta - 1$. Whenever y is L-colored, that partial coloring g on G is an L-coloring on $G - C - w$. Recall that by Lemma 2.3.3, vertices in $N(y) \cap N(w) \cap C$ must have the same L_g list. In particular, $L_g(v) = L_g(u)$.

Assume $L_f(w) \cap L_f(x) \neq \emptyset$ and $L_f(y) \cap L_f(z) \neq \emptyset$. If $|(L_f(w) \cap L_f(x)) \cup (L_f(y) \cap L_f(z))| \geq 2$, then we can find two different colors c and c' where we can color w, x with c and y, z with c'. We can now color every vertex in $C - N(y) \cap N(w)$ first since each vertex is adjacent to two uncolored vertices (y and w). We can then color the vertices in $N(y) \cap N(w) \cap C$ to complete an L-coloring of G since every vertex in $N(y) \cap N(w) \cap C$ is safe.

Now assume $L_f(w) \cap L_f(x) = L_f(y) \cap L_f(z) = \{c\}$ for some color c. In this case, $|(L_f(w) - L_f(x)) \cup (L_f(x) - L_f(w)) - L_f(v)| \geq 1$, which implies that there is a color c' in $L_f(w) \cup L_f(x)$ that is not in $L_f(v)$. So now we can color y and z with c and color either w or x with c' first. We can color uncolored vertices in $C - N(y)$, since each vertex is adjacent to two uncolored vertices (u and v). We can then finish off the L-coloring by coloring each vertex in $N(y)$, since each vertex in $N(y) \cap C$ is safe.

The remaining case is when $L_f(w) \cap L_f(x) = \emptyset$ or $L_f(y) \cap L_f(z) = \emptyset$. Without loss of generality, assume $L_f(w) \cap L_f(x) = \emptyset$, which implies $|L_f(w) \cup L_f(x) - L_f(v)| \geq \Delta + 1 - (\Delta - 1) = 2$. If also $L_f(y) \cap L_f(z) = \emptyset$, then by the same reasoning $|L_f(y) \cup L_f(z) - L_f(v)| \geq 2$. Now, color y or z with a color $c' \in L_f(y) \cup L_f(z) - L_f(v)$, and color w or x with a color c that is not in $L_f(v) \cup \{c'\}$. Such color c exists since $|L_f(w) \cup L_f(x) - L_f(v)| \geq 2$.

If $L_f(y) \cap L_f(z) \neq \emptyset$, then there exists a color c' in $L_f(y) \cap L_f(z)$. Now color y and z with c', and color w or x with a color c that is not in $L_f(v) \cup \{c'\}$. Such color c exists since $|L_f(w) \cup L_f(x) - L_f(v)| \geq 2$.

Either way, we can color every vertex in $C - u - v$, since each vertex is adjacent to two uncolored vertices (u and v). We can then color u due to the following two properties: (1) u is adjacent to an uncolored vertex (v) and (2) either there exists a color in $N(u)$ that is not in the list of u or there exists a repeated color in $N(u)$. Now since v is safe, we can L-color v, and we have extended a partial L-coloring f of $G - C - w - y$ to G. This completes the proof of (iii). □

Proof. of (iv). If the claim fails, then some color in $L(v) - S$ appears in a set P of at least 6 vertices. Consider the set Q of neighbors of vertices in P that are not in $C \cup \{w\}$. Note that the neighbors of vertices in Q that are in P partition P, since each vertex in P has exactly one neighbor not in $C \cup \{w\}$. By (iii), since a vertex outside of $C \cup \{w\}$ has at most 4 neighbors in $N(w) \cap C$, there must be at least 2 vertices in Q. Also, Q must be an independent set. Otherwise, if there is an edge with two endpoints in Q, then the endpoints of this edge will receive different colors in an L-coloring f of $G - C - w$, which exists by the minimality of G. Now, the vertices in P cannot have the same list under L_f, which is a contradiction to Lemma 2.3.3.

For any edge e with both endpoints in Q, the graph $G-C-w+e$ has fewer vertices than G, has maximum degree Δ, and has clique number at most $\Delta - 1$. By the minimality of G, there is an L-coloring of $G - C - w + e$, which is a contradiction since the two endpoints of e will receive different colors. Thus, adding an edge e with endpoints in Q must create a Δ-clique in $G - C - w$.

If $v \in Q$ has $d_P(v) \geq 3$, this is impossible since v cannot have $\Delta - 2$ neighbors outside P. Hence $|Q| \geq 3$ and $d_P(v) \leq 2$ for all $v \in Q$. Note that three vertices $x, y, z \in Q$ must have at least $\Delta - 3$ common neighbors not in $C \cup \{w\}$. These common neighbors and x, y, z induce a copy of $E_3 \vee K_6$, which is d_1-choosable, which is a contradiction. $\square$

Definition 2.3.5. Let C be a $(\Delta - 1)$-clique of G, and let $w \notin C$ be a vertex such that $|N(w) \cap C| \geq 6$. The *core* of $N(w) \cap C$ is the set S of $\Delta - 2$ colors that are in $L(v)$ for every $v \in N(w) \cap C$. For a vertex $v \in N(w) \cap C$, the *special color* of v is the color in $L(v) - S$, and the *external neighbor* of v is the one vertex that is adjacent to v that is not in $C \cup \{w\}$.

2.4 A Decomposition of G

In this section, we will construct a decomposition of G that will allow us to analyze G in smaller pieces. This will facilitate the probabilistic argument in the next section. Here is a definition and a couple lemmas from [22].

Definition 2.4.1. Given a graph H and a list assignment L on H, let the *pot* of H, denoted $\mathrm{Pot}(L)$, be $\bigcup_{v \in V(H)} L(v)$.

Lemma 2.4.2 (Small Pot Lemma [22])**.** *Let H be a graph and $f : V(H) \to \mathbb{N}$ with $f(v) < |H|$ for all $v \in V(H)$. If H is not f-choosable, then H has a list assignment L where $|L(v)| = f(v)$ for each vertex v such that $|Pot(L)| < |H|$.*

Lemma 2.4.3. *For any graph B with $\delta(B) \geq \frac{|B|}{2} + 1$ and $\omega(B) \leq |B| - 2$, the graph $K_1 \vee B$ is d_1-choosable.*

Proof. By the Small Pot Lemma, it suffices to prove that all list assignments L on $K_1 \vee B$ with $|L(v)| = f(v) - 1$ for each vertex v with $|\mathrm{Pot}(L)| \leq |B|$ are L-colorable. Let L be such a list assignment on $K_1 \vee B$.

First, suppose B contains disjoint nonadjacent pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$. Since $|L(x_i)| + |L(y_i)| \geq |B| + 2$, we have $|L(x_i) \cap L(y_i)| \geq 2$ for each i. Color x_1 and y_1 with $c_1 \in L(x_1) \cap L(y_1)$ and color x_2 and y_2 with $c_2 \in L(x_2) \cap L(y_2) - c_1$. By the minimum degree condition on B, each component of $B - \{x_1, y_1, x_2, y_2\}$ has a vertex joined to $\{x_1, y_1\}$ or $\{x_2, y_2\}$. Hence we can complete the coloring to all of B and then to the K_1. Thus L is good.

So, we may assume there are no disjoint nonadjacent pairs. Now let K be a maximum clique in B. Then we know $|K| \leq |B| - 2$ so we can pick $x, y \in B - K$. The only possibility is that there is $z \in K$ such that both x and y are joined to $K - z$. Since K is maximum x is not adjacent to y and hence B is a $K_{|B|-3} \vee E_3$. By Lemma 2.2.1, $|B| \leq 4$. Since $d_B(y) = |B| - 3$, this violates our minimum degree condition on B. □

Now we actually construct a decomposition using a definition on page 158 of [37].

Definition 2.4.4. A vertex v of G is *d-sparse* if the subgraph induced by its neighborhood contains fewer than $\binom{\Delta}{2} - d\Delta$ edges. Otherwise, v is *d-dense*.

Lemma 2.4.5. *We can partition $V(G)$ into sets $S, D_1, \ldots, D_l$ and specify vertices $w_1, \ldots, w_l$ so that*

(i) each vertex of S is d-sparse;

(ii) each D_i contains a vertex w_i such that $D_i - w_i$ is a clique of size at least $\Delta - 8\Delta^{9/10} + 1$;

(iii) no vertex outside of D_i has more than $\frac{3\Delta}{4}$ neighbors in D_i and w_i has at least $\frac{3\Delta}{4}$ neighbors in D_i.

Proof. Let $C_1, \ldots, C_s$ be the maximal cliques in G with at least $\frac{3\Delta}{4} + 1$ vertices. Suppose $|C_i| \leq |C_j|$ and $C_i \cap C_j \neq \emptyset$. Then $|C_i \cap C_j| \geq |C_i| + |C_j| - (\Delta + 1) \geq 6$. It follows from Lemma 2.2.1 that $|C_i - C_j| \leq 1$. Now suppose C_i intersects C_j and C_k. By the above, $|C_i \cap C_j| \geq \frac{3\Delta}{4}$. Hence $|C_i \cap C_j \cap C_k| \geq \frac{\Delta}{2} \geq 6$. By Lemma 2.2.1 we see that $\omega(G[C_i \cup C_j \cup C_k]) \geq |C_i \cup C_j \cup C_k| - 1$ which is impossible since each of C_i, C_j, C_k are maximal. Hence $\bigcup_{i \in [s]} C_i$ can be partitioned into sets $F_1, \ldots, F_r$ so that each F_j is either one of the C_i or one of the C_i and an extra vertex w_i with at least $\frac{3\Delta}{4}$ neighbors in C_i.

Put $d = \Delta^{9/10}$ and let $D_1, \ldots, D_l$ be all the F_j such that some vertex in F_j is d-dense and let S be $V(G) - \bigcup_{i \in [l]} D_i$. Then (iii) follows by construction. It remains to check (i) and (ii).

We show that if $v \in V(G)$ is d-dense, then it is in a $(\Delta - 8d + 2)$-clique. Since we know that any $v \in S$ is either in no $(\frac{3\Delta}{4} + 1)$-clique (and hence in no $(\Delta - 8d + 2)$-clique) or is d-sparse, (i) follows. Also, since each F_j contains a d-dense vertex, (ii) follows as well.

So, suppose $v \in V(G)$ is d-dense but in no $(\Delta - 8d + 2)$-clique. Then applying Lemma 2.4.3 repeatedly, we get a sequence $y_1, \ldots, y_{8d} \in N(v)$ such that

$$|N(y_i) \cap (N(x) - \{y_1, \ldots, y_{i-1}\})| \leq \frac{1}{2}(\Delta + 1 - i).$$

Hence the number of non-edges in v's neighborhood is at least

$$\frac{1}{2} \sum_{i=1}^{8d} (\Delta - i) > d\Delta.$$

□

Definition 2.4.6. Let $K_i = \begin{cases} C_i & \text{if } D_i = C_i \\ C_i \cap N(w_i) & \text{if } D_i = C_i \cup \{w_i\} \end{cases}$.

Definition 2.4.7. Using the notation used in Lemma 2.4.5, partition the set of C_i into the following three sets (if some C_i can be either (ii) or (iii), then just choose an arbitrary one):

(i) $\mathcal{P}_1$: the set of C_i such that $|C_i| \leq \Delta - 2$;

(ii) $\mathcal{P}_2$: the set of C_i such that $|C_i| = \Delta - 1$ and every vertex outside C_i has at most $\Delta^{0.29}$ neighbors in C_i;

(iii) $\mathcal{P}_3$: the set of C_i such that $|C_i| = \Delta - 1$ and some vertex w_i' outside C_i has more than $\Delta^{0.29}$ neighbors inside C_i; let $K_i' = N(w_i') \cap C_i$. If w_i is defined, then let $w_i' = w_i$.

Now we prove a structural lemma that will be crucial in the following sections. We use a lemma from [22] to prove the lemma needed.

Lemma 2.4.8 ([22])**.** *Let H be a d_0-choosable graph such that $F := K_1 \vee H$ is not d_1-choosable and let L be a bad d_1-assignment on F minimizing $|Pot(L)|$. If some nonadjacent pair in H has intersecting lists, then $|Pot(L)| \leq |H| - 1$.*

Lemma 2.4.9. *Each $v \in C_i$ of G has at most one neighbor outside of C_i with more than 4 neighbors in C_i, and no such neighbor if v has degree $\Delta - 1$.*

Proof. Suppose there exists $v \in C_i$ with two neighbors $w_1, w_2 \in V(G) - C_i$, each with 5 or more neighbors in C_i. Put $Q := G[\{w_1, w_2\} \cup C_i - v]$, so that v is joined to Q and hence $K_1 \vee Q$ is an induced subgraph of G. We will show that $K_1 \vee Q$ must be d_1-choosable. Note that Q is d_0-choosable since it contains a K_4 without one edge. Let L be a bad d_1-assignment on $K_1 \vee Q$ minimizing $|\mathrm{Pot}(L)|$.

First, suppose there are different $z_1, z_2 \in C_i$ such that $\{w_1, z_1\}$ and $\{w_2, z_2\}$ are independent. By the Small Pot Lemma 2.4.2, $|\mathrm{Pot}(L)| \leq |Q|$. Thus $|L(w_1)| + |L(z_1)| \geq 4 + |Q| - 3 > |\mathrm{Pot}(L)|$ and therefore w_1 and w_2 have intersecting lists. Applying Lemma 2.4.8 shows that $|\mathrm{Pot}(L)| \leq |Q| - 1$.

Now $|L(w_j)| + |L(z_j)| \geq 4 + |Q| - 3 \geq |\mathrm{Pot}(L)| + 2$. Hence $|L(w_j) \cap L(z_j)| \geq 2$. Pick $x \in N(w_1) \cap \{C_i - v - z_2\}$. Then after coloring each pair $\{w_1, z_1\}$ and $\{w_2, z_2\}$ with a different color, we can finish the coloring because we saved a color for x and two colors for v.

By maximality of C_i, neither w_1 nor w_2 can be adjacent to all of C_i hence it must be the case that there is $y \in C_i$ such that w_1 and w_2 are joined to $C_i - y$. If w_1 and w_2 aren't adjacent, then G contains $K_6 \vee E_3$ contradicting Lemma 2.2.1. Hence C_i intersects the larger clique $\{w_1, w_2\} \cup C_i - \{y\}$, this is impossible by the definition of C_i.

When v is low, an argument similar to the above shows that there can be no z_1 in C_i with $\{w_1, z_1\}$ independent, and hence $C_i \cup \{w_1\}$ is a clique contradicting maximality of C_i. $\square$

2.5 An L-Coloring of G

Now we will show the existence of a partial L-coloring f that can be extended to the entire graph G in a greedy fashion, which contradicts the assumption that G is a counterexample. We will do this by applying the naive coloring procedure (Definition 2.5.1), Algorithm 2.5.3, and then Algorithm 2.6.1.

The two goals of this section are to prove that such a partial L-coloring f exists and to demonstrate how to manipulate the coloring f on $G - \bigcup_{C_i \in \mathcal{P}_3} C_i$ to get ready for Section 2.6, which is where f is extended to all of G.

For the sake of presentation, let us abuse notation and redefine G' to be the disjoint union of two copies of G, and then add an edge between the two copies of each vertex of degree $\Delta - 1$. The advantage of this is that G' is now Δ-regular. Also, extend the list assignment L of G to a list assignment of G' in the natural way. If there is an L-coloring of G', then G has an L-coloring, which contradicts the assumption that G is a minimum counterexample.

We first apply the *naive coloring procedure* from [37] to obtain a partial L-coloring of G'.

Definition 2.5.1. The *naive coloring procedure* is defined as the following:

(i) For each vertex v, choose a color $c \in L(v)$ uniformly at random and propose c to be its color.

(*ii*) For each vertex v in $G' - \bigcup_{C_i \in \mathcal{P}_3} C_i$, if there is no vertex in $N(v)$ that has the same proposed color c as v, then accept c to be the color of v.

Using the Lovász Local Lemma, we will show that with positive probability, the naive coloring procedure will produce a partial coloring f of G' in which none of the bad events happen. The bad events are defined in a way that if none of them happen, then we can extend the partial L-coloring f to an L-coloring of $G' - \bigcup_{C_i \in \mathcal{P}_3} C_i$ in a greedy fashion.

For $C_i \in \mathcal{P}_3$, recall that w_i' is a vertex outside of C_i with more than $\Delta^{0.29}$ neighbors inside C_i and $K_i' = N(w_i') \cap C_i$.

Definition 2.5.2. The *bad events* are the following events:

(*i*) For a sparse vertex v, let $\mathcal{S}_v$ be the event that v is not safe.

(*ii*) For $C_i \in \mathcal{P}_1 \cup \mathcal{P}_2$, let $\mathcal{E}_i$ be the event that K_i does not contain two uncolored safe vertices.

(*iii*) For $C_i \in \mathcal{P}_3$, let $\mathcal{E}_i$ be the event that K_i' does not contain two vertices that can be colored using their special colors.

To apply the Lovász Local Lemma, we need to bound the dependencies among the events and bound the probability of each event. A bad event associated with C_i depends only on the colors of the vertices that are distance at most 2 away from C_i. This implies that if two cliques have a path of length at most 4 connecting them, then the associated events are not (mutually) independent. This implies that a bad event is mutually independent to all but at most Δ^5 events since each clique has less than Δ vertices. The task of proving that the probability of each (bad) event is at most Δ^{-6} will be done in the following (sub)sections.

Assuming none of the bad events happen, we will obtain a partial L-coloring f of $G' - \bigcup_{C_i \in \mathcal{P}_3} (C_i - w_i')$ using Algorithm 2.5.3.

Algorithm 2.5.3.

(i) For each vertex v in $G' - \bigcup_{C_i \in \mathcal{P}_3}(C_i - w_i')$ that is not safe, color v with a color in $L(v)$.

(ii) For each vertex v in $G' - \bigcup_{C_i \in \mathcal{P}_3}(C_i - w_i')$ that is safe, color v with a color in $L(v)$.

(iii) Uncolor each vertex w_i' that is colored.

Note that every vertex in $G' - \bigcup_{C_i \in \mathcal{P}_3} C_i$ that is not safe must be either w_i or in some $C_i \in \mathcal{P}_1 \cup \mathcal{P}_2$, and therefore is adjacent to at least two uncolored safe vertices; this means we can find a color in its list to use on the vertex.

2.5.1 $Pr(\mathcal{S}_v) \leq \Delta^{-6}$

Recall that v has at least $\Delta^{1+\frac{9}{10}}$ nonadjacent pairs of vertices in its neighborhood. Let $A = \{x \in N(v) : |L(x) \cap L(v)| \geq \frac{2}{3}\Delta\}$ and $B = N(v) - A$. Note that for $x, y \in A$, we have $|L(x) \cap L(y)| \geq \frac{\Delta}{3}$ and for $x \in B$ we have $|L(x) - L(v)| \geq \frac{\Delta}{3}$. Let b be the number of nonadjacent pairs in $N(v)$ that intersect B, so that $G'[A]$ contains at least $\Delta^{1+\frac{9}{10}} - b$ nonadjacent pairs and $b \leq |B|\Delta$. Let A_v be the random variable that counts the number of colors that appear at least twice in $N(v)$. Let B_v be the random variable that counts the number of colors that appear in $N(v)$ that are not in the list of $L(v)$. Let $Z_v = A_v + B_v$ so that $E[Z_v] = E[A_v] + E[B_v]$. We will prove that $E[Z_v]$ is high, and then use Azuma's Inequality to prove that with high probability, Z_v is concentrated around its mean.

For A_v, let $x, y \in A$ be nonadjacent. We will actually calculate the number of colors that appear exactly twice in $N(v)$. Since $|L(x) \cap L(y)| \geq \frac{\Delta}{3}$, the probability that x, y get the same color and retain it and this color is not used on the rest of $N(v)$ is at least $\frac{\Delta}{3}(\Delta - 1)^{-2}(1 - \frac{1}{\Delta - 1})^{|N(v) \cup N(x) \cup N(y)|} \geq \Delta^{-1}3^{-5}$. Thus, $E[A_v] \geq (\Delta^{1+\frac{9}{10}} - b)\Delta^{-1}3^{-5}$.

For B_v, let $x \in B$. Since $|L(x) - L(v)| \geq \frac{\Delta}{3}$, the probability that x gets a color not in $L(v)$ and retains it and is not used on the rest of $N(v)$ is at least $\frac{\Delta}{3}\frac{1}{\Delta - 1}(1 - (\Delta - 1)^{-1})^{|N(v) \cup N(x)|} \geq 3^{-4}$. Thus $E[B_v] \geq \frac{|B|}{3^4} \geq \frac{b}{3^4\Delta}$. Hence, $E[Z_v] \geq \Delta^{\frac{9}{10}}3^{-5}$.

Now we use Azuma's Inequality to show that the probability that Z_v deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of Z_w change by at most

c_w when changing the color of w.

Changing the color of w from α to β will only affect Z_v if some neighbor of w that is in $N(v)$ receives either α or β. This occurs with probability at most $\frac{2d_w}{\Delta-1}$, where d_w is the number of neighbors of w that are in $N(v)$. Therefore, by changing the color of w, the conditional expectation of Z_v changes by at most $c_w = \frac{4d_w}{\Delta-1}$. Since the d_w sum is at most Δ^2, the sum of these c_w is at most 5Δ. As each c_w is at most 5, we see that the sum of c_w^2 is at most 25Δ.

Hence, the sum of all the c_w is at most 25Δ. Applying Azuma's Inequality yields $Pr(\mathcal{S}_v) \leq \Delta^{-6}$ for sufficiently large Δ. See Calculation 2.7.9.

2.5.2 $Pr(\mathcal{E}_i) \leq \Delta^{-6}$ for $C_i \in \mathcal{P}_1$

Let C_i' be a subset of C_i with one less vertex where every two vertices in C_i' have at least $|C_i| - 3$ colors in common in their lists; such a C_i' exists by Lemma 2.3.2. Note that $|C_i'| = |C_i| - 1 \geq \Delta - 8\Delta^{9/10} \geq 0.92\Delta$ for $\Delta \geq 10^{20}$. Let $\mathcal{T}_i$ be a maximum set of disjoint P_3 where the center vertex is in C_i' and each of the other two vertices is not in C_i' and has at most 4 neighbors in C_i'.

Claim 2.5.4. *There are at least* 0.1314Δ *such* P_3.

Proof. Consider a maximal set of P_3. Let A be the central vertices and let B be the endpoints of these P_3. Each vertex in B has at most 3 neighbors in $C_i' - A$ and by Lemma 2.4.9 and maximality, each vertex in $C_i' - A$ has at most 2 neighbors in $G' - C_i' - B$. Thus, $6|A| = 3|B| \geq ||C_i' - A, B|| \geq |C_i'| - |A|$. Hence, $|A| \geq \frac{|C_i'|}{7} \geq \frac{0.92\Delta}{7} \geq 0.13142\Delta$. □

Consider a set T_i of vertices of a set of 0.1314Δ such P_3. For some fixed P_3, we want to bound the probability that the center vertex c is uncolored and safe, and the colors used on the two end vertices, a and b, are used on none of the rest of T_i. To do so, we distinguish three cases.

Case 1. When $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$.

For $\alpha \in L(a) - L(c)$, $y \in C'_i - T_i - N(b)$, $\beta \in L(b) \cap L(y)$, $z \in C'_i - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different and y, z, c are all different, let $A_{\alpha,\beta,\gamma,y,z}$ be the event that all of the following holds:

(i) α is used on a and none of the rest of $N(a) \cup T_i$;

(ii) β is used on b and y and none of the rest of $N(b) \cup N(y) \cup T_i$;

(iii) γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by Calculation 2.7.1,

$$\begin{aligned} Pr(A_{\alpha,\beta,\gamma,y,z}) &\geq (\Delta - 1)^{-5} \left(1 - \frac{1}{\Delta - 1}\right)^{|T_i \cup N(a)| + |T_i \cup N(b) \cup N(y)| + |T_i|} \\ &\geq \Delta^{-5} \left(1 - \frac{1}{\Delta - 1}\right)^{3.3942\Delta} \geq \Delta^{-5} e^{-3.3943}. \end{aligned}$$

The $A_{\alpha,\beta,\gamma,y,z}$ are disjoint for different sets of indices. Since

$$|L(a) - L(c)| \geq \frac{\Delta}{3} - 1 \geq 0.3332\Delta,$$

we have 0.3332Δ choices for α. For y, we have at least

$$|C'_i| - |T_i \cap C'_i| - |N(b) \cap C'_i| - 1 \geq 0.92\Delta - 0.1314\Delta - 5 \geq 0.7885\Delta$$

choices. For each y, we have

$$0.92\Delta - 2 + \frac{2}{3}\Delta - (\Delta - 1) - 1 = 0.92\Delta - \frac{\Delta}{3} - 2 \geq 0.5866\Delta$$

choices for β since $|L(y) \cap L(c)| \geq |C_i| - 3$. There are

$$|C'_i| - |T_i| - 2 \geq 0.92\Delta - 0.1314\Delta - 2 \geq 0.7885\Delta$$

choices for z. Since $|L(z) \cap L(c)| \geq |C_i| - 3$, there are $0.92\Delta - 4 \geq 0.9199\Delta$ choices for γ.

Thus, the probability that $A_{\alpha,\beta,\gamma,y,z}$ holds for some choice of indices is at least

$$\Delta^{-5}e^{-3.3943} \cdot 0.3332\Delta \cdot (0.7885\Delta)^2 \cdot 0.5866\Delta \cdot 0.9199\Delta \geq 0.00375.$$

Case 2. When $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| < \frac{2}{3}\Delta$.

For $\alpha \in L(a) - L(c)$, $\beta \in L(b) - L(c)$, $z \in C_i' - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different and $z \neq c$, let $A_{\alpha,\beta,\gamma,z}$ be the event that all of the following holds:

(i) α is used on a and none of the rest of $N(a) \cup T_i$;

(ii) β is used on b and none of the rest of $N(b) \cup T_i$;

(iii) γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by Calculation 2.7.1,

$$\begin{aligned} Pr(A_{\alpha,\beta,\gamma,z}) &\geq (\Delta-1)^{-4}\left(1 - \frac{1}{\Delta - 1}\right)^{|T_i \cup N(a)| + |T_i \cup N(b)| + |T_i|} \\ &\geq \Delta^{-4}\left(1 - \frac{1}{\Delta - 1}\right)^{2.3942\Delta} \geq \Delta^{-4}e^{-2.3943}. \end{aligned}$$

The $A_{\alpha,\beta,\gamma,z}$ are disjoint for different sets of indices. Similarly to Case 1, there are at least 0.3332Δ choices for α, at least 0.3332Δ choices for β, and at least 0.9199Δ choices for γ for each of the at least 0.7885Δ choices for z.

Thus, the probability that $A_{\alpha,\beta,\gamma,z}$ holds for some choice of indices is at least

$$\Delta^{-4}e^{-2.3943} \cdot (0.3332\Delta)^2 \cdot 0.7885\Delta \cdot 0.9199\Delta \geq 0.00734.$$

Case 3. When $|L(a) \cap L(c)| \geq \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$.

For $x \in C_i' - T_i - N(a)$, $\alpha \in L(a) \cap L(c)$, $y \in C_i' - T_i - N(b)$, $\beta \in L(b) \cap L(y)$, $z \in C_i' - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different and x, y, z, c are all different, let $A_{\alpha,\beta,\gamma,x,y,z}$

be the event that all of the following hold:

(i) α is used on a and x and none of the rest of $N(a) \cup N(x) \cup T_i$;

(ii) β is used on b and y none of the rest of $N(b) \cup N(y) \cup T_i$;

(iii) γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by Calculation 2.7.1,

$$\begin{aligned} Pr(A_{\alpha,\beta,\gamma,x,y,z}) &\geq (\Delta-1)^{-6}\left(1-\frac{1}{\Delta-1}\right)^{|T_i \cup N(a) \cup N(x)|+|T_i \cup N(b) \cup N(y)|+|T_i|} \\ &\geq \Delta^{-6}\left(1-\frac{1}{\Delta-1}\right)^{4.3942\Delta} \geq \Delta^{-6}e^{-4.3943}. \end{aligned}$$

The $A_{\alpha,\beta,\gamma,x,y,z}$ are disjoint for different sets of indices. Similarly to Case 1, there are at least 0.5866Δ choices for α for each of the at least 0.7885Δ choices for x, at least 0.5866Δ choices for β for each of the at least 0.7885Δ choices for y, and at least 0.9199Δ choices for γ for each of the at least 0.7885Δ choices for z.

Thus, the probability that $A_{\alpha,\beta,\gamma,x,y,z}$ holds for some choice of indices is at least

$$\Delta^{-6}e^{-4.3943} \cdot (0.7885\Delta)^3 \cdot (0.5866\Delta)^2 \cdot 0.9199\Delta \geq 0.00191.$$

Since we have 0.1314Δ triples, the expected number of uncolored safe vertices X_i is at least $0.00191 \cdot 0.1314\Delta \geq 2.5 \cdot 10^{-4}\Delta$.

Now we use Azuma's Inequality to show that the probability that X_i deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of X_i change by at most c_v when changing the color of v.

If $v \in T_i \cup C_i'$, then $c_v \leq 2$ since changing the color on v affects X_i by at most 2 for any given assignment of colors to the remaining vertices. Thus, the sum of the c_v^2 is at most $4|T_i \cup C_i'| \leq 4(0.1314\Delta + \Delta - 2) \leq 8\Delta$.

If $v \in V(G') - T_i - C_i'$, then changing the color of v from α to β will only affect X_i if some neighbor of v that is in $T_i \cup C_i'$ receives either α or β. This occurs with probability at most $\frac{2d_v}{\Delta-1}$, where d_v is the number of neighbors of v that are in $T_i \cup C_i'$. Therefore, by changing the color of v, the conditional expectation of X_i changes by at most $c_v = \frac{4d_v}{\Delta-1}$. Since the d_v sum is at most $\Delta\left(\Delta - 2 + 0.2628\Delta\right) \leq 2\Delta^2$, the sum of these c_v is at most $\frac{4}{\Delta}2\Delta^2 = 8\Delta$. As each c_v is at most 4, we see that the sum of c_v^2 is at most 32Δ.

Hence, the sum of all the c_v is at most 40Δ. Applying Azuma's Inequality yields $Pr(\mathcal{E}_i) \leq \Delta^{-6}$ for sufficiently large Δ. See Calculation 2.7.2.

2.5.3 $Pr(\mathcal{E}_i) \leq \Delta^{-6}$ for $C_i \in \mathcal{P}_2$

This subsection is similar to the previous subsection, except a linear (in terms of Δ) number of P_3 is not guaranteed. Let C_i' be a subset of C_i with one less vertex where every two vertices in C_i' have at least $|C_i| - 3$ colors in common in their lists; such a C_i' exists by Lemma 2.3.2. Let $\mathcal{T}_i$ be a maximum set of disjoint P_3 where the center vertex is in C_i' and each of the other two vertices is not in C_i' and has at most 4 neighbors in C_i'. Since at most one of the two endpoints can have more than 4 neighbors in C_i by Lemma 2.4.9, it follows that $|\mathcal{T}_i| \geq \frac{\Delta-1}{\Delta^{0.29}+4}$. By Calculation 2.7.3, the number of P_3 is at least $0.9999\Delta^{0.71}$. Consider a set T_i of vertices of $0.9999\Delta^{0.71}$ such P_3 that are in $\mathcal{T}_i$.

For some such fixed path, we want to bound the probability that the center vertex c is uncolored and safe, and the colors used on the two end vertices, a and b, are used on none of the rest of T_i. To do so, we distinguish three cases.

Case 1. When $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$.

For $\alpha \in L(a) - L(c)$, $y \in C_i' - T_i - N(b)$, $\beta \in L(b) \cap L(y)$, $z \in C_i' - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different and y, z, c are all different, let $A_{\alpha,\beta,\gamma,y,z}$ be the event that all of the following hold:

(*i*) α is used on a and none of the rest of $N(a) \cup T_i$;

(*ii*) β is used on b and y and none of the rest of $N(b) \cup N(y) \cup T_i$;

(*iii*) γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by Calculation 2.7.4,

$$\begin{aligned} Pr(A_{\alpha,\beta,\gamma,y,z}) &\geq (\Delta-1)^{-5}\left(1-\frac{1}{\Delta-1}\right)^{|T_i \cup N(a)|+|T_i \cup N(b) \cup N(y)|+|T_i|} \\ &\geq \Delta^{-5}\left(1-\frac{1}{\Delta-1}\right)^{3\Delta+2.9998\Delta^{0.71}} \geq \Delta^{-5}e^{-3.1}. \end{aligned}$$

The $A_{\alpha,\beta,\gamma,y,z}$ are disjoint for different sets of indices. Since

$$|L(a)-L(c)| \geq \frac{\Delta}{3} - 1 \geq 0.3332\Delta,$$

we have 0.3332Δ choices for α. For y, we have at least

$$|C_i'| - |T_i \cap C_i'| - |N(b) \cap C_i'| \geq 0.92\Delta - 0.9999\Delta^{0.71} - 5 \geq 0.9199\Delta$$

choices. For each y, we have about

$$\frac{2}{3}\Delta + 0.92\Delta + 1 - 3 - (\Delta - 1) - 1 = 0.92\Delta - \frac{\Delta}{3} - 2 \geq 0.5866\Delta$$

choices for β. There are

$$|C_i'| - |T_i| - 2 \geq 0.92\Delta - 0.9999\Delta^{0.71} - 2 \geq 0.9199\Delta$$

choices for z. Since $|L(z) \cap L(c)| \geq |C_i| - 3$, there are $0.92\Delta - 4 \geq 0.9199\Delta$ choices for γ.

Thus, the probability that $A_{\alpha,\beta,\gamma,y,z}$ holds for some choice of indices is at least

$$\Delta^{-5}e^{-3.1} \cdot 0.3332\Delta \cdot (0.9199\Delta)^3 \cdot 0.5866\Delta \geq 0.00685.$$

Case 2. When $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| < \frac{2}{3}\Delta$.

For $\alpha \in L(a) - L(c)$, $\beta \in L(b) - L(c)$, $z \in C'_i - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different and $c \neq z$, let $A_{\alpha,\beta,\gamma,z}$ be the event that all of the following hold:

(i) α is used on a and none of the rest of $N(a) \cup T_i$;

(ii) β is used on b and none of the rest of $N(b) \cup T_i$;

(iii) γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by Calculation 2.7.4,

$$\begin{aligned} Pr(A_{\alpha,\beta,\gamma,z}) &\geq (\Delta - 1)^{-4}\left(1 - \frac{1}{\Delta - 1}\right)^{|T_i \cup N(a)| + |T_i \cup N(b)| + |T_i|} \\ &\geq \Delta^{-4}\left(1 - \frac{1}{\Delta - 1}\right)^{2\Delta + 2.9998\Delta^{0.71}} \geq \Delta^{-4}e^{-2.1}. \end{aligned}$$

The $A_{\alpha,\beta,\gamma,z}$ are disjoint for different sets of indices. Similarly to Case 1, there are at least 0.3332Δ choices for α, at least 0.3332Δ choices for β, and at least 0.9199Δ choices for γ for each of the at least 0.9199Δ choices for z.

Thus, the probability that $A_{\alpha,\beta,\gamma,z}$ holds for some choice of indices is at least

$$\Delta^{-4}e^{-2.1} \cdot (0.3332\Delta)^2 \cdot (0.9199\Delta)^2 \geq 0.01150.$$

Case 3. When $|L(a) \cap L(c)| \geq \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$.

For $x \in C'_i - T_i - N(a)$, $\alpha \in L(a) \cap L(c)$, $y \in C'_i - T_i - N(b)$, $\beta \in L(b) \cap L(y)$, $z \in C'_i - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different and c, z, y, x are all different, let $A_{\alpha,\beta,\gamma,x,y,z}$ be the event that all of the following hold:

(i) α is used on a and x and none of the rest of $N(a) \cup N(x) \cup T_i$;

(ii) β is used on b and y and none of the rest of $N(b) \cup N(y) \cup T_i$;

(iii) γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by Calculation 2.7.4,

$$\begin{aligned} Pr(A_{\alpha,\beta,\gamma,x,y,z}) &\geq (\Delta-1)^{-6}\left(1-\frac{1}{\Delta-1}\right)^{|T_i \cup N(a) \cup N(x)|+|T_i \cup N(b) \cup N(y)|+|T_i|} \\ &\geq \Delta^{-6}\left(1-\frac{1}{\Delta-1}\right)^{4\Delta+2.9998\Delta^{0.71}} \geq \Delta^{-6}e^{-4.1}. \end{aligned}$$

The $A_{\alpha,\beta,\gamma,x,y,z}$ are disjoint for different sets of indices. Similarly to Case 1, there are at least 0.5866Δ choices for α for each of the at least 0.9199Δ choices for x, at least 0.5866Δ choices for β for each of the at least 0.9199Δ choices for y, and at least 0.9199Δ choices for γ for each of the at least 0.9199Δ choices for z.

Thus, the probability that $A_{\alpha,\beta,\gamma,x,y,z}$ holds for some choice of indices is at least

$$\Delta^{-6}e^{-4.1} \cdot (0.9199\Delta)^4 \cdot (0.5866\Delta)^2 \geq 0.00408.$$

Since we have $0.9999\Delta^{0.71}$ triples, the expected number of uncolored safe vertices X_i is at least $4.0 \cdot 10^{-3}\Delta^{0.71}$.

Now we use Azuma's Inequality to show that the probability that X_i deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of X_i change by at most c_v when changing the color of v.

If $v \in T_i \cup C_i'$, then $c_v \leq 2$ since changing the color on v affects X_i by at most 2 for any given assignment of colors to the remaining vertices. Thus, the sum of the c_v^2 is at most $4|T_i \cup C_i'| \leq 4(\Delta - 2 + 0.9999\Delta^{0.71}) \leq 4.001\Delta$.

If $v \in V(G') - T_i - C_i'$, then changing the color of v from α to β will only affect X_i if some neighbor of v that is in $T_i \cup C_i'$ receives either α or β. This occurs with probability at most

$\frac{2d_v}{\Delta-1}$, where d_v is the number of neighbors of v that are in $T_i \cup C_i'$. Therefore, by changing the color of v, the conditional expectation of X_i changes by at most $c_v = \frac{4d_v}{\Delta-1}$. Since the d_v sum is at most $2(\Delta - 1 - 0.9999\Delta^{0.71}) + (\Delta - 1)0.9999\Delta^{0.71} \leq \Delta^{1.71}$ (see Calculation 2.7.5), the sum of these c_v is at most $\frac{4}{\Delta}\Delta^{1.71}$. As each c_v is at most 4, we see that the sum of c_v^2 is at most $4\Delta^{0.71}$.

Hence, the sum of all the c_v is at most $4.001\Delta + 4\Delta^{0.71} \leq 4.01\Delta$ for $\Delta \geq 10^{20}$. Applying Azuma's Inequality yields $Pr(\mathcal{E}_i) \leq \Delta^{-6}$ for sufficiently large Δ. See Calculation 2.7.6.

2.6 $Pr(\mathcal{E}_i) \leq \Delta^{-6}$ for $C_i \in \mathcal{P}_3$

At this point, we have an L-coloring f of $G' - \bigcup_{C_i \in \mathcal{P}_3} (C_i \cup \{w_i'\})$. Recall that a C_i corresponding to this case has a vertex w_i' outside of C_i that has at least $\Delta^{0.29}$ neighbors inside C_i and $K_i' = N(w_i') \cap C_i$. We will extend the coloring f to all of G' by Algorithm 2.6.1.

Algorithm 2.6.1.

(i) For each $C_i \in \mathcal{P}_3$, while it has less than 2 colored vertices in K_i',

  (1) For each special color c where no vertex of C_i is colored with c, for at most one vertex $v \in K_i'$ with $c \in L(v)$, if its external neighbor is in $\bigcup_{C_i \in \mathcal{P}_3}(C_i - K_i')$, then let $f(v) = c$.

  (2) For each special color c where no vertex of C_i is colored with c, for at most one vertex $v \in K_i'$ with $c \in L(v)$, if its external neighbor is colored, then let $f(v) = c$.

(ii) For each vertex in $\bigcup_{C_i \in \mathcal{P}_3}(C_i - K_i')$, color it with a color in its list.

(iii) For each uncolored vertex of G' that is not safe, color it with a color in its list.

(iv) For each uncolored vertex of G' that is safe, color it with a color in its list.

Let us assume that (i) terminates and thus each K_i' has two colored vertices x_i, y_i; this implies that K_i' contains at least two uncolored safe vertices, namely, the vertices that do not contain the special colors of x_i and y_i. It follows that (ii) is possible since each such vertex is adjacent to at least two uncolored vertices, which are in K_i'. A vertex corresponding to (iii) can also be colored since it is adjacent to the at least two uncolored safe vertices in the corresponding K_i'.

The remainder of this section will prove that (i) terminates. We only need to consider the set of K_i' where there is at most one vertex in K_i' that has an external neighbor in $\bigcup_{C_i \in \mathcal{P}_3}(C_i - K_i')$; let $\mathcal{K}$ be the set of such K_i'. Let T_i be a maximum set of vertices in $K_i' \in \mathcal{K}$ such that every vertex in T_i has a different special color, and each vertex in T_i has its external neighbor in $G' - \left(\bigcup_{C_i \in \mathcal{P}_3} C_i\right)$. Note that the external neighbors of T_i must all be distinct. Partition $\mathcal{K}$ into two sets $\mathcal{K}_1$ and $\mathcal{K}_2$ so that for $K_i' \in \mathcal{K}$, the set K_i' is in $\mathcal{K}_1$ if and only if $|T_i| \geq \frac{\Delta^{0.29}}{5} - 40$.

Claim 2.6.2. *There exists $Z \subset V(G')$ where $G'[Z]$ is a 1-factor consisting of one edge from $G'[K_i']$ for each $K_i' \in \mathcal{K}_2$.*

Proof. For each $K_i' \in \mathcal{K}_2$, vertices of at most one special color have their external neighbors in $\bigcup_{C_i \in \mathcal{P}_3}(C_i - K_i')$. Since there are at least $\frac{\Delta^{0.29}}{5}$ special colors by Lemma 2.3.4, for each $K_i' \in \mathcal{K}_2$, there exists at least 40 vertices with different special colors that have their external neighbors in $\bigcup_{K_i' \in \mathcal{K}_2} K_i'$; let R_i be 40 of these vertices for each $K_i' \in \mathcal{K}_2$. Choose two vertices from each R_i uniformly at random. We will apply the local lemma. Let E_e be the (bad) event that both endpoints of an edge e with endpoints in different K_i' is chosen. Thus, $Pr(E_e) \leq \left(\frac{2}{39}\right)^2$. E_{e_1} is mutually independent from E_{e_2} unless e_1 and e_2 have at least one endpoint in the same K_i'. Thus, E_e is mutually independent to all but at most 80 other events. Since $e\left(\frac{2}{39}\right)^2 81 < 1$, we are done. □

For each vertex in Z, color it with its special color; this is possible since no vertex in Z has a colored neighbor. Now each $K_i' \in \mathcal{K}_2$ has at least two uncolored safe vertices. We finish this section by finally showing that $Pr(\mathcal{E}_i) \leq \Delta^{-6}$.

Claim 2.6.3. *With high probability, for each $K'_i \in \mathcal{K}_1$, there are at least two vertices in K'_i where the special color of each vertex is available, and their external neighbors are colored.*

Proof. We will actually show that two vertices we are looking for are in T_i. Let $\mathcal{U}_i$ be the external neighbors of $\mathcal{T}_i$, which is a maximum set of vertices in T_i that satisfy the following conditions:

(i) the external neighbor of $x \in \mathcal{T}_i$ retains a color that is not the special color of x;

(ii) every external neighbor of a vertex in $\mathcal{T}_i$ has a distinct color.

We will first show that the expectation of $|\mathcal{T}_i|$ is high, and then we will show that $|\mathcal{T}_i|$ is concentrated around its expectation.

The probability that an external neighbor y of $x \in \mathcal{T}_i$ will not receive the special color of x is at least $1 - \frac{1}{\Delta-1}$, and the probability that the at most $\Delta - 1$ neighbors of y do not receive the color y received is at least $\left(1 - \frac{1}{\Delta-1}\right)^{\Delta-1}$, which is at least $\frac{1}{e}$. Since $|T_i| \geq \frac{\Delta^{0.29}}{5} - 40$, it follows that $E[|\mathcal{T}_i|] \geq \left(\frac{\Delta^{0.29}}{5} - 40\right) \cdot \frac{1}{e}$.

Now we use Azuma's Inequality to show that the probability that $|\mathcal{T}_i|$ deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of $|\mathcal{T}_i|$ change by at most c_v when changing the color of v.

If $v \in \mathcal{U}_i - C_i$, then $c_v \leq 2$ since changing the color on v affects $|\mathcal{T}_i|$ by at most 2 for any given assignment of colors to the remaining vertices. Thus, the sum of the c_v^2 is at most $4|\mathcal{T}_i|$, which is about $\frac{4\Delta^{0.29}}{5}$.

If $v \in V(G') - \mathcal{U}_i - C_i$, then changing the color of v from α to β will only affect $|\mathcal{T}_i|$ if some neighbor of v that is in $\mathcal{U}_i$ receives either α or β. This occurs with probability at most $\frac{2d_v}{\Delta-1}$, where d_v is the number of neighbors of v that are in $\mathcal{U}_i$. Therefore, by changing the color of v, the conditional expectation of $|\mathcal{T}_i|$ changes by at most $c_v = \frac{4d_v}{\Delta-1}$. Since the d_v sum is at most $\left(\frac{\Delta^{0.29}}{5} - 40\right)(\Delta - 1)$, the sum of these c_v is at most $\frac{4\Delta^{0.29}}{5}$. As each c_v is at most 4, we see that the sum of c_v^2 is at most $\frac{16\Delta^{0.29}}{5}$.

Hence, the sum of all the c_v is at most $4\Delta^{0.29}$. Applying Azuma's Inequality yields $Pr(\mathcal{E}_i) \leq \Delta^{-6}$ for sufficiently large Δ. See Calculation 2.7.8. $\square$

2.7 Calculations

Calculation 2.7.1.

$$\left(1 - \frac{1}{\Delta - 1}\right)^{2.3942\Delta} \geq e^{-2.3943}$$

Proof.

$$\left(1 - \frac{1}{\Delta - 1}\right)^{2.3942\Delta} \geq \left(1 - \frac{1}{\Delta - 1}\right)^{2.3943(\Delta-2)} \geq e^{-2.3943}$$

$\square$

Calculation 2.7.2. Azuma's Inequality for $\mathcal{E}_{1,i}$,

$$2\exp\left(\frac{-\left(2.5 \cdot 10^{-4}\Delta - 2\right)^2}{80\Delta}\right) \leq \Delta^{-6}$$

Proof.

$$\begin{aligned}
\Delta \geq 10^{12} \quad &\Rightarrow \quad 10^8 \cdot 200 \ln \Delta \leq 2.5\Delta \\
&\Leftrightarrow \quad 500\Delta \ln \Delta \leq \left(2.5 \cdot 10^{-4}\right)^2 \Delta^2 \\
&\Rightarrow \quad (80 \ln 2)\Delta + 6 \cdot 80\Delta \ln \Delta \leq \left(2.5 \cdot 10^{-4}\right)^2 \Delta^2 + 4 - 4\left(2.5 \cdot 10^{-4}\right)\Delta \\
&\Leftrightarrow \quad 2\Delta^6 \leq \exp\left(\frac{\left(2.5 \cdot 10^{-4}\Delta - 2\right)^2}{80\Delta}\right)
\end{aligned}$$

$\square$

Calculation 2.7.3. For number of P_3 for $\mathcal{E}_{2,i}$, when proving

$$\frac{\Delta - 1}{\Delta^{0.29} + 4} \geq 0.9999\Delta^{1-0.29}$$

Proof. $\frac{\Delta-1}{\Delta+4\Delta^{0.71}}$ is an increasing function. For $\Delta = 10^{20}$,

$$0.9999 \leq \frac{10^{20}-1}{10^{20}+4\cdot 10^{20\cdot 0.7}} = \frac{\Delta-1}{\Delta+4\Delta^{0.71}} \Leftrightarrow \frac{\Delta-1}{\Delta^{0.29}+4} \geq 0.9999\Delta^{0.71}$$

□

Calculation 2.7.4.

$$\left(1-\frac{1}{\Delta-1}\right)^{2\Delta+2.9998\Delta^{0.71}} \geq e^{-2.1}$$

Proof.

$$\left(1-\frac{1}{\Delta-1}\right)^{2\Delta+2.9998\Delta^{0.71}} \geq \left(1-\frac{1}{\Delta-1}\right)^{2.1(\Delta-2)} \geq e^{-2.1}$$

□

Calculation 2.7.5.

$$2(\Delta-1-0.9999\Delta^{0.71})+(\Delta-1)0.9999\Delta^{0.71} \leq \Delta^{1.71}$$

Proof.

$$\begin{aligned}
\Delta \geq 10^{20} \quad &\Rightarrow \quad 0.0001\cdot(10^{20})^{0.71} \geq 15848931924\\
&\Rightarrow \quad 2 \leq 0.0001\Delta^{0.71}\\
&\Leftrightarrow \quad 2\Delta + 0.9999\Delta^{1.71} \leq \Delta^{1.71}\\
&\Rightarrow \quad 2(\Delta-1-0.9999\Delta^{0.71})+(\Delta-1)0.9999\Delta^{0.71} \leq 2\Delta+0.9999\Delta^{1.71}
\end{aligned}$$

□

Calculation 2.7.6. Azuma's Inequality for $\mathcal{E}_{2,i}$,

$$2\exp\left(\frac{-\left(4.0\cdot 10^{-3}\cdot\Delta^{0.71}-2\right)^2}{8.02\Delta}\right) \leq \Delta^{-6}$$

Proof.

$$\begin{aligned}
\Delta \geq 10^{20} \quad &\Rightarrow \quad 50\Delta \ln \Delta \leq 16 \cdot 10^{-6} \Delta^{1.42} \\
&\Rightarrow \quad (8.02 \ln 2)\Delta + 6 \cdot 8.02\Delta \ln \Delta \leq \left(4.0 \cdot 10^{-3}\right)^2 \Delta^{1.42} + 4 - 4\left(4.0 \cdot 10^{-3}\right)\Delta^{0.71} \\
&\Leftrightarrow \quad 2\Delta^6 \leq \exp\left(\frac{\left(4.0 \cdot 10^{-3}\Delta^{0.71} - 2\right)^2}{8.02\Delta}\right)
\end{aligned}$$

□

Calculation 2.7.7.

$$\frac{\Delta^{0.29}}{5} - 40 \geq 1.9993\Delta^{0.29}$$

Proof.

$$1.9993 \leq \frac{1}{5} - \frac{40}{\left(10^{20}\right)^{0.29}} \Leftrightarrow \frac{\Delta^0.29}{5} - 40 \geq 1.9993\Delta^{0.29}$$

□

Calculation 2.7.8. Azuma's Inequality for $\mathcal{E}_{3,i}$, Claim 2.6.3,

$$2\exp\left(\frac{-\left(\frac{0.1999\Delta^{0.29}}{e} - 2\right)^2}{8\Delta^d}\right) \leq \Delta^{-6}$$

Proof.

$$\begin{aligned}
\Delta^{0.29} \geq \frac{50e^2 \ln \Delta}{0.1999^2} \quad &\Leftrightarrow \quad 50\Delta^{0.29} \ln \Delta \leq 0.1999^2 \left(\frac{\Delta^{0.29}}{e}\right)^2 \\
&\Rightarrow \quad (8 \ln 2)\Delta^{0.29} + 6 \cdot 8\Delta^{0.29} \ln \Delta \leq \left(0.1999\frac{\Delta^{0.29}}{e} - 2\right)^2 \\
&\Leftrightarrow \quad 2\Delta^6 \leq \exp\left(\frac{\left(\frac{0.1999\Delta^{0.29}}{e} - 2\right)^2}{8\Delta^{0.29}}\right)
\end{aligned}$$

□

Calculation 2.7.9. For $\mathcal{S}_v$,

$$2\exp\left(\frac{-\left(3^{-5}\Delta^{\frac{9}{10}}-2\right)^2}{50\Delta}\right)\le\Delta^{-6}$$

Proof.

$$\begin{aligned}
50\Delta\le 3^{-10}\Delta^{9/5} \quad &\Rightarrow \quad \frac{\Delta^{\frac{9}{10}}}{25}+120\Delta+300\Delta\ln\Delta\le 3^{-10}\Delta^{\frac{9}{5}}\\
&\Rightarrow \quad (50\ln 2)\Delta+6\cdot 50\Delta\ln\Delta\le\left(3^{-5}\Delta^{\frac{9}{10}}-2\right)^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad = 3^{-10}\Delta^{\frac{9}{5}}-4\cdot 3^{-5}\Delta^{\frac{9}{10}}+4\\
&\Leftrightarrow \quad 2\Delta^6\le\exp\left(\frac{\left(3^{-5}\Delta^{\frac{9}{10}}-2\right)^2}{50\Delta}\right)
\end{aligned}$$

$\square$

Chapter 3

Choosability of Toroidal Graphs

3.1 Introduction

Thomassen [46] proved that planar graphs are 5-choosable, and Voigt [49] constructed a planar graph that is not 4-choosable. It is known that [36, 53, 52, 24] that planar graphs without k-cycles for any one $k \in \{3, 4, 5, 6, 7\}$ are 4-choosable. There is also a vast literature on forbidding cycles in a planar graph to ensure that it is 3-choosable; we direct the readers to [4] for a thorough survey.

Regarding toroidal graphs, Böhme, Mohar, and Stiebitz [3] showed that they are 7-choosable, and they characterized when the choosability of a toroidal graph is 7 by proving that a toroidal graph G has $\chi_\ell(G) = 7$ if and only if $K_7 \subseteq G$. Cai, Wang, and Zhu [11] establish several tight results on the choosabiltiy of a toroidal graph with no short cycles. In particular, they proved that a toroidal graph G with no 7-cycles is 6-choosable, and $\chi_\ell(G) = 6$ if and only if $K_6 \subseteq G$. They also proved that a toroidal graph G with no 6-cycles is 5-choosable, and they conjectured that $\chi_\ell(G) = 5$ if and only if $K_5 \subseteq G$.

We disprove this conjecture by constructing an infinite family of toroidal graphs containing neither K_5 nor 6-cycles that are not even 4-colorable. (See Theorem 3.4.1.) It is worth mentioning that this infinite family of graphs are embeddable on any surface, orientable or non-orientable, except the plane and the projective plane. This shows that for the family of graphs embeddable on some surface, forbidding a cycle of length 6 and K_5 is not enough to ensure 4-choosability for any surface except the plane and the projective plane. Recall that forbidding a cycle of length 6 is enough to ensure 4-choosability for a planar graph.

Therefore, the following question by Kostochka [31] is natural:

Question 3.1.1. *Is every projective planar graph containing neither K_5 nor 6-cycles 4-choosable?*

The main result of this section is a different weakening of the original conjecture suggested by Zhu [55]:

Theorem 3.1.2. *A toroidal graph containing neither K_5^- nor 6-cycles is 4-choosable.*

The graph K_5^- is K_5 minus one edge. This theorem is sharp in the sense that forbidding only one of a K_5^- or 6-cycles in a toroidal graph does not guarantee that it is 4-choosable.

In the figures throughout this section, the white vertices do not have incident edges besides the ones drawn, and the black vertices may have other incident edges.

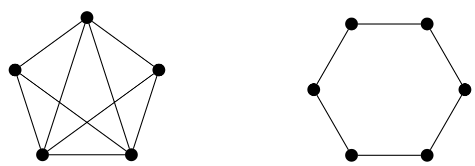

Figure 3.1: Forbidden configurations.

In Section 3.2, we prove many structural lemmas needed in Section 3.3, which is where we prove Theorem 3.1.2 using discharging. In Section 3.4, we display the sharpness examples of Theorem 3.1.2.

3.2 Lemmas

From now on, let G be a counterexample to Theorem 3.1.2 with the fewest number of vertices, and fix some embedding of G. It is easy to see that the minimum degree of (a vertex of) G is at least 4 and G is connected.

The *neighborhood* of a vertex v, denoted $N(v)$, is the set of vertices adjacent to v, and let $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted $d(v)$, is $|N(v)|$. The *degree* of a face f, denoted $d(f)$, is the length of f. A *k-vertex*, *k^+-vertex*, *k-face*, *k^+-face* is a vertex of degree k, a vertex of degree at least k, a face of degree k, and a face of degree at least k, respectively.

A graph is *degree-choosable* if there is an L-coloring for each list assignment L where $|L(v)| \geq d(v)$ for each vertex v. The following is a very well-known fact.

Theorem 3.2.1. *A graph is degree-choosable unless each maximal* 2*-connected subgraph is either a complete graph or an odd cycle.*

A set $S \subseteq V(G)$ of vertices is *k-regular* if every vertex in S has degree k in G. A *chord* is an edge between two non-consecutive vertices on a cycle. Let W_4 be a K_5 missing two edges that are not incident to each other.

Lemma 3.2.2. *$V(G)$ does not contain any of the following:*

(*i*) *A* 4*-regular set S where $G[S]$ is a cycle of even length.*

(*ii*) *A* 4*-regular set S where $G[S]$ is a cycle with one chord.*

(*iii*) *A set S with four vertices of degree* 4 *and one vertex of degree* 5 *where $G[S]$ is W_4.*

(*iv*) *A set S where $G[S]$ is a* 5*-face with one vertex of degree* 1*.*

(*v*) *A set S where $G[S]$ is a* 6*-face with one vertex of degree* 1*.*

Proof. Assume for the sake of contradiction that G contains a 4-regular set S described in either (i) or (ii). By the minimality of G, there exists an L-coloring f of $G - S$. For $v \in S$, let $L_f(v) = L(v) \setminus \{f(u) : u \in N(v) \setminus S\}$. By Lemma 3.2.1, since $|L_f(v)|$ is at least the degree of v in $G[S]$, it follows that there exists an L_f-coloring g of $G[S]$. By combining f and g, we obtain an L-coloring of G, which contradicts that G is a counterexample. (iii) follows from (ii) since (iii) contains (ii) as a subgraph. (iv) and (v) also cannot exist since G has minimum degree at least 4. □

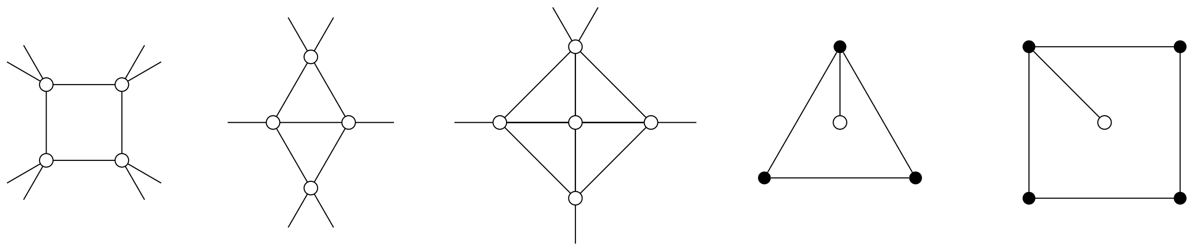

Figure 3.2: Forbidden configurations of G.

A 6-face is *degenerate* if some vertex y is incident to it twice; namely, it is of the form $xyzayw$ (see Figure 3.3). A list of faces of a vertex v is *consecutive* if it is a sublist of the list of faces incident to v in cyclic order.

Proposition 3.2.3. *If f is a 6-face of G where $wxyz$ are consecutive vertices on f, then the following holds:*

(*i*) *f must be a degenerate 6-face.*

(*ii*) *If xz is not an edge, then wy is an edge and y is incident to f twice.*

(*iii*) *If $w \neq z$, then either x or y is incident to f twice.*

(*iv*) *f cannot appear consecutively in the list of consecutive faces of a vertex.*

Proof. (*i*) follows from Lemma 3.2.2 (*v*). It is easy to check (*ii*), (*iii*), and (*iv*). $\square$

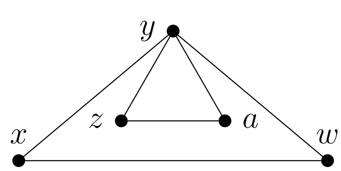

Figure 3.3: A degenerate 6-face.

Proposition 3.2.4. *Given a 4-face vu_2xu_3 and $u_1 \notin \{v, u_2, u_3, x\}$, if u_1vu_2y is a 4-face for some vertex y, then $y = u_3$.*

Proof. Note that $y \notin \{v, u_1, u_2\}$, and if $y = x$, then $d(u_2) = 2 < 4$, which contradicts the minimum degree of G. Now, $vu_1yu_2xu_3$ is a 6-cycle, unless $y = u_3$. $\square$

Claim 3.2.5. *If f_1, f_2, f_3 are consecutive faces of a vertex v with $d(f_1) = d(f_3) = 3 \neq d(f_2)$, then $d(f_2) \geq 6$.*

Proof. Let u_1, u_2, u_3, u_4 be neighbors of v in cyclic order so that f_1 is vu_1u_2 and f_3 is vu_3u_4. If f_2 is a 4-face u_2vu_3x, then $vu_1u_2xu_3u_4$ is a 6-cycle, unless $x \in \{u_1, u_4\}$. Yet, if $x \in \{u_1, u_4\}$, then either $d(u_2) = 2$ or $d(u_3) = 2$, which contradicts the minimum degree of G. If f_2 is a 5-face u_2vu_3xy, then G has a 6-cycle, unless $\{x, y\} = \{u_1, u_4\}$. If $x = u_4$ or $y = u_1$, then either $d(u_3) = 2$ or $d(u_2) = 2$. Thus, $x = u_1$ and $y = u_4$, which implies $u_1u_3, u_1u_4, u_2u_4 \in E(G)$. Yet, now $K_5^- \subseteq G[N[v]]$. □

Claim 3.2.6. *If f_1, f_2, f_3, f_4 are consecutive faces of a vertex v with $d(f_1) = d(f_2) = 3$ and $d(f_3) = 4$, then $d(f_4) \geq 6$.*

Proof. Let u_1, u_2, u_3, u_4 be the neighbors of v in cyclic order so that f_1 is vu_1u_2, f_2 is vu_2u_3, and f_3 is u_3vu_4x for some x. If $x \notin \{u_1, u_2\}$, then $u_1u_2u_3xu_4v$ is a 6-cycle, which is a contradiction. If $x = u_2$, then $d(u_3) = 2$, which is a contradiction. Therefore, $x = u_1$.

Note that if either u_3u_4 or u_2u_4 is an edge, then $K_5^- \subseteq G[N[v]]$. Also, vu_4 and u_4u_1 cannot be consecutive edges on the boundary of f_4 since this implies $d(u_4) = 2$. If f_4 is a 3-face vu_4x, then $x \notin \{u_1, u_2, u_3\}$. Yet, $vxu_4u_1u_3u_2$ is a 6-cycle. If f_4 is a 4-face vu_4xy, then $x \notin \{u_1, u_2, u_3\}$. If $y \notin \{u_1, u_2, u_3\}$, then $vyxu_4u_1u_2$ is a 6-cycle. If $y = u_1$, then $vu_4xyu_2u_3$ is a 6-cycle. If $y = u_2$, then $u_4xyu_3vu_1$ is a 6-cycle. If $y = u_3$, then $u_4xyvu_2u_1$ is a 6-cycle. If f_4 is a 5-face vu_4xyz, then $x, y \notin \{u_1, u_2, u_3\}$. If $z \notin \{u_1, u_2, u_3\}$, then u_4xyzvu_1 is a 6-cycle. If $z = u_1$, then u_4xyzu_2v is a 6-cycle. If $z = u_2$, then u_4xyzvu_1 is a 6-cycle. If $z = u_3$, then u_4xyzvu_1 is a 6-cycle. □

Corollary 3.2.7. *If f_1, f_2, f_3, f_4 are consecutive faces of a 5-vertex v with $d(f_1) = d(f_2) = 3$ and $d(f_4) = 3$, then $d(f_3) \geq 7$.*

Proof. Let u_1, u_2, u_3, u_4, u_5 be the neighbors of v in cyclic order so that f_1 is u_1vu_2, f_2 is u_2vu_3, and f_4 is u_4vu_5. By Claim 3.2.5, $d(f_3) \geq 6$. Assume for the sake of contradiction that $d(f_3) = 6$. If u_3u_4 is not an edge, then by Proposition 3.2.3 (ii), v must be incident to f_3 twice. This implies that f_3 is either $u_3vu_4u_5vu_1$ or $u_3vu_4u_1vu_5$. In the former, $d(u_4) = 2$, and in the latter, $u_3u_5u_4u_1u_2v$ is a 6-cycle. □

Claim 3.2.8. *There is no* 5*-vertex* v *with* $d(f_1) = d(f_2) = d(f_4) = 4$ *and* $d(f_3) = 3$ *where* f_1, f_2, f_3, f_4 *are consecutive faces of* v*.*

Proof. Let u_1, u_2, u_3, u_4, u_5 be the neighbors of v in cyclic order so that f_1 is u_1vu_2x, f_2 is u_2vu_3y, and f_3 is u_3vu_4, and f_4 is u_4vu_5z for some x, y, z. Note that $y \neq u_4$ since otherwise $d(u_3) = 2$, and $z \neq u_3$ since otherwise $d(u_4) = 2$.

Assume $y \notin \{u_1, u_5\}$. By considering f_1 and f_2 and Proposition 3.2.4, $x = u_3$. If $z \notin \{u_1, u_3\}$, then $u_4zu_5vu_1u_3$ is a 6-cycle. Thus, $z = u_1$. Yet, now $u_4u_1vu_2yu_3$ is a 6-cycle.

Assume $y = u_1$. If $z \notin \{u_1, u_3\}$, then $u_4zu_5vu_1u_3$ is a 6-cycle. Thus, $z = u_1$. If $x \notin \{u_3, u_4\}$, then $u_4u_1xu_2vu_3$ is a 6-cycle. Yet, if $x = u_3$, then $u_4vu_5u_1u_2u_3$ is a 6-cycle, and if $x = u_4$, then $u_4u_3vu_5u_1u_2$ is a 6-cycle.

Assume $y = u_5$. If $z \notin \{u_2, u_3\}$, then $u_4zu_5u_2vu_3$ is a 6-cycle. Thus, $z = u_2$. If $x \notin \{u_3, u_4\}$, then $u_1xu_2u_4u_3v$ is a 6-cycle. Yet, if $x = u_3$, then $u_1u_3u_4u_2u_5v$ is a 6-cycle, and if $x = u_4$, then $u_1u_4u_3u_5u_2v$ is a 6-cycle. □

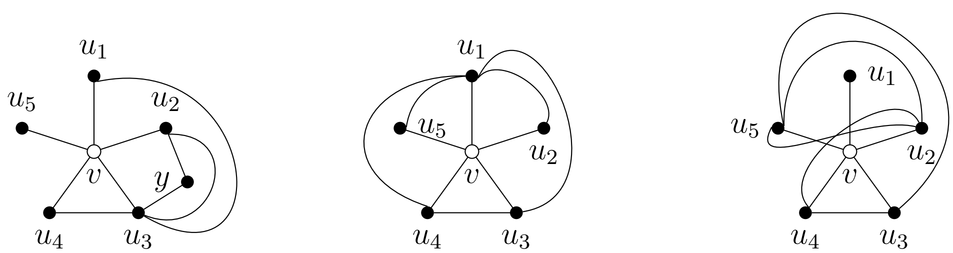

Figure 3.4: Cases for Claim 3.2.8.

Claim 3.2.9. *There is no* 5*-vertex* v *that is incident to only* 4*-faces.*

Proof. Let u_1, u_2, u_3, u_4, u_5 be the neighbors of v in cyclic order so that u_4vu_5x is a 4-face for some x.

Assume $x \notin \{u_1, u_2, u_3\}$. By considering the two 4-faces adjacent to vu_4 and Proposition 3.2.4, $u_3u_5, u_4u_5 \in E(G)$. By considering the two 4-faces adjacent to vu_5 and Proposition 3.2.4, $u_1u_4 \in E(G)$. Now, $u_1u_4xu_5u_3v$ is a 6-cycle.

Assume $x = u_2$. By considering the two 4-faces adjacent to vu_5 and Proposition 3.2.4, $u_4u_5, u_4u_1 \in E(G)$. By considering the two 4-faces adjacent to vu_4 and Proposition 3.2.4, $u_3u_5 \in E(G)$. Now, $vu_1u_4u_2u_5u_3$ is a 6-cycle.

The only cases left are $x \in \{u_3, u_1\}$. Without loss of generality, assume $x = u_3$. By considering the two 4-faces adjacent to vu_5 and Proposition 3.2.4, $u_4u_5, u_4u_1 \in E(G)$. By considering the two 4-faces adjacent to vu_1 and Proposition 3.2.4, $u_5u_1, u_5u_2 \in E(G)$. Yet, $vu_2u_5u_1u_4u_3$ is a 6-cycle. □

A 4-vertex v is *special* if v is incident to a 4-face and exactly two 3-faces.

Corollary 3.2.10. *The two 3-faces incident to a special vertex v must be adjacent to each other.*

Proof. If the two 3-faces are nonadjacent, then Claim 3.2.5 says no 4-face is incident to v. □

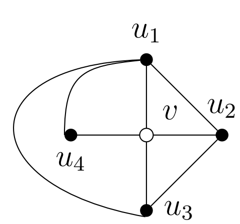

Figure 3.5: A special vertex.

Claim 3.2.11. *Each 4-face is incident to at most one special vertex.*

Proof. Let u_1, u_2, u_3, u_4 be the neighbors of a special vertex v in cyclic order so that vu_1u_2 and vu_2u_3 are the two 3-faces incident to v, and u_3vu_4x is a 4-face for some x. If $x \notin \{u_1, u_2\}$, then $u_1u_2u_3xu_4v$ is a 6-cycle. If $x = u_2$, then $d(u_3) = 2$. Therefore, $x = u_1$.

Note that if either u_3u_4 or u_2u_4 is an edge, then $K_5^- \subseteq G[N[v]]$. If u_1 is a special vertex, then u_1u_2x must be a 3-face for some $x \in \{u_3, u_4\}$, otherwise $u_1xu_2u_3vu_4$ is a 6-cycle. Since $x = u_4$ creates a K_5^-, it must be that $x = u_3$, but this implies that $d(u_2) = 3$. If u_3 is a special vertex, then u_2u_3x must be a 3-face for some $x \in \{u_1, u_4\}$, otherwise $u_2xu_3u_1u_4v$ is a 6-cycle. Since $x = u_4$ creates a K_5^-, it must be that $x = u_1$, but this implies that $d(u_2) = 3$. If u_4 is a special vertex, then since vu_4u_1 cannot be a 3-face, it must be that u_4u_1x is a 3-face for some $x \in \{u_2, u_3\}$, otherwise $u_1xu_4vu_3u_2$ is a 6-cycle. Yet either choice of x creates a K_5^-. Hence none of u_1, u_3, u_4 can be a special vertex, and thus there is only at most special vertex. □

Claim 3.2.12. *If f_1, f_2, f_3, f_4 are consecutive faces of a 4-vertex v with $d(f_1) = 3$, $d(f_2) = d(f_4) = 4$ and $d(f_3) \geq 5$, then neither f_2 nor f_4 is incident to a special vertex.*

Proof. Let u_1, u_2, u_3, u_4 be the neighbors of v in cyclic order so that vu_1u_2 is the 3-face f_1 incident to v. Let the 4-face f_2 be u_2vu_3x and let the other 4-face f_4 be u_1vu_4y. If $x = u_1$, then $d(u_2) = 2$, which is a contradiction. If $y = u_2$, then $d(u_1) = 2$. If $x = u_4$ and $y \notin N[v]$, then $u_1u_2vu_3u_4y$ is a 6-cycle. If $y = u_3$ and $x \notin N[v]$, then $u_2u_1vu_4u_3x$ is a 6-cycle. So either $x = u_4$ and $y = u_3$ or $x, y \notin N[v]$. Note that v cannot be a special vertex since it is incident to two 4-faces.

Assume $x = u_4$ and $y = u_3$. If u_2 is a special vertex, then u_1u_2z must be a 3-face for some $z \neq v$. If $z \notin \{u_3, u_4\}$, then $zu_1vu_3u_4u_2$ is a 6-cycle. If $z = u_3$, then $K_5^- \subseteq G[N[v]]$. If $z = u_4$, then $d(u_2) = 3$. Note that u_3, u_4 are not special vertices since each is incident to two 4-faces. Therefore, the f_2 is not incident to a special vertex, and by similar logic, f_4 is not incident to a special vertex.

Assume $x, y \notin N[v]$. If $x = y$, then x cannot be a special vertex since it is incident to two 4-faces. Without loss of generality, assume u_1u_2z is a 3-face for some $z \neq v$. If $z \notin \{x, u_3\}$, then $zu_1vu_3xu_2$ is a 6-cycle. If $z = x$, then $d(u_2) = 3$. If $z = u_3$, then $K_5^- \subseteq G[N[v] \cup \{x\}]$. Since u_1u_2z cannot be a 3-face, it follows that both u_2 and u_1 cannot be special vertices.

If u_3 is a special vertex, then u_3xz must be a 3-face for some $z \neq v$. If $z \notin \{u_1, u_2\}$, then $u_1u_2xzu_3v$ is a 6-cycle. If $z = u_1$, then $u_1u_2xu_4vu_3$ is a 6-cycle. If $z = u_2$, then $u_2u_1vu_4xu_3$ is a 6-cycle. Therefore, neither f_2 nor f_4 is incident to a special vertex.

If $x \neq y$, then both u_1, u_2 cannot be special vertices since u_1u_2z cannot be a 3-face for some $z \neq v$; this is because if $z \notin \{x, u_3\}$ then $vu_1zu_2xu_3$ is a 6-cycle, and if $z \notin \{y, u_4\}$ then $u_1zu_2vu_4y$ is a 6-cycle. If xu_2z is a 3-face for some z, then $z \in \{v, u_1, u_3\}$, otherwise $zu_2u_1vu_3x$ is a 6-cycle. If $z = u_1$, then $d(u_2) = 3$, and if $z = u_3$ then $d(x) = 2$. If $z = v$, then $zxu_2u_1yu_4$ is a 6-cycle. If xu_3z is a 3-face for some z, then $z \in \{u_1, u_2, v\}$, otherwise, $u_1u_2xzu_3v$ is a 6-cycle. If $z \in \{v, u_2\}$, then either $d(u_3) = 2$ or $d(u_2) = 3$. If $z = u_1$, then $u_1yu_4vu_3x$ is a 6-cycle. Therefore, f_2 is not incident to a special vertex, and by similar logic, f_4 is also not incident to a special vertex. $\square$

Claim 3.2.13. *If f_1, f_2, f_3, f_4 are consecutive faces of a non-special 4-vertex v where $d(f_1) = 3$ and $d(f_3) = 4$, then one of the following holds:*

(i) $d(f_i) \geq 6$ and $d(f_j) \geq 5$ where $\{i, j\} = \{2, 4\}$;

(ii) $d(f_i) \geq 6$ and $d(f_j) = 4$ and f_3 is not incident to a special vertex where $\{i, j\} = \{2, 4\}$.

Proof. Let u_1, u_2, u_3, u_4 be the neighbors of v in cyclic order so that f_1 is vu_1u_2 and f_3 is u_4vu_3x. Assume $x \notin \{u_1, u_2\}$. Consider the face f_2. If f_2 is a 3-face, then $u_1u_2u_3xu_4v$ is a 6-cycle. If f_2 is a 4-face u_2vu_3y, then by Proposition 3.2.4, $y = u_4$. Yet, now $vu_1u_2u_4xu_3$ is a 6-cycle. If f_2 is a 5-face u_2vu_3yz, then $vu_1u_2zyu_3$ is a 6-cycle, unless $u_1 \in \{z, y\}$. If $u_1 = z$, then $d(u_2) = 2$. If $u_1 = y$, then $vu_2u_1yxu_4$ is a 6-cycle. Therefore, $d(f_2) \geq 6$, and by symmetry, $d(f_4) \geq 6$.

Without loss of generality, assume $x = u_2$ and consider f_4. Note that f_2 cannot be a 3-face since this implies that $d(u_3) = 2$. Since v is not special, this implies that f_4 cannot be a 3-face. If f_4 is a 4-face u_1vu_4y, then by Proposition 3.2.4, $y = u_3$. Yet, now $K_5^- \subseteq G[N[v]]$. If f is a 5-face u_1vu_4yz, then $u_1u_2vu_4yz$ is a 6-cycle, unless $u_2 \in \{y, z\}$. If $u_2 = z$, then $d(u_1) = 2$, and if $u_2 = y$, then $d(u_4) = 2$. Therefore, $d(f_4) \geq 6$. If $d(f_2) \geq 5$, then (i) is satisfied. If $d(f_2) = 4$, then (ii) is satisfied since u_2, u_3, u_4 are each incident to at least two 4-faces, none of them can be special. $\square$

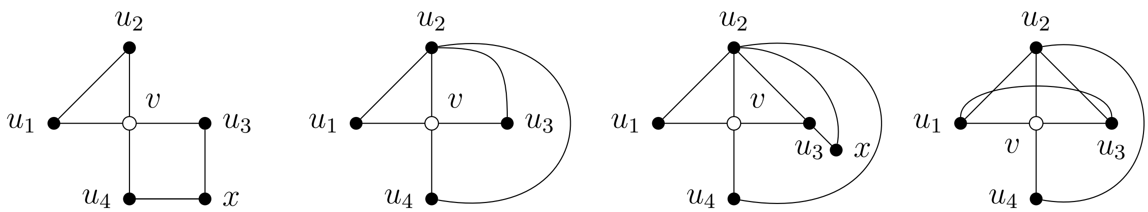

Figure 3.6: Pictures for Claim 3.2.13 and Claim 3.2.14.

Claim 3.2.14. *If f_1, f_2, f_3, f_4 are consecutive faces of a 4-vertex v where $d(f_1) = d(f_2) = 3$, $d(f_3) = 5$, and $d(f_4) \geq 5$, then $d(f_4) \geq 7$ and f_3 is incident to a 5^+-vertex.*

Proof. Let u_1, u_2, u_3, u_4 be the neighbors of v in cyclic order so that f_1 is u_1vu_2, f_2 is u_2vu_3, and f_3 is u_4vu_3xy for some x, y. Note that if $x = u_2$, then $d(u_3) = 2$.

Assume $x = u_1$. If $d(u_1) = d(u_2) = d(u_3) = 4$, then this contradicts Lemma 3.2.2 (ii). Thus, some vertex has higher degree, and therefore f_3 is incident to a 5^+-vertex. If f_4 is a 6-face, then since v cannot be incident to f_4 twice, it must be that u_1u_4 is an edge by Proposition 3.2.3. Yet, $K_5^- \subseteq G[N[v]]$. If f_4 is a 5-face zu_1vu_4w, then $u_1u_2vu_4wz$ is a 6-cycle, unless $u_2 \in \{z, w\}$. If $u_2 = z$, then $d(u_1) = 2$ and if $u_2 = w$, then $d(u_4) = 2$.

Assume $x \notin \{u_1, u_2\}$. Note that u_2 is a 5-vertex incident to f_3. If f_4 is a 6-face, then since v cannot be incident to f_4 twice, it must be that u_1u_4 is an edge by Proposition 3.2.3. Now, $u_1u_4u_2xu_3v$ is a 6-cycle. If f_4 is a 5-face zu_1vu_4w, then $u_1u_2vu_4wz$ is a 6-cycle, unless $u_2 \in \{z, w\}$. If $u_2 = z$, then $d(u_1) = 2$ and if $u_2 = w$, then $d(u_4) = 2$. $\square$

Claim 3.2.15. *If a 4-vertex v is incident to only 4-faces, then there are at least two incident 4-faces that are not incident to special vertices.*

Proof. Let u_1, u_2, u_3, u_4 be the neighbors of v in cyclic order so that f_1 is u_1vu_2x for some x, f_2 is u_2vu_3y for some y, and f_3 is u_3vu_4z for some z. Without loss of generality, either

$y \notin N[v]$ or $y = u_4$. Since each vertex in $N[v]$ is incident to at least two 4-faces, no vertex in $N[v]$ can be special. If $y = u_1$, then by Proposition 3.2.4, $z = u_2$. Thus, f_1 and f_2 are not incident to special vertices. If $y \notin N[v]$, then by Proposition 3.2.4, $x = u_3$ and $z = u_2$. Now, f_1, f_2, and f_3 are not incident to special vertices. $\square$

For $i \in \{3, 4\}$, a vertex v is *i-bad* if $d(v) = 4$ and v is incident to exactly i 3-faces. A vertex is *bad* if it is either 3-bad or 4-bad; a vertex is *good* if it is neither bad nor special. A face f is *great* if $d(f) \geq 7$.

Figure 3.7: A 3-bad vertex (left) and a 4-bad vertex (right).

Claim 3.2.16. *A face that is not incident to a 4-bad vertex v but is adjacent to a 3-face incident to v is a great face.*

Proof. Let u_1, u_2, u_3, u_4 be the neighbors of v in cyclic order as in Figure 3.7. By symmetry, we just need to show that a face f that is adjacent to u_1u_2 but is not incident to v is a great face. Note that if either u_1u_3 or u_2u_4 is an edge, then $K_5^- \subseteq G[N[v]]$. If f is a 3-face u_1u_2x, then $x \in \{u_3, u_4\}$, otherwise $xu_1vu_4u_3u_2$ is a 6-cycle. Yet, if $x \in \{u_3, u_4\}$, then $K_5^- \subseteq G[N[v]]$. If f is a 4-face u_1u_2xy, then $\{x, y\} = \{u_3, u_4\}$, otherwise G has a 6-cycle. Since $x \neq u_4$ and $y \neq u_3$, it must be that $x = u_3$ and $y = u_4$, which implies that $d(u_3) = 3$. Also, f cannot be a 5-face since f along with v would form a 6-cycle. If f is a 6-face where x, u_1, u_2, y are consecutive vertices on f, then, by Proposition 3.2.3 (ii), either $xu_2 \in E(G)$ or $u_1y \in E(G)$. In all cases, we get a 6-cycle or a K_5^-. $\square$

Claim 3.2.17. *A face that is not incident to a 3-bad vertex v but is adjacent to a 3-face incident to v cannot be a 3-face.*

Proof. Let f_0, f_1, f_2, f_3 be consecutive faces of v and let u_1, u_2, u_3, u_4 be the neighbors of v in cyclic order as in Figure 3.7. Note that f_0 cannot be a 3-face, otherwise v would be a 4-bad vertex. Assume f_2 was adjacent to a 3-face u_3u_4x that is not f_1, f_3. If $x \notin \{u_1, u_2\}$, then $xu_4u_1vu_2u_3$ is a 6-cycle. If $x \in \{u_1, u_2\}$, then either $d(u_3) = 3$ or $d(u_4) = 3$.

Without loss of generality, assume f_3 is adjacent to a 3-face u_2u_3x that is not f_2. If $x \notin \{u_1, u_4\}$, then $xu_2vu_1u_4u_3$ is a 6-cycle. If $x = u_4$, then $d(u_3) = 3$. If $x = u_1$, then $K_5^- \subseteq G[N[v]]$. $\square$

Corollary 3.2.18. *Each* 3*-bad vertex v is incident to either a great face or a degenerate* 6*-face.*

Proof. Let f_0 be the face incident to v that is not a 3-face. By Claim 3.2.5, $d(f_0) \geq 6$. If f_0 is a 6-face, it must be a degenerate 6-face, otherwise, f_0 is a great face. $\square$

Corollary 3.2.19. *If a* 3*-bad vertex v is incident to a degenerate* 6*-face f, then a face that is not incident to v but is adjacent to a face incident to v must be a great face.*

Proof. Let $u_1, u_2 \in N(v)$ so that u_1, v, u_2 are consecutive vertices of f. Since v cannot be incident to f twice, by Proposition 3.2.3 (ii), it must be that $u_1u_2 \in E(G)$. The rest of the proof is identical to Claim 3.2.16. $\square$

Corollary 3.2.20. *If a* 3*-bad vertex v is incident to a great face f, then a face that is not incident to v but is adjacent to a* 3*-face incident to v has length at least* 6*.*

Proof. Let f_0, f_1, f_2, f_3 be consecutive faces of v and let u_1, u_2, u_3, u_4 be the neighbors of v in cyclic order as in Figure 3.7. Let f be the face adjacent to u_3u_4 that is not f_2. By Claim 3.2.17, f cannot be a 3-face. If f is a 4-face u_3u_4xy, then $\{x, y\} = \{u_1, u_2\}$, otherwise G has a 6-cycle. If either $y = u_2$ or $x = u_1$, then either $d(u_3) = 2$ or $d(u_4) = 2$. If $x = u_2$ and $y = u_1$, then $K_5^- \subseteq G[N[v]]$. Note that f cannot be a 5-face since f along with v would form a 6-cycle.

Without loss of generality, let f be the face adjacent to u_2u_3 that is not f_3. By Claim 3.2.17, f cannot be a 3-face. If f is a 4-face u_2xyu_3 for some x, y, then $u_4vu_2xyu_3$ is a 6-cycle, unless $u_4 \in \{x, y\}$. Since $u_4 = y$ implies $d(u_3) = 3$, it must be that $u_4 = x$. If $x = u_1$, then $K_5^- \subseteq G[N[v]]$, and if $x \neq u_1$, then $u_4xu_3u_2vu_1$ is a 6-cycle. Note that f cannot be a 5-face since f along with v would form a 6-cycle. □

Corollary 3.2.21. *Given a* 3*-bad vertex* v *incident to a great face, let* u_1, u_2, u_3, u_4 *be the neighbors of* v *in cyclic order so that* u_1vu_2 *is not a* 3*-face. If* $d(u_3) = d(u_4) = 4$*, then each face that is not incident to* v *but is adjacent to a* 3*-face incident to* v *is great.*

Proof. Let $x \in N(u_2) \setminus \{u_1, v, u_3\}$ and let $y \in N(u_3) \setminus \{u_2, v, u_4\}$. For $i \in [3]$, let f_i be the face that is incident to an edge u_iu_{i+1} that is not a 3-face. By Corollary 3.2.20, we know that $d(f_i) \geq 6$. Assume for the sake of contradiction that f_i is a 6-face for some $i \in [3]$. Assume $i = 1$. By Proposition 3.2.3 (*ii*), either xu_1 is an edge or u_2 is incident to f_1 twice. Yet, by Proposition 3.2.3 (*iv*), u_2 cannot be incident to f_1 twice, so xu_1 must be an edge. If $x \neq u_4$, then $xu_1vu_4u_3u_2$ is a 6-cycle. If $x = u_4$, then $K_5^- \subseteq G[N[v]]$. By symmetry, this also solves the case when $i = 3$.

Assume $i = 2$. If $x = y$, then $x \notin N[v]$. Now, $u_1u_2xu_3u_4v$ is a 6-cycle. If $x \neq y$, then by Proposition 3.2.3 (*iii*), either u_2 or u_3 is incident to f_2 twice. In either case, this contradicts Proposition 3.2.3 (*iv*). □

Corollary 3.2.22. *No two bad vertices are adjacent to each other.*

Proof. Follows from Claim 3.2.16 and Claim 3.2.17. □

For $i \in \{1, 2\}$, a 5^+-vertex u is *i-responsible for* an adjacent bad vertex v if uv is incident to i 3-faces. A 5^+-vertex u is *responsible for* a bad vertex v if u is either 1-responsible or 2-responsible for v. A 4-vertex u is *responsible for* an adjacent bad vertex v if uv is incident to two 3-faces. Note that a vertex might be responsible for several bad vertices, and several vertices might be responsible for the same bad vertex.

Corollary 3.2.23. *Each vertex v is responsible for at most $\lfloor \frac{d(v)}{2} \rfloor$ bad vertices.*

Proof. If v is responsible for a vertex u, one of the two faces incident to the edge vu must be a 3-face vux. By Corollary 3.2.22, x cannot be a bad vertex. By Claim 3.2.16 and Claim 3.2.17, the face incident to xv that is not xvu has length at least 6, and this finishes the proof. □

Corollary 3.2.24. *Each vertex v is 2-responsible for at most $\lfloor \frac{d(v)}{3} \rfloor$ bad vertices.*

Proof. Let $x_1, \ldots, x_{d(x)}$ be the neighbors of v in cyclic order. If v is 2-responsible for x_i, then both faces incident to the edge vx_i must be 3-faces. By Claim 3.2.16 and Claim 3.2.17, the face incident to v, x_{i+1}, x_{i+2} cannot be a 3-face, thus, v cannot be 2-responsible for x_{i+1} and x_{i+2}. By the same argument, v cannot be 2-responsible for x_{i-1}, x_{i-2}. □

3.3 Discharging

Recall that an embedding of G was fixed, and let $F(G)$ be the set of faces of G. In this section, we will prove that G cannot exist by assigning an *initial charge* $\mu(z)$ to each $z \in V(G) \cup F(G)$, and then applying a discharging procedure to end up with *final charge* $\mu^*(z)$ at z. We prove that the final charge has positive total sum, whereas the initial charge sum is at most zero. The discharging process will preserve the total charge sum, and hence we find a contradiction to conclude that G does not exist.

For each vertex $v \in V(G)$, let $\mu(v) = d(v) - 6$, and for each face $f \in F(G)$, let $\mu(f) = 2d(f) - 6$. The total initial charge is at most zero since

$$\sum_{z \in V(G) \cup F(G)} \mu(z) = \sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in V(F)} (2d(f) - 6) = 6|E(G)| - 6|V(G)| - 6|F(G)| \leq 0.$$

The final equality holds by Euler's formula.

The rest of this section will prove that the sum of the final charge after the discharging phase is positive.

Recall that a 4-vertex v is *special* if v is incident to a 4-face and exactly two 3-faces. A 4-vertex v is *bad* if is incident to three or four 3-faces; a vertex is *good* if it is neither bad nor special. For $i \in \{1, 2\}$, a 5^+-vertex u is *i-responsible for* an adjacent bad vertex v if uv is incident to i 3-faces. A 5^+-vertex u is *responsible for* a vertex v if u is either 1-responsible or 2-responsible for v. A 4-vertex u is *responsible for* an adjacent bad vertex v if uv is incident to two 3-faces. A face f is *great* if $d(f) \geq 7$.

Here are the discharging rules:

(R1) Each 4-face sends charge 1 to each incident special vertex, $\frac{1}{5}$ to each incident 5^+-vertex, and distributes its remaining initial charge uniformly to each incident non-special 4-vertex.

(R2) Each 5-face sends charge $\frac{4}{7}$ to each incident 5^+-vertex and distributes its remaining initial charge uniformly to each incident 4-vertex.

(R3) Each 6^+-face distributes its initial charge uniformly to each incident vertex.

(R4) Each good 4-vertex u sends its excess charge to each vertex v where u is responsible for v.

(R5) Each 5^+-vertex u sends charge 1 to each vertex v where u is 2-responsible for v.

(R6) Each 5^+-vertex u sends charge $\frac{2}{7}$ to each vertex v where u is 1-responsible for v.

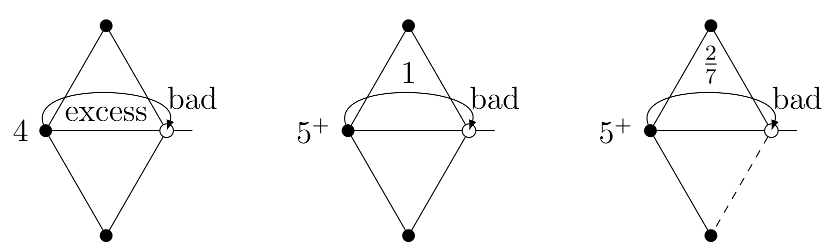

Figure 3.8: Discharging Rules

We will first show that each 4-face has nonnegative final charge. It is trivial that each 6^+-face has nonnegative final charge. Then, we will show that each vertex has nonnegative final charge. Moreover, we will show that each bad vertex and each 5^+-vertex that is not adjacent to a bad vertex has positive final charge.

Claim 3.3.1. *Each 4-face f has nonnegative final charge. Moreover, f sends charge at least $\frac{3}{5}$ to each incident 4-vertex if f is not incident to a special vertex, and f sends charge at least $\frac{2}{5}$ to each incident non-special 4-vertex if f is incident to a special vertex.*

Proof. By Claim 3.2.11, f is incident to at most one special vertex. By Lemma 3.2.2, there are at most three vertices of degree 4 incident to f. Since $\frac{1}{5} < \frac{2}{5} < \frac{3}{5}$, the worst case is when f has many incident 4-vertices. If f is not incident to a special vertex, then $\mu^*(f) \geq 2-3\cdot\frac{3}{5}-\frac{1}{5} = 0$. If f is incident to a special vertex, then $\mu^*(f) \geq 2-1-2\cdot\frac{2}{5}-\frac{1}{5} = 0$. □

Claim 3.3.2. *Each 5-face f has nonnegative final charge. Moreover, f sends charge at least $\frac{6}{7}$ to each incident 4-vertex if f is incident to a 5^+-vertex, and f sends charge at least $\frac{4}{5}$ to each incident 4-vertex if f is not incident to a 5^+-vertex.*

Proof. Since $\frac{4}{7} < \frac{4}{5} < \frac{6}{7}$, the worst case is when f has many incident 4-vertices. If f is incident to a 5^+-vertex, then $\mu^*(f) \geq 4 - 4 \cdot \frac{6}{7} - \frac{4}{7} = 0$. If f is not incident to a 5^+-vertex, then $\mu^*(f) \geq 4 - 5 \cdot \frac{4}{5} = 0$. □

Note that each (degenerate) 6-face sends charge 1 to each incident vertex, and each great face f sends charge $\frac{\mu(f)}{d(f)} = \frac{2d(f)-6}{d(f)} \geq \frac{8}{7}$ to each incident vertex.

Claim 3.3.3. *Each 6^+-vertex v has positive final charge. Moreover, if v is not adjacent to a bad vertex, then it has positive final charge.*

Proof. By Claim 3.2.16, Corollary 3.2.19, and Corollary 3.2.20, for each vertex v is responsible for, there exist two faces of length at least 6 incident to v that will each send charge at least 1 to v. If v is responsible for r vertices, then, $\mu^*(v) \geq \frac{2\cdot r}{2} - 1 \cdot r = 0$.

If v is not adjacent to a bad vertex, then v is not responsible for any vertex. Also v cannot be incident to only 3-faces since this would create a 6-cycle. Now, since v is incident to a 4^+-face, v has positive final charge. □

Claim 3.3.4. *Each 5-vertex v has nonnegative final charge. Moreover, if v is not adjacent to a bad vertex, then it has positive final charge.*

Proof. By Claim 3.2.23, v is responsible for at most two vertices, and by Claim 3.2.24, v is 2-responsible for at most one vertex. If v is 2-responsible for a vertex and 1-responsible for a vertex, then there must be two great faces incident to v by Corollary 3.2.7. Thus, $\mu^*(v) \geq -1 - 1 - \frac{2}{7} + 2 \cdot \frac{8}{7} = 0$. If v is 2-responsible for a vertex and is not 1-responsible for any vertex, then v is incident to at least two 6^+-faces by Claim 3.2.16, Corollary 3.2.19, and Corollary 3.2.20. Thus, $\mu^*(v) \geq -1 - 1 + 2 \cdot 1 = 0$.

If v is not 2-responsible for any vertex, then v is 1-responsible for at most two vertices. If v is 1-responsible for at least one vertex, then v is incident to at least two 6^+-faces, by Claim 3.2.16, Corollary 3.2.19, and Corollary 3.2.20. Thus, $\mu^*(v) \geq -1 - 2 \cdot \frac{2}{7} + 2 \cdot 1 > 0$.

The only case left is when v is not responsible for any vertex. If there are three consecutive faces f_1, f_2, f_3 where $d(f_1) = d(f_3) = 3 \neq d(f_2)$, then $d(f_2) \geq 6$ by Claim 3.2.5. Since the other two faces cannot be both 3-faces, $\mu^*(v) \geq -1 + 1 + \frac{1}{5} > 0$.

If there are consecutive faces f_1, f_2, f_3, f_4 where $d(f_1) = d(f_2) = 3$ and $d(f_3) = 4$, then $d(f_4) \geq 6$ by Claim 3.2.6. Thus, $\mu^*(v) \geq -1 + 1 + \frac{1}{5} > 0$. Thus, given consecutive faces f_0, f_1, f_2, f_3 where $d(f_1) = d(f_2) = 3$, then both $d(f_0), d(f_3) \geq 5$. Thus, $\mu^*(v) \geq -1 + 2 \cdot \frac{4}{7} > 0$.

Now, let v be incident to at most one 3-face. If v is incident to one 3-face, then by Claim 3.2.8, there exists a 5^+-face incident to v. Thus, $\mu^*(v) \geq -1 + \frac{4}{7} + 3 \cdot \frac{1}{5} > 0$. Note that v cannot be incident to only 4-faces by Claim 3.2.9. Thus, v is incident to at least one 5^+-face and at four 4^+-faces. Thus, $\mu^*(v) \geq -1 + 4 \cdot \frac{2}{5} + \frac{4}{5} > 0$.

Note that if v is not adjacent to a bad vertex, then v is not responsible for any vertex. Also v cannot be incident to only 3-faces since this would create a 6-cycle. Now, since v is incident to a 4^+-face, v has positive final charge. $\square$

Claim 3.3.5. *Each good 4-vertex v has nonnegative final charge.*

Proof. Note that v is incident to at most two 3-faces, otherwise v is a bad vertex. If v is incident to two 3-faces that are not adjacent to each other, then the other two faces have length at least 6 by Claim 3.2.5. Thus, $\mu^*(v) \geq -2 + 2 \cdot 1 = 0$. If v is incident to two 3-faces that are adjacent to each other and v is responsible for at least one vertex, then the other two faces must be 6^+-faces by Claim 3.2.16, Corollary 3.2.19, and Corollary 3.2.20. Thus, $\mu^*(v) \geq -2 + 2 \cdot 1 = 0$.

Assume v is incident to two 3-faces that are adjacent to each other and v is not responsible for any vertex. If v is incident to a 4-face, then v is a special vertex. If v is incident to a 5-face, then by Claim 3.2.14, v is also incident to a 7^+-face and the 5-face is incident to a 5^+-vertex. Thus, $\mu^*(v) \geq -2 + \frac{8}{7} + \frac{6}{7} = 0$. If v is incident to 6^+-faces, then, $\mu^*(v) \geq -2 + 2 \cdot 1 = 0$.

Assume v is incident to one 3-face f_1, where f_1, f_2, f_3, f_4 are consecutive faces of v. If $d(f_3) \geq 5$, and the other faces are 4-faces, then by Claim 3.2.12, $\mu^*(v) \geq -2 + \frac{4}{5} + 2 \cdot \frac{3}{5} = 0$. If $d(f_3) \geq 5$, and the other faces are not both 4-faces, then $\mu^*(v) \geq -2 + \frac{4}{5} + \frac{4}{5} + \frac{2}{5} = 0$. If $d(f_3) = 4$, then by Claim 3.2.13, either $\mu^*(v) \geq -2 + \frac{2}{5} + 1 + \frac{4}{5} > 0$ or $\mu^*(v) \geq -2 + \frac{3}{5} + \frac{2}{5} + 1 = 0$.

Assume v is incident to only 4^+-faces. If a 5^+-face is incident to v, then $\mu^*(v) \geq -2 + \frac{4}{5} + 3 \cdot \frac{2}{5} = 0$. If v is incident to only 4-faces, then by Claim 3.2.15, at least two of the 4-faces give charge at least $\frac{3}{5}$. Thus, $\mu^*(v) \geq -2 + 2 \cdot \frac{3}{5} + 2 \cdot \frac{2}{5} = 0$. □

Claim 3.3.6. *Each special vertex v has nonnegative final charge.*

Proof. A special vertex v is incident to two 3-faces and a 4-face. By Claim 3.2.6, the fourth face must be a 6^+-face. Thus, $\mu^*(v) \geq -2 + 1 + 1 = 0$. □

Claim 3.3.7. *Each 3-bad vertex v incident to a great face has positive final charge.*

Proof. Let u_1, u_2, u_3, u_4 be the neighbors of v in cyclic order so that $u_1 v u_4$ is not a 3-face. According to Lemma 3.2.2, either $d(u_2) = d(u_3) = 4$ and $d(u_1), d(u_4) \geq 5$ or $d(u_i) \geq 5$ for some $i \in \{2, 3\}$. In the former, u_2, u_3 sends charge at least $\frac{2}{7}$ since they are incident to two 7^+-faces by Corollary 3.2.21. Thus, $\mu^*(v) \geq -2 + \frac{8}{7} + 2 \cdot \frac{2}{7} + 2 \cdot \frac{2}{7} > 0$. In the latter, $\mu^*(v) \geq -2 + \frac{8}{7} + 1 > 0$. □

Claim 3.3.8. *Each 3-bad vertex v incident to a degenerate 6-face has positive final charge.*

Proof. Let u_1, u_2, u_3, u_4 be the neighbors of v in cyclic order so that u_1, v, u_4 is not a 3-face. For $i \in \{2, 3\}$, if $d(u_i) = 4$, then it sends charge at least $\frac{2}{7}$ since it is incident to two 7^+-faces by Corollary 3.2.19. According to Lemma 3.2.2, either $d(u_2) = d(u_3) = 4$ and $d(u_1), d(u_4) \geq 5$, or $d(u_i) \geq 4$ and $d(u_j) \geq 5$ for $\{i, j\} = \{2, 3\}$. In the former case, $\mu^*(v) \geq -2 + 1 + 2 \cdot \frac{2}{7} + 2 \cdot \frac{2}{7} > 0$. In the latter case, $\mu^*(v) \geq -2 + 1 + 1 + \frac{2}{7} > 0$. □

Claim 3.3.9. *Each 4-bad vertex v has positive final charge.*

Proof. According to Lemma 3.2.2, at least two vertices in $N(v)$ must have degree at least 5. Note that each 4-vertex in $N(v)$ sends charge $\frac{2}{7}$ since they are not incident to 6-faces by Claim 3.2.16. Thus, $\mu^*(v) \geq -2 + 2 \cdot 1 + 2 \cdot \frac{2}{7} > 0$. □

Since each bad vertex has positive final charge, there are no bad vertices. Since each 5^+-vertex v that is not adjacent to bad vertices has positive final charge, it must be the case that G is 4-regular. Since there is no K_5^-, there is no K_5, and by Theorem 3.2.1, we know that G is 4-choosable, which contradicts the assumption that G is a counterexample.

3.4 Sharpness Examples

In this section, we show that Theorem 3.1.2 is sharp by showing that we must forbid both K_5^- and 6-cycles. It is worth mentioning that both infinite families of graphs is embeddable on any surface, orientable or non-orientable, except the plane and projective plane. Note that Theorem 3.4.1 disproves a conjecture in [11].

Theorem 3.4.1. *For each $k \geq 6$, there exists an infinite family of toroidal graphs without ℓ-cycles for any $6 \leq \ell \leq k$ with chromatic number 5.*

Proof. Let H_i be a complete graph on 5 vertices minus an edge where x_i and z_i are the vertices of degree 3. Create G_s in the following way: given s copies of H_i, identify x_i and z_{i+1} for $i \in [s-1]$, and also add the edge $x_s z_1$.

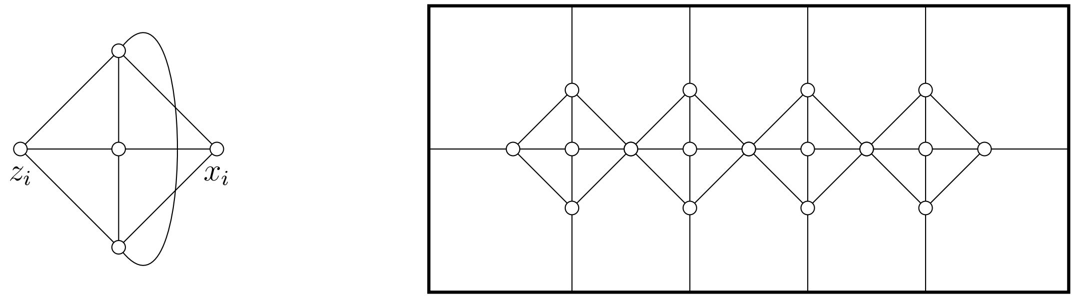

Figure 3.9: The graphs H_i (left) and an embedding of G_4 on the torus (right).

Let $s \geq \lceil \frac{k}{2} \rceil$ and consider G_s. It is easy to check that any 4-coloring of G_s must assign the same color to all the identified vertices as well as x_s, z_1, which is a contradiction since $x_s z_1$ is an edge. This shows G_s is not 4-colorable, which further implies that it is not 4-choosable. For each cycle in G_s, if it uses the the edge $x_s z_1$, then it must have length at least $2s+1$, which is at least $k+1$. All other cycles are contained within a copy of H_i, and has length at most 5. It is easy to check that there is no K_5 and that there is a 5-coloring of G_s. Note that G_s is toroidal, as seen in Figure 3.9. $\square$

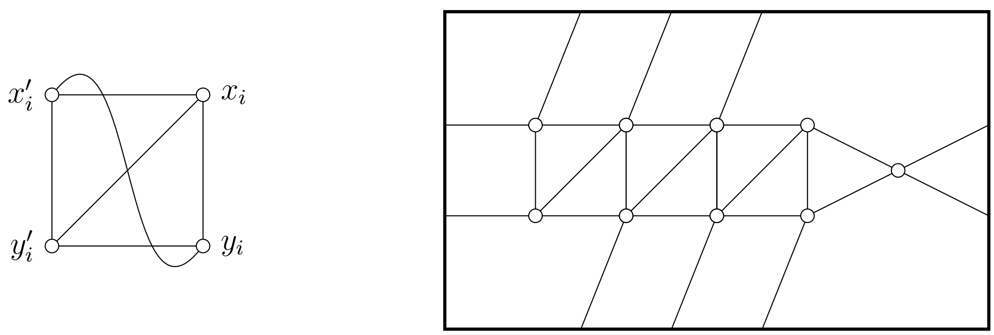

Figure 3.10: The graphs H_i' (left) and an embedding of G_1' on the torus (right).

Theorem 3.4.2. *There exists an infinite family of hamiltonian toroidal graphs without K_5^- with chromatic number* 5*.*

Proof. Let H_i' be a complete graph on 4 vertices where x_i, x_i', y_i, y_i' are the vertices of H_i. Create G_s' in the following way: given $2s+1$ copies of H_i', identify x_i with x_{i+1}' and identify y_i with y_i' for $i \in [2s]$, and also add a vertex z and the edges zx_1', zy_1', zx_{2s+1}', and zy_{2s+1}'.

It is easy to check that any 4-coloring of G'_s must assign different colors to the neighbors of z, which implies that G'_s is not 4-colorable; this further implies that it is not 4-choosable. It is easy to check that G'_s is hamiltonian and there is no K_5^-. Note that G'_s is toroidal, as seen in Figure 3.10. $\square$

Chapter 4

Choosability with Separation for Planar Graphs

4.1 Introduction

A graph G is said to be (k,d)-choosable if there is an L-coloring for each list assignment L where $|L(v)| \geq k$ for each vertex v and $|L(x) \cap L(y)| \leq d$ for each edge xy.

This concept is known as choosability with separation, since the second parameter may force the lists on adjacent vertices to be somewhat separated. If G is (k,d)-choosable, then G is also (k',d')-choosable for all $k' \geq k$ and $d' \leq d$. A graph is (k,k)-choosable if and only if it is k-choosable. Clearly, all graphs are $(k,0)$-choosable for $k \geq 1$. Thus, for a graph G and each $1 \leq k < \chi_\ell(G)$, there is some threshold $d \in \{0,\ldots,k-1\}$ such that G is (k,d)-choosable but not $(k,d+1)$-choosable.

This concept of choosability with separation was introduced by Kratochvíl, Tuza, and Voigt [33]. They used the following, more general definition. A graph G is (p,q,r)-*choosable*, if for every list assignment L with $|L(v)| \geq p$ for each $v \in V(G)$ and $|L(u) \cap L(v)| \leq p - r$ whenever u, v are adjacent vertices, G is q-tuple L-colorable. Since we consider only $q = 1$, we use a simpler notation. They investigate this concept for both complete graphs and sparse graphs. The study of dense graphs were extended to complete bipartite graphs and multipartite graphs by Füredi, Kostochka, and Kumbhat [26, 25].

Thomassen [46] proved that planar graphs are 5-choosable, and hence they are $(5,d)$-choosable for all d. Voigt [50] constructed a non-4-choosable planar graph, and there are also examples of non-$(4,3)$-choosable planar graphs. Kratochvíl, Tuza, and Voigt [33] showed that all planar graphs are $(4,1)$-choosable. The question of whether all planar graphs are

$(4,2)$-choosable or not was raised in the same paper and it still remains open.

Voigt [49] also constructed a non-3-choosable triangle-free planar graph. Škrekovski [43] observed that there are examples of triangle-free planar graphs that are not $(3,2)$-choosable, and posed the question of whether or not every planar graph is $(3,1)$-choosable; Kratochvíl, Tuza and Voigt [33] proved the following partial case of this question:

Theorem 4.1.1 ([33]). *Every triangle-free planar graph is* $(3,1)$*-choosable.*

We strengthen Theorem 4.1.1 by showing an alternative proof that uses a method developed by Thomassen; we also use this method to prove Theorem 4.1.2 below. Our inspiration was Thomassen's proof [47] that every planar graph of girth 5 is 3-choosable. We also prove the following two different partial cases:

Theorem 4.1.2. *Every planar graph without* 4*-cycles is* $(3,1)$*-choosable.*

Theorem 4.1.3. *Every planar graph with no* 5*-cycle and no* 6*-cycle is* $(3,1)$*-choosable.*

These results are similar in nature to other results on the choosability of planar graphs when certain cycles are forbidden (see a survey of Borodin [4]). One of the motivations is Steinberg's Conjecture that states that all planar graphs containing no 4- or 5-cycles are 3-colorable [45]. We construct a planar graph without cycles of length 4 and 5 that is not $(3,2)$-choosable, to show that Steinberg's Conjecture cannot be extended to $(3,2)$-choosability.

Theorems 4.1.1 and 4.1.2 are shown in Sections 2 and 3, respectively. Theorem 4.1.3 uses a discharging technique, and is showed in Section 4.

4.1.1 Preliminaries and Notation

Always L is a list assignment on the vertices of a graph G. In our proofs of Theorems 4.1.1 and 4.1.2, we use list assignments where vertices can have lists of different sizes. A $(*,1)$-*list* *assignment* is a list assignment L where $|L(v)| \geq 1$ and $|L(u) \cap L(v)| \leq 1$ for every pair of adjacent vertices u, v. A vertex v is an *Ld-vertex* when $|L(v)| = d$.

Given a graph G and a cycle $K \subset G$, an edge uv of G is a *chord* of K if $u, v \in V(K)$, but uv is not an edge of K. For an integer $k \geq 2$, a path $v_0 v_1 \ldots v_k$ is a *k-chord* if $v_0, v_k \in V(K)$ and $v_1, \ldots, v_{k-1} \notin V(K)$. If G is a plane graph, then let $\mathrm{Int}_K(G)$ be the subgraph of G consisting of the vertices and edges drawn inside the closed disc bounded by K, and let $\mathrm{Ext}_K(G)$ be the subgraph of G obtained by removing all vertices and edges drawn inside the open disc bounded by K. In particular, $K = \mathrm{Int}_K(G) \cap \mathrm{Ext}_K(G)$.

Note that each k-chord of K belongs to exactly one of $\mathrm{Int}_K(G)$ or $\mathrm{Ext}_K(G)$. If the cycle K is the outer face of G and Q is a k-chord of K, then let C_1 and C_2 be the two cycles in $K \cup Q$ that contain Q. Then the subgraphs $G_1 = \mathrm{Int}_{C_1}(G)$ and $G_2 = \mathrm{Int}_{C_2}(G)$ are the *Q-components* of G.

A graph G is *H-free* if it does not contain a copy of H as a subgraph.

4.2 Forbidding 3-cycles

In this section, we prove Theorem 4.1.1 as a corollary of the following theorem. Observe that any $(3,1)$-list assignment on a triangle-free plane graph satisfies the conditions of the following theorem.

Theorem 4.2.1. *Let G be a triangle-free plane graph with outer face F with a subpath $P \subset F$ containing at most two vertices, and let L be a $(*,1)$-list assignment such that the following conditions are satisfied:*

(i) $|L(v)| \geq 3$ *for* $v \in V(G) \setminus V(F)$,

(ii) $|L(v)| \geq 2$ *for* $v \in V(F) \setminus V(P)$,

(iii) $|L(v)| = 1$ *for* $v \in V(P)$,

(iv) no two vertices with lists of size two are adjacent in G,

(v) the subgraph induced by $V(P)$ is L colorable.

Then G is L-colorable.

Proof. Let G be a counterexample where $|V(G)| + |E(G)|$ is as small as possible. By the minimality of G, we assume that $|L(u) \cap L(v)| = 1$ for every edge $uv \in E(G) \setminus E(P)$. If otherwise, then we can remove the edge uv to obtain an L-coloring of $G - uv$, which is also an L-coloring of G. It is also clear that G is connected.

We quickly prove that G is 2-connected. Suppose v is a cut-vertex of G. There exist nontrivial connected induced subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{v\}$. Assume by symmetry that $P \subseteq G_1$. By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on $V(G_2)$ where $L'(u) = L(u)$ if $u \neq v$ and $L'(v) = \{\varphi(v)\}$; the lists L' satisfy the hypothesis on G_2. By the minimality of G, the graph G_2 has an L'-coloring ψ where $\psi(v) = \varphi(v)$, so φ and ψ form an L-coloring of G.

Since G is 2-connected, the outer face is bounded by a cycle. In the following claims, we prove that the cycle on F does not have chords or certain types of 2-chords.

Claim 4.2.2. *F does not contain any chords.*

Proof. Suppose for the sake of contradiction that $Q = uv$ is a chord of F. Let G_1 and G_2 be the two Q-components of G. Assume by symmetry that $P \subseteq G_1$. By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on $V(G_2)$ where for $x \in V(G_2)$, $L'(x) = \varphi(x)$ if $x \in \{u, v\}$ and $L'(x) = L(x)$ otherwise. By the minimality of G, there exists an L'-coloring ψ of G_2 with $\psi(u) = \varphi(u)$ and $\psi(v) = \varphi(v)$; together ψ and φ form an L-coloring of G. □

A 2-chord $v_0v_1v_2$ of F is *bad* if v_0 or v_2 is an $L2$-vertex. An $L3$-vertex $x \in V(F)$ is *good* if there is no bad 2-chord of F containing x.

Claim 4.2.3. *G has a good vertex.*

Proof. Suppose that F has no good vertex, so all $L3$-vertices in F are contained in a bad chord. Since G is 2-connected and triangle-free, $|V(F)| \geq 4$. Hence F contains at least one $L3$-vertex. Among all $L3$-vertices in F, let v_0 be an $L3$-vertex with a bad 2-chord $Q = v_0v_1v_2$ such that the size of the Q-component G_2 not containing P is minimized.

Let u be the neighbor of v_2 on F that is in G_2. Since G is triangle-free, the vertices u and v_0 are distinct. Since Q is a bad 2-chord, v_2 is an $L2$-vertex and hence u is an $L3$-vertex. Since F has no good $L3$-vertices, there is a bad 2-chord $Q' = uu_1u_2$ of F where u_2 is an L2-vertex. Since G is triangle-free, $u_1 \neq v_1$. Therefore, Q' is contained in G_2 and the Q'-component not containing P is properly contained within G_2, contradicting our extremal choice. □

Let $v_0v_1v_2$ be a path in F where v_1 is a good vertex. There exists a color c in $L(v_1)$ that does not appear in $L(v_0) \cup L(v_2)$. We will color v_1 with c and extend that coloring to $G - v_1$. Let $G' = G - v_1$, and let L' be the list assignment on $V(G')$ where $L'(u) = L(u) \setminus \{c\}$ for vertices u adjacent to v_1 in G, and $L'(u) = L(u)$ otherwise.

The neighbors of v_1 are $L'2$-vertices in G', and we verify that G' satisfies our hypotheses. Since G is triangle-free, the neighbors of v_1 form an indepenent set. Since v_1 is a good vertex, the $L'2$-vertices in G' form an independent set. By minimality of G, the graph G' has an L'-coloring φ. This L'-coloring φ extends to an L-coloring of G by assigning $\varphi(v_1) = c$. □

4.3 Forbidding 4-cycles

In this section, we prove Theorem 4.1.2 using a strengthened hypothesis. Observe that any $(3,1)$-list assignment on a C_4-free planar graph satisfies the conditions of the following theorem.

Theorem 4.3.1. *Let G be a C_4-free plane graph with outer face F with a subpath P of F containing at most three vertices, and let L be a $(*,1)$-list assignment such that the following conditions are satisfied:*

(i) $|L(v)| \geq 3$ *for* $v \in V(G) \setminus V(F)$*,*

(ii) $|L(v)| \geq 2$ *for* $v \in V(F) \setminus V(P)$*,*

(iii) $|L(v)| = 1$ *for* $v \in V(P)$*,*

(iv) no two $L2$*-vertices are adjacent in* G*,*

(v) the subgraph induced by $V(P)$ *is* L *colorable,*

(vi) no vertex with list of size two is adjacent to two vertices of P*.*

Then G *is* L*-colorable.*

Proof. Let G be a counterexample where $|V(G)| + |E(G)|$ is as small as possible. Moreover, we assume that the sum of the sizes of the lists is also as small as possible subject to the previous condition. By the minimality of G, we assume that for every edge $uv \in E(G)\setminus E(P)$, $|L(u) \cap L(v)| = 1$. If otherwise, then we can remove the edge uv to obtain an L-coloring of $G - uv$, which is also an L-coloring of G. It is also clear that G is connected.

Moreover, we show G is 2-connected. Suppose v is a cut-vertex of G. There exist nontrivial connected induced subgraphs G_1 and G_2 such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{v\}$. Suppose P is contained within exactly one of G_1 or G_2; by symmetry $P \subseteq G_1$. By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on $V(G_2)$ where $L'(u) = L(u)$ if $u \neq v$ and $L'(v) = \{\varphi(v)\}$. By the minimality of G, there exists an L'-coloring of G_2 and this coloring combined with φ gives an L-coloring of G. When P is not contained within only one of G_1 or G_2, we have $v \in V(P)$. By the minimality of G, both G_1 and G_2 are L-colorable and these colorings agree on v which gives an L-coloring of G.

In Claims 4.3.2 through 4.3.6, we determine certain structural properties of our counterexample G. A vertex v is a *middle vertex* if it has degree two in P. Observe that since G is C_4-free, any two vertices have at most one common neighbor.

Claim 4.3.2. *G does not contain a triangle with nonempty interior.*

Proof. Suppose not and assume $T = pqr$ is a triangle with nonempty interior in G. Let $G_1 = \text{Ext}_T(G)$ and $G_2 = \text{Int}_T(G)$. Since T has nonempty interior, $|V(G_1)| < |V(G)|$, and there exists an L-coloring φ of G_1. Let G' be obtained from G_2 by removing the edge rp and let L' be a list assignment on $V(G')$ where $L'(v) = \{\varphi(v)\}$ if $v \in \{p, q, r\}$ and $L'(v) = L(v)$ otherwise. The hypothesis applies to G' and L' with pqr as the path on three precolored vertices on the outer face of G'. Since $|E(G')| < |E(G)|$, there exists an L'-coloring ψ of G' which combined with φ forms an L-coloring of G. This contradicts G being a counterexample. □

If $|V(F)| \leq 4$, then $|V(F)| = 3$ since G contains no 4-cycles. By Claim 4.3.2, $G = F$ and it is easy to check Theorem 4.3.1 for graphs with at most three vertices. Thus, $|V(F)| \geq 5$.

A chord $Q = uv$ is *bad* if one of the Q-components is a triangle uvx where $|L(x)| = 2$. Otherwise, the chord Q is *good*.

Claim 4.3.3. *F contains only bad chords.*

Proof. For a good chord $Q = uv$, let G_1 and G_2 be the Q-components such that $|V(G_1) \cap V(P)| \geq |V(G_2) \cap V(P)|$. If F contains a good chord, select a good chord Q that minimizes $|V(G_2)|$. Since Q is good, the vertices u and v are at distance at least three apart in the path $F \cap G_2$. Assume $v \notin V(P)$.

By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on $V(G_2)$ where $L'(x) = \{\varphi(x)\}$ if $x \in \{u, v\}$ and $L'(x) = L(x)$ otherwise. Since uv is a chord, Since G_2 contains fewer vertices of P than G_1, the graph G_2 has at most three $L'1$-vertices, and they form a path of length at most two on the outer face of G_2.

Since we only changed the lists on u and v in G_2, the $L'2$-vertices remain an independent set. The only condition that remains to be verified is that every $L'2$-vertex in G_2 has at most one $L'1$-neighbor.

Suppose there exists an $L'2$-vertex $x \in V(G_2)$ adjacent to two $L'1$-vertices. Since x is not adjacent to two $L2$-vertices, one of these vertices must be v, which is not an $L1$-vertex. Since G has no 4-cycles, these two $L'1$-vertices must be adjacent, so x is adjacent to u and v. Since $|L(x)| = 2$, if either ux or vx is a chord, then it must be a good chord, so this contradicts the choice of Q. Hence both ux and vx are edges of F. Moreover, Claim 4.3.2 implies that G_2 is exactly the triangle uvx, which contradicts that Q is a good chord.

Hence there exists an L'-coloring ψ of G_2 that agrees with φ on Q, and these colorings together form an L-coloring of G. □

Claim 4.3.4. *F contains only bad chords uv where $u, v \notin V(P)$.*

Proof. Suppose for a contradiction that uv is a bad chord and $u \in V(P)$. Let $z \in V(F)$ be a common neighbor of u and v forming the bad chord. Since $|L(u)| = 1$, $L(u) \subset L(v)$ and $L(u) \subset L(z)$. Hence $L(v) \cap L(z) = L(u)$. By the minimality of G, there exists an L-coloring of $G - vz$. However, it is also an L-coloring of G. □

A 2-chord $Q = v_0v_1v_2$ of a cycle K is *separating* if $v_0v_2 \notin E(K)$. We now eliminate the possibility of F containing certain separating 2-chords.

Claim 4.3.5. *F does not contain a separating 2-chord $v_0v_1v_2$ where $|L(v_2)| = 2$ and v_0 is not a middle vertex.*

Proof. For a separating 2-chord $Q = v_0v_1v_2$ where v_2 is an $L2$-vertex and v_0 is not a middle vertex, let G_1 and G_2 be the Q-components of G where G_1 contains the vertices of P. If such a 2-chord exists, select Q to minimize $|V(G_2)|$.

By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on G_2 where $L'(v_i) = \{\varphi(v_i)\}$ for $i \in \{0, 1, 2\}$ and $L'(x) = L(x)$ for $x \in V(G_2) \setminus V(Q)$. The $L'1$-vertices of G_2 are exactly v_0, v_1, and v_2.

Since the $L'2$-vertices are also $L2$-vertices, the hypothesis holds for G_2 and L' as long as every $L'2$-vertex in G_2 has at most one neighbor in Q. Since v_2 in an $L2$-vertex it is not adjacent to any other $L2$-vertices. If some $L'2$-vertex x is adjacent to both v_1 and v_0, then the separating 2-chord v_2v_1x contradicts our extremal choice of Q.

Hence by the minimality of G there exists an L'-coloring ψ of G_2 which agrees with φ on Q and together these colorings form an L-coloring of G. □

Claim 4.3.6. *F does not contain a separating 2-chord $v_0v_1v_2$ where $|L(v_2)| = 3$, $v_0 \in V(P)$, and v_0 is not a middle vertex.*

Proof. Suppose there exists a separating 2-chord $Q = v_0v_1v_2 \subset G$ where $|L(v_2)| = 3$, $v_0 \in V(P)$, and v_0 is not a middle vertex. Let G_1 and G_2 be the Q-components of G where G_1 contains the vertices of P.

By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment on G_2 such that $L'(v_i) = \{\varphi(v_i)\}$ for $i \in \{0, 1, 2\}$ and $L'(x) = L(x)$ for $x \in V(G_2) \setminus V(Q)$. The $L'1$-vertices in G_2 are exactly those in Q.

Since all $L'2$-vertices in G_2 are also $L2$-vertices, we must verify that every $L'2$-vertex in G_2 has at most one neighbor in Q. If an $L'2$-vertex u has two neighbors, then one of them must be v_1 since G is C_4-free. However, at least one of the 2-chords v_0v_1u or v_2v_1u is separating and contradicts Claim 4.3.5.

Hence there exists an L'-coloring ψ of G_2 which agrees with φ on Q and together these colorings form an L-coloring of G. □

Our investigation of chords and 2-chords is complete. We now investigate the lists of adjacent vertices along the outer face in Claims 4.3.7 and 4.3.8.

Claim 4.3.7. *If $v_0v_1v_2$ is a path in F where $|L(v_1)| = 2$, then $L(v_1) \cap L(v_0) \neq L(v_1) \cap L(v_2)$.*

Proof. Suppose that there exists a path $v_0v_1v_2$ in F where $L(v_1) = \{a, b\}$ and $L(v_1) \cap L(v_0) = L(v_1) \cap L(v_2) = \{a\}$. We will find an L-coloring of G where v_1 is assigned the color b.

Let L' be the list assignment on $G - v_1$ where $L'(u) = L(u) \setminus \{b\}$ if $uv_1 \in E(G)$ and

$L'(u) = L(u)$ otherwise. Let G' be obtained from $G - v_1$ by removing edges between $L'2$-vertices with disjoint lists. We will verify that G' and L' satisfy the hypothesis.

If u is a neighbor of v_1 with $b \in L(u)$, then u is not in F since by Claim 4.3.3 G contains no chord uv_1. Hence, the vertices that had the color b removed are now $L'2$-vertices, all $L'2$-vertices are on the outer face of G', and the $L'1$-vertices are exactly the vertices in P.

It remains to show that the $L'2$-vertices are independent in G' and no $L'2$-vertex has two neighbors in P. The $L2$-vertices in G still form an independent set in G'. The $L'2$-vertices that are neighbors of v_1 form an independent set since their L'-lists are pairwise disjoint (their L-lists previously contained b and cannot share more colors). If an $L2$-vertex u is adjacent to an $L'2$-vertex x that is a neighbor of v_1, then since $u \notin \{v_0, v_2\}$, the path uxv_1 is a separating 2-chord contradicting Claim 4.3.5. Similarly, if a neighbor x of v_1 is adjacent to two vertices u_0, u_1 of P, then at least one of them, say u_1, is not a middle vertex, and when $u_1 \notin \{v_0, v_2\}$ the path v_1xu_1 is a separating 2-chord contradicting Claim 4.3.5. If $u_1 \in \{v_0, v_2\}$, then since G contains no 4-cycles, the vertices u_0 and u_1 are adjacent and $v_1xu_0u_1$ is a 4-cycle.

Thus the hypothesis holds on G' and L', so by the minimality of G there exists an L'-coloring φ of G' which extends to an L-coloring of G with $\varphi(v_1) = b$. $\square$

Claim 4.3.8. *If $v_0v_1v_2$ is a path in F where $|L(v_1)| = 3$, then v_0 and v_2 are $L2$-vertices, and the only $L2$-vertices adjacent to v_1.*

Proof. For a path $v_0v_1v_2$ where $|L(v_1)| = 3$, we consider how many of v_0 and v_2 are $L2$-vertices.

Suppose that neither v_0 nor v_2 is an $L2$-vertex. By Claim 4.3.3, G contains no good chord, and G contains no bad chord v_1u since v_0 and v_2 are not $L2$-vertices. Thus, all neighbors of v_1 other than v_0 and v_2 are $L3$-vertices. Select a color $a \in L(v_1)$ and let L' be the list assignment on G where $L'(z) = L(z)$ for $z \in V(G) \setminus \{v_1\}$ and $L'(v_1) = L(v_1) \setminus \{a\}$. If v_1 is adjacent to two vertices of P, they are v_0 and v_2, and $F = P \cup \{v_1\}$. This contradicts that $|V(F)| \geq 5$. Thus, the hypothesis holds on G with lists L' and by the minimality of G guarantees an L'-coloring of G, which is an L-coloring of G.

Now suppose that v_2 is an $L2$-vertex and v_0 is not. By Claim 4.3.3, G contains no good chord, and if G contains a bad chord v_1u it is with a triangle v_1uv_2, and we can write $u = v_3$ as the other neighbor of v_2 on F; in this case, v_3 is an $L3$-vertex since it is adjacent to v_2. Thus, all neighbors of v_1 other than v_0 and v_2 are $L3$-vertices.

Let a be the color in $L(v_1) \cap L(v_2)$. Let G' be obtained from G by removing the edge v_1v_2 and L' be the list assignment where $L'(v_1) = L(v_1) \setminus \{a\}$ and $L'(x) = L(x)$ for $x \in V(G) \setminus \{v_1\}$. Since the only $L2$-vertex adjacent in G to v_1 is v_2, and they are not adjacent in G', the $L'2$-vertices form an independent set in G'. Moreover, Claim 4.3.4 implies that v_1 has at most one neighbor in P. Hence G' satisfies the hypothesis, and by the minimality of G there exists an L'-coloring of G'. By the construction of L' and G', φ is also an L-coloring of G.

Thus, for a path $v_0v_1v_2$ in F with v_1 an $L3$-vertex, v_0 and v_1 are both $L2$-vertices. Since every bad chord v_1u has u adjacent to v_0 or v_2, the vertex u is an $L3$-vertex. Thus Claim 4.3.3 implies that v_0 and v_2 are the only $L2$-vertices adjacent to v_1. □

By the minimality of the sum of the sizes of the lists, we can assume that $|V(P)| \geq 1$ by removing colors if necessary. Let $p_0v_1v_2v_3 \ldots v_tv_{t+1} \ldots$ be vertices of F in cyclic order where $p_0 \in V(P)$, $\{v_1, \ldots, v_t\} = V(F) \setminus V(P)$, and thus $v_{t+1} \in V(P)$.

Claims 4.3.7 and 4.3.8 together imply that for all $i \in \{1, \ldots, t\}$, the vertex v_i is an $L2$-vertex when i is odd; otherwise v_i is an $L3$-vertex. Furthermore, v_t is an $L2$-vertex, so t is odd.

Select a set $X \subseteq \{v_2, v_3, v_4\}$ and a partial L-coloring φ of X by the following rules:

(X1) If v_2v_4 is not a bad chord, then let $c \in L(v_2) \setminus (L(v_1) \cup L(v_3))$ and:

- (X1a) If there is no common neighbor w of v_2 and v_3 such that $c \in L(v_2) \cap L(w)$, then let $X = \{v_2\}$ and $\varphi(v_2) = c$.

(X1b) If there is a common neighbor w of v_2 and v_3 such that $c \in L(v_2) \cap L(w)$, then let $X = \{v_2, v_3\}$, $\varphi(v_2) = c$, and $\varphi(v_3) = b$ where b is the unique color in $L(v_3) \setminus L(v_4)$.

(X2) If v_2v_4 is a bad chord, then let $X = \{v_2, v_3, v_4\}$. If v_4 and v_5 have a common neighbor w, then let $\varphi(v_4) \in L(v_4) \setminus (L(v_5) \cup L(w))$; otherwise let $\varphi(v_4) \in L(v_4) \setminus L(v_5)$. Finally, select $\varphi(v_2) \in L(v_2) \setminus (L(v_1) \cup \{\varphi(v_4)\})$ and $\varphi(v_3) \in L(v_3) \setminus \{\varphi(v_4)\}$ such that $\varphi(v_2) \neq \varphi(v_3)$

Observe that X and φ are well-defined, since there is always a choice for φ satisfying those rules. See Figure 4.1 for diagrams of these cases.

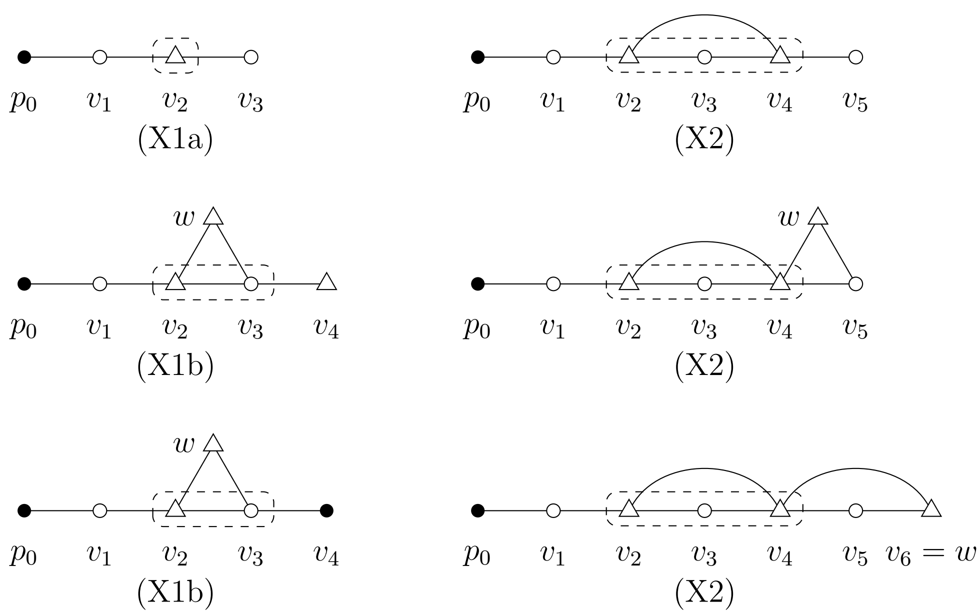

Figure 4.1: Cases (X1) and (X2). A black circle is an $L1$-vertex, a white circle is an $L2$-vertex, and a triangle is an $L3$-vertex. The dashed box indicates X.

Let L' be a list assignment on $G - X$ where

$$L'(v) = L(v) \setminus \{\varphi(x) : x \in X \text{ and } xv \in E(G)\}$$

for all $v \in V(G) \setminus X$. Let G' be obtained from $G - X$ by removing edges among vertices with disjoint L'-lists except the edges of P.

Below, we verify that G', L', and P satisfy the assumptions of Theorem 4.3.1. Then by the minimality of G, there is an L'-coloring ψ of G'. By the definition of L', the colorings φ and ψ together form an L-coloring of G, a contradiction.

Let N be the set of vertices u where $|L(u)| > |L'(u)|$. Necessarily, every vertex of N has a neighbor in X. Observe that X and φ are chosen such that $L(u) = L'(u)$ for all $u \in V(F) \setminus X$. Hence $N \subseteq V(G) \setminus V(F)$ and every vertex in N is an $L3$-vertex.

Since G is C_4-free, any pair of vertices has at most one common neighbor. When $|X| = 3$, we are in the case (X2), and the chord v_2v_4 implies that no vertex in N is adjacent to v_3, and a vertex adjacent to v_2 and v_4 would form a 4-cycle with v_3. When $|X| = 2$, there is at most one vertex in N having two neighbors in X. This is possible only in the case (X1b), and the colors $\varphi(v_2)$ and $\varphi(v_3)$ are chosen so that the common neighbor is an $L'2$- or $L'3$-vertex. Therefore $|L'(v)| \geq 2$ for every vertex $u \in N$.

If two vertices $x, y \in N$ are adjacent in G', the color $c \in L(x) \cap L(y)$ is also in $L'(x) \cap L'(y)$ and hence the colors $a \in L(x) \setminus L'(x)$ and $b \in L(y) \setminus L'(y)$ are distinct. Thus, x is adjacent to some $v_i \in X$ where $\varphi(v_i) = a$, and y is adjacent to some $v_j \in X$ where $\varphi(v_j) = b$. In every case above, any two distinct vertices in X that have neighbors not in X are also adjacent, so xv_iv_jy is a 4-cycle. Thus, N is an independent set.

Suppose that there is an edge $uv \in E(G')$ where $u \in N$ and $v \in V(F) \setminus X$ where $|L'(v)| = |L(v)| = 2$. If the 2-chord xuv is separating, we find a contradiction by Claim 4.3.5. If the 2-chord is not separating, then x and v are consecutive in F, and exactly one is in X.

First, we consider the case when $xuv = v_2uv_1$. If $L(v_1) \cap L(u) = L(p_0)$, then the edge v_1u does not restrict the colors assigned to v_1 and u by an L-coloring, so G is not minimal; thus $L(v_1) \cap L(u) \neq L(p_0)$. Hence the vertices v_1, u, and v_2 all share a common color, and this color was not removed from the list $L(u)$, so $|L'(u)| = 3$.

When $xuv \neq v_2uv_1$, then $xuv = v_iuv_{i+1}$, where i is maximum such that $v_i \in X$. However, the cases (X1a), (X1b), and (X2) all consider whether v_i and v_{i+1} have a common neighbor, and avoid using any color in common if v_{i+1} is an $L2$-vertex. Therefore, u is an $L'3$-vertex, so the $L'2$-vertices in G' form an independent set.

Finally, we verify that no $L'2$-vertex in G' has two neighbors in P. Since G' is obtained from G by deletions of edges and vertices, it suffices to check the condition only for vertices in N. If $v \in N$ has a neighbor $x \in X$, then v is not adjacent to two vertices of P by Claims 4.3.5 and 4.3.6 and G being C_4-free.

Therefore, G', L', and P satisfy the assumptions of Theorem 4.3.1. $\square$

4.4 Forbidding 5- and 6-cycles

The goal of this section is to prove Theorem 4.1.3. We prove a slightly stronger statement.

Theorem 4.4.1. *Let G be a plane graph without 5- or 6-cycles and let $p \in V(G)$. Let L be a $(*,1)$-list assignment such that*

- $|L(p)| = 1$,
- $|L(v)| = 3$ *for* $v \in V(G) - p$.

Then G is L-colorable.

This strengthening allows us to assume that a minimum counterexample is 2-connected, since we can iteratively color a graph by its blocks using at most one precolored vertex at each step.

Our proof uses a discharging technique. In Section 4.4.1, we define a family of *prime graphs* and prove in Section 4.4.3 that a minimum counterexample is prime. The proof is then completed in Section 4.4.2, where we define a discharging process and prove that prime graphs do not exist, and hence a minimum counterexample does not exist.

4.4.1 Configurations

We introduce some notation for a plane graph G. Let $V(G)$, $E(G)$, and $F(G)$ be the set of vertices, edges, and faces, respectively. For $v \in V(G)$, let $d(v) = |N(v)|$ where $N(v)$ is the set of vertices adjacent to v. For $f \in F(G)$, let $d(f)$ be the length of f.

For a C_5- and C_6-free plane graph, the subgraph of the dual graph induced by the 3-faces has no component with more than three vertices. A *facial* K_4 is a set of three pairwise adjacent 3-faces. We say four vertices xz_1yz_2 form a *diamond* if xz_1yz_2 is a 4-cycle formed by two adjacent 3-faces xyz_i for $i \in \{1, 2\}$. If a 3-face is not adjacent to another 3-face, then it is *isolated*.

A vertex is *low* if it has degree three; otherwise it is *high*. A 3-face is *bad* if it is incident to a low vertex; otherwise it is *good*. A face is *small* if it has length three or four. A face is *large* if it has length at least seven. A 4-face is *special* if is is incident to p and *normal* otherwise.

Definition 4.4.2. For a plane graph G, a list assignment L from Theorem 4.4.1, and $p \in V(G)$ with $|L(p)| = 1$, the pair (G, L) is *prime* if

- G is 2-connected
- $d(v) \geq 3$ for every $v \in V(G) - p$
- $d(p) \geq 2$

and in addition (G, L) contains none of the configurations (C1)–(C16) below:

(C1) A 3-face containing p.

(C2) A normal 4-face where all incident vertices are low.

(C3) A 3-face incident to at most one high vertex.

(C4) A diamond xz_1yz_2 where $d(z_1) = d(z_2) = 3$ and $d(x) = d(y) = 4$.

(C5) A diamond xz_1yz_2 where $d(y) = 4$ and $d(x) = 3$.

(C6) A facial K_4 $wxyz$ where w is the internal vertex, and at least one of x, y, and z has degree at most 4.

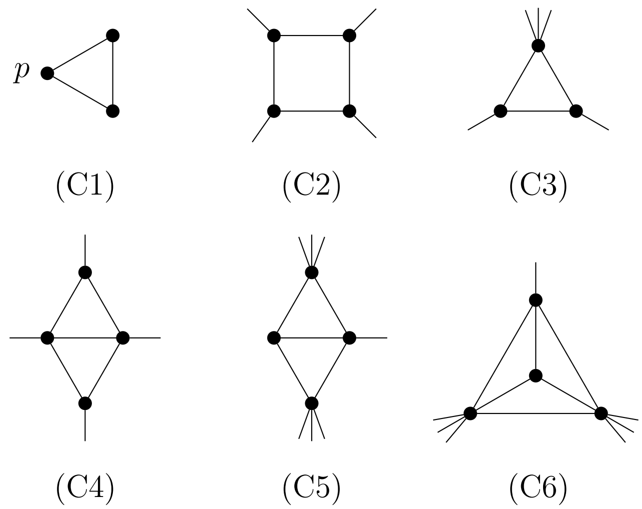

Figure 4.2: Simple reducible configurations.

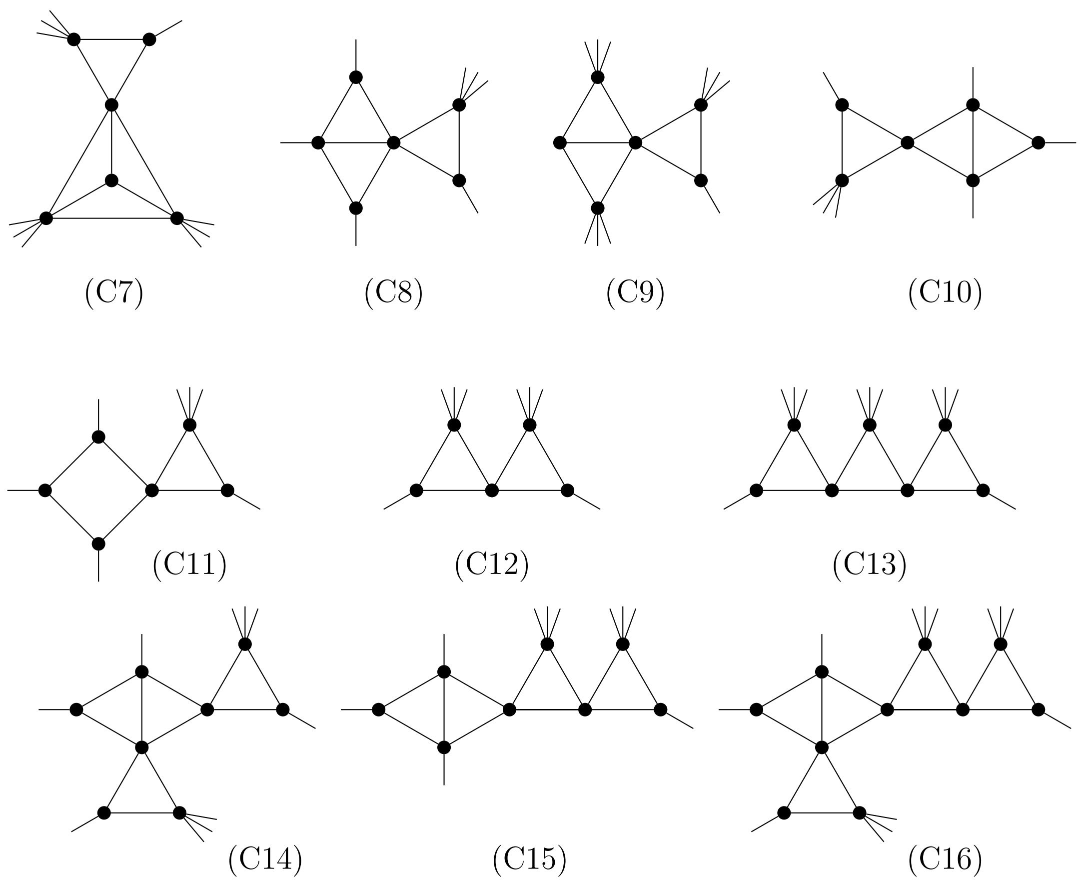

Figure 4.3: Compound reducible configurations.

(C7) A facial K_4 $wxyz$ where w is the internal vertex, the vertex z has degree exactly five, and the other two neighbors u, v of z bound a bad 3-face zvu.

(C8) A diamond xz_1yz_2 where $d(z_1) = d(z_2) = 3$, $d(x) = 4$, $d(y) = 5$, and the other two neighbors u, v of y form a bad 3-face yvu.

(C9) A diamond xz_1yz_2 where $d(x) = 3$, $d(y) = 5$, and the other two neighbors u, v of y form a bad 3-face yvu.

(C10) A diamond xz_1yz_2 where $d(z_2) = 3$, $d(x) = d(y) = d(z_1) = 4$, and the other two neighbors u, v of z_1 form a bad 3-face z_1vu.

(C11) A bad 3-face xyz and a normal 4-face $wuvx$ where $d(x) = 4$ and x is the only high vertex incident to the 4-face.

(C12) Two 3-faces xyz and xuv where $d(x) = 4$ and $d(y) = d(v) = 3$.

(C13) Three 3-faces xyz, xuv, vpq, where $d(y) = d(p) = 3$ and $d(x) = d(v) = 4$.

(C14) A diamond xz_1yz_2 where xy is an edge, $d(z_1) = 3$, $d(y) = 4$, $d(x) = 5$, and $d(z_2) = 4$, where x and z_2 are each incident to a bad 3-face.

(C15) A diamond xz_1yz_2 where xy is an edge, $d(z_1) = 3$ and $d(y) = d(x) = d(z_2) = 4$, where z_2 is incident to a good 3-face z_2uv with $d(v) = 4$ and v is incident to another bad 3-face.

(C16) A diamond xz_1yz_2 where xy is an edge, $d(z_1) = 3$, $d(x) = 5$, and $d(y) = d(z_2) = 4$, where x is incident to a bad 3-face and z_2 is incident to a good 3-face z_2uv with $d(v) = 4$ and v is incident to another bad 3-face.

The configurations (C1)–(C6) are called *simple*. See Figure 4.2. Other configurations can be built from simple ones by replacing an edge with one endpoint in the configuration by a bad 3-face; we call these *compound*. See Figure 4.8 for a sketch of creating compound configurations. For convenience, we list compound configurations used in our proof. See Figure 4.3. Reducibility is proved in Lemma 4.4.20 from Section 4.4.3.

Observe that a prime graph G has no 5- or 6-faces since no 5- or 6-cycles exist and G is 2-connected.

4.4.2 Discharging

In this section, we prove the following proposition.

Proposition 4.4.3. *No pair (G, L) is prime.*

We shall prove that a prime (G, L) does not exist by assigning an initial *charge* $\mu(z)$ to each $z \in V(G) \cup F(G)$ with strictly negative total sum, then applying a discharging process to end up with charge $\mu^*(z)$. We prove that since (G, L) does not contain any configuration in (C1)–(C16), then μ^* has nonnegative total sum. The discharging process will preserve the total charge sum, and hence we find a contradiction and G does not exist.

For every vertex $v \in V(G) - p$ let $\mu(v) = 2d(v) - 6$, for p let $\mu(p) = 2d(p)$, and for every face $f \in F(G)$, let $\mu(f) = d(f) - 6$. The total initial charge is negative by

$$\begin{aligned}\sum_{z \in V(G) \cup F(G)} \mu(z) &= \sum_{v \in V(G) - p} (2d(v) - 6) + 2d(p) + \sum_{f \in V(F)} (d(f) - 6) \\ &= 6|E(G)| - 6|V(G)| - 6|F(G)| + 6 = -6.\end{aligned}$$

The final equality holds by Euler's formula.

In the rest of this section we will prove that the sum of the final charge after the discharging phase is nonnegative. Instead of looking at each individual face, we look at groups of adjacent 3-faces.

Note that since G has no 5-cycles and 6-cycles, no 4-face is adjacent to a 3- or 4-face (and hence every face adjacent to a 4-face has length at least seven). If a vertex v with

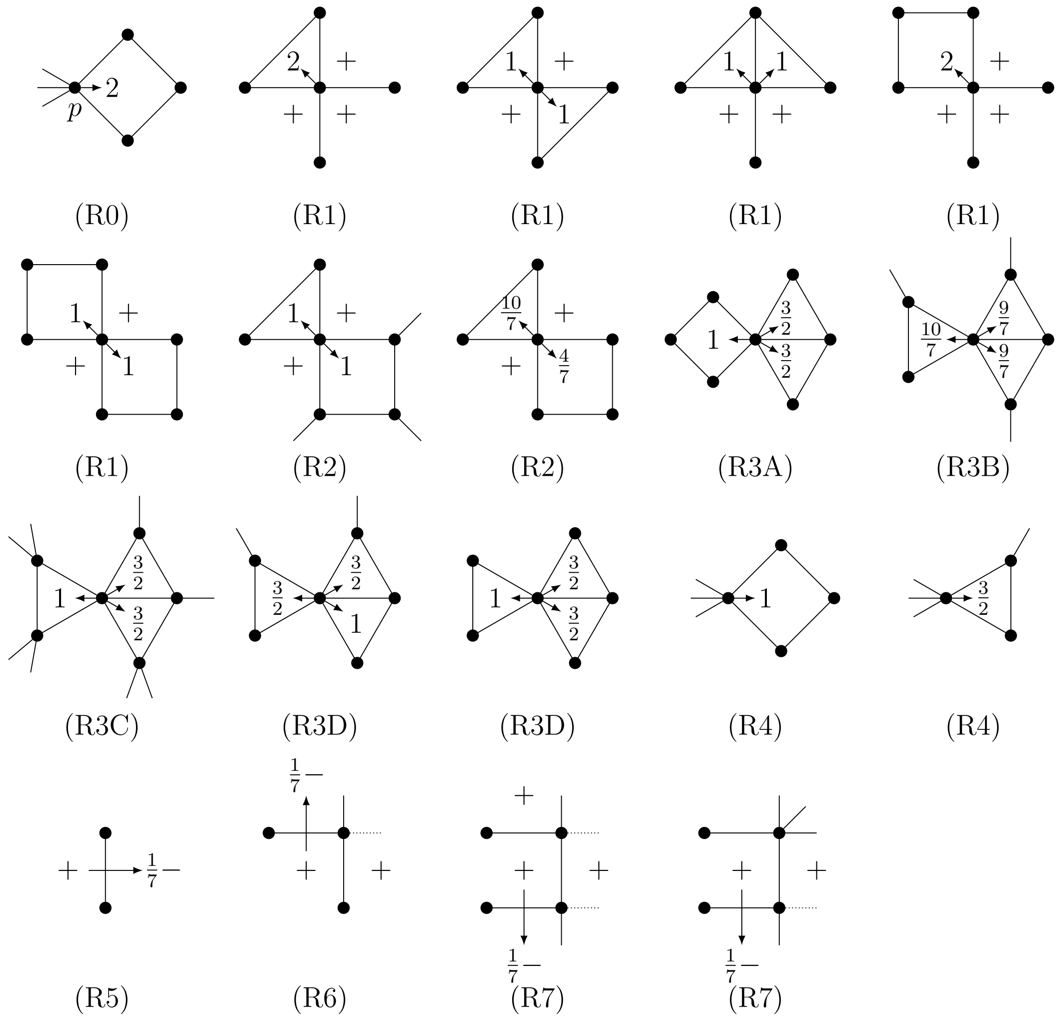

Figure 4.4: Discharging rules.

$d(v) \geq 4$ is incident to ℓ_3 3-faces and ℓ_4 4-faces, then $d(v) \geq \frac{3}{2}\ell_3 + 2\ell_4$. Thus, every vertex v is incident to at most $2d(v)/3$ small faces.

We begin by discharging from vertices with positive charge to small faces with negative charge. The precolored vertex p transfers charge according to rule (R0).

(R0) p sends charge 2 to every (special) incident 4-face.

For a vertex $v \in V(G) - p$ with $d(v) \geq 4$, exactly one of the discharging rules (R1)–(R4) applies; rules (R0)–(R4) are called *vertex rules.*

(R1) If $d(v) = 4$ and v is not incident to both a 3-face and a normal 4-face, then v distributes its charge uniformly to each incident 3-face or normal 4-face.

(R2) If $d(v) = 4$ and v is incident to a 3-face t and a normal 4-face f, then:

(R2A) If f is incident to exactly one high vertex, then v gives charge 1 to f and 1 to t.

(R2B) If f is incident to more than one high vertex, then v gives charge $\frac{4}{7}$ to f and $\frac{10}{7}$ to t.

(R3) If $d(v) = 5$, then:

(R3A) If v is incident to a normal 4-face, then v gives charge 1 to each normal 4-face and distributes its remaining charge uniformly to each incident 3-face.

(R3B) If v is incident to three bad 3-faces, then v gives charge $\frac{10}{7}$ to the isolated 3-face and $\frac{9}{7}$ to each 3-face in the diamond.

(R3C) If v is incident to only one bad 3-face that is in a diamond with another 3-face incident to v, then v gives charge $\frac{3}{2}$ to both 3-faces in the diamond, and if v is incident to another 3-face t, then v gives charge 1 to t.

(R3D) Otherwise, v gives charge $\frac{3}{2}$ to each incident bad 3-face and distributes its remaining charge uniformly to each incident non-bad 3-face.

(R4) If $d(v) \geq 6$, then v gives charge 1 to each incident normal 4-face and charge $\frac{3}{2}$ to each incident 3-face.

After applying the vertex rules, we say a face is *hungry* if it is a negatively-charged small face, or it is a 3-face in a negatively-charged diamond.

We now discharge from large faces to hungry faces; the rules (R5)–(R7) are *face rules*. Let f be a face with $d(f) \geq 7$ and let $f_0, f_1, f_2, \ldots, f_{d(f)} = f_0$ be the faces adjacent to f in counterclockwise order. Observe that f has charge at least $d(f)/7$, and so f could send charge $\frac{1}{7}$ to each adjacent face. Each of the rules below could apply to f and an adjacent face f_i, to decide where the charge $\frac{1}{7}$ associated with f_i should go.

(R5) If f_i is hungry, then f gives charge $\frac{1}{7}$ to f_i.

(R6) If f_i is not hungry, f_{i+1} is hungry, and the vertex incident to f, f_i, and f_{i+1} has degree at most four, then f gives charge $\frac{1}{7}$ to f_{i+1} instead of f_i.

(R7) If f_i is not hungry, f_{i-1} is hungry, the vertex incident to f, f_{i-1}, and f_i has degree at most four, and either the vertex incident to f, f_i, and f_{i+1} has degree at least five or f_{i+1} is not hungry, then f gives charge $\frac{1}{7}$ to f_{i-1} instead of f_i.

We now show that the discharging rules result in a nonnegative charge sum $\sum_{v\in V(G)} \mu^*(v)+\sum_{f\in F(G)} \mu^*(f) \geq 0$, contradicting our previously computed sum of -6. First, we prove that the final charge μ^* is nonnegative on every vertex. Then, we prove that the final charge μ^* is nonnegative on every large face and every 4-face. A set S of 3-faces is *connected* if they induce a connected subgraph of the dual graph. Since G contains no 5- or 6-cycles, a connected set of 3-faces is either a facial K_4, a diamond, or an isolated 3-face. We will show that for every connected set S of 3-faces, the final charge sum $\sum_{f\in S} \mu^*(f)$ is nonnegative.

Claim 4.4.4. *For each vertex $v \in V(G)$, the final charge $\mu^*(v)$ is nonnegative.*

Proof. If $v = p$, then rule (R0) applies then p is incident to at most $d(p)/2$ 4-faces since 4-faces cannot share an edge. So $\mu^*(v) \geq 2d(p) - 2d(p)/2 \geq 0$.

Assume $v \in V(G) \setminus p$. Recall $d(v) \geq 3$. If $d(v) = 3$, then $\mu(v) = 0$ and no charge is sent from this vertex.

If $d(v) = 4$, then $\mu(v) = 2$ and (R1) or (R2) applies. Consider the four faces incident to v. Since G avoids 5- and 6-cycles, at most two of these faces are small. If v is not incident to both a 3- and 4-face, then (R1) applies and v sends all charge uniformly to each small face; hence $\mu^*(v) = 0$. If v is incident to both a 3- and 4-face, then (R2) applies and v sends total charge two to the two small faces (either as $1 + 1$ for (R2A) or $\frac{4}{7} + \frac{10}{7}$ for (R2B)).

If $d(v) = 5$, then $\mu(v) = 4$ and (R3) applies. There are five faces incident to v, and since G avoids 5- and 6-cycles, at most three of these faces are small. If v is incident to three bad 3-faces, then two of the faces are adjacent so these faces partition into a bad 3-face and a diamond; (R3B) applies and a total charge of four is sent from v, so $\mu^*(v) = 0$. If v is not incident to three bad 3-faces, v is incident to at most two 3- or 4-faces; (R3A), (R3C), or (R3D) applies, and v sends at most charge three, $\mu^*(v) \geq 0$.

If $d(v) \geq 6$, then (R4) applies. Since G avoids 5- and 6-cycles, v is incident to at most $\frac{2d(v)}{3}$ small faces. Since $\mu(v) = 2d(v) - 6$ and v sends charge at most $\frac{2d(v)}{3} \cdot \frac{3}{2} = d(v)$, the final charge on v is $\mu^*(v) \geq d(v) - 6 \geq 0$. □

Claim 4.4.5. *For each face $f \in F(G)$ with $d(f) \geq 7$, the final charge $\mu^*(f)$ is nonnegative.*

Proof. Let the faces adjacent to f be listed in clockwise order as $f_1, f_2, \dots$ as in the discharging rules. Observe that each adjacent face f_i satisfies at most one of the rules (R5), (R6), and (R7), and hence f sends charge $\frac{1}{7}$ at most $d(f)$ times, leaving $\mu^*(f) \geq \mu(f) - \frac{d(f)}{7} \geq 0$. □

Claim 4.4.6. *For each 4-face $f \in F(G)$, the final charge $\mu^*(f)$ is nonnegative.*

Proof. If f is a special 4-face, (R0) applies. Thus $\mu^*(f) = -2 + 2 = 0$. So assume that f is a normal 4-face.

Observe $\mu(f) = -2$, and all faces adjacent to f have length at least seven. Since G contains no (C2), the normal 4-face f is incident to at least one high vertex.

If f is incident to exactly one high vertex v, then v sends charge at least 1 to f by (R1), (R2A), or (R4). The four incident faces each send charge at least $\frac{1}{7}$ by (R5). Three of the four faces adjacent to f each send an additional $\frac{1}{7}$ by (R6). Thus, $\mu^*(f) \geq -2+1+4\cdot\frac{1}{7}+3\cdot\frac{1}{7} = 0$.

If f is incident to exactly two high vertices u and v, then u and v each send charge at least $\frac{4}{7}$ by (R1), (R2B), (R3A), or (R4). The four incident faces each send charge $\frac{1}{7}$ by (R5). Two of the four faces adjacent to f each send an additional $\frac{1}{7}$ by (R6). Thus, $\mu^*(f) \geq -2 + 2 \cdot \frac{4}{7} + 4 \cdot \frac{1}{7} + 2 \cdot \frac{1}{7} = 0$.

If f is incident to at least three high vertices, then each high vertex sends charge at least $\frac{4}{7}$ by (R1), (R2B), (R3A), or (R4). The four faces adjacent to f each send charge at least $\frac{1}{7}$ by (R5). Thus, $\mu^*(f) \geq -2 + 3 \cdot \frac{4}{7} + 4 \cdot \frac{1}{7} > 0$. □

We now show the total charge sum over the 3-faces is nonnegative by showing the charge sum is nonnegative on each connected set of 3-faces, starting with facial K_4's (Claim 4.4.7), then diamonds (Claim 4.4.8), and finally isolated 3-faces (Claim 4.4.9).

Observe that if t is a good 3-face, then t receives charge at least 1 from every incident vertex by the vertex rules.

Claim 4.4.7. *For each facial K_4, the sum of the final charge of the three 3-faces is nonnegative.*

Proof. Let $wxyz$ be a facial K_4 where w is the internal vertex. Since G contains no (C6), all vertices $v \in \{x, y, z\}$ have degree $d(v) \geq 5$. Since G contains no (C7), any vertex $v \in \{x, y, z\}$ with $d(v) = 5$ is not incident to another bad 3-face outside the facial K_4. Thus, each vertex $v \in \{x, y, z\}$ sends charge at least $2 \cdot \frac{3}{2}$ by (R3D) or (R4) to the 3-faces in the facial K_4, and $\mu^*(wxy) + \mu^*(wyz) + \mu^*(wzx) \geq -9 + 3 + 3 + 3 = 0$. □

Claim 4.4.8. *For each diamond, the sum of the final charge of the two 3-faces is nonnegative.*

Proof. Let the diamond have vertices xz_1yz_2 where xyz_i is a 3-face f_i for $i \in \{1, 2\}$. We assume $d(x) \leq d(y)$. Note that if the diamond does not have nonnegative charge after the vertex rules, then both faces f_1 and f_2 are hungry.

By our earlier observation, if both faces f_1 and f_2 are good, then they each receive charge at least 3 from the incident vertices, and the diamond has nonnegative charge after the vertex rules. We now consider which faces in f_1 and f_2 are bad.

Case 1: Exactly one 3-face f_i is bad. In this case, we will assume $d(z_1) \leq d(z_2)$, so it must be f_1 that is bad, while f_2 is good. Thus, $d(y) \geq d(x) \geq 4$, and $d(z_2) \geq 4$. Since f_1 is bad, $d(z_1) = 3$. For $v \in \{x, y\}$, let f_v be the face incident to v that follows f_1 and f_2 in counterclockwise order around v.

If $d(y) \geq 5$ then y contributes at least $\frac{3}{2} + 1$ by (R3D), $2 \cdot \frac{9}{7}$ by (R3B), or at least $2 \cdot \frac{3}{2}$ by (R3A), (R3C), or (R4). If $d(y) = 4$ then y contributes at least 2 by (R1), but now (R6) also applies to the face f_y, adding an extra contribution of $\frac{1}{7}$. The contribution to the diamond is at least $2 + \frac{1}{7}$ from the vertex rules applied to y and (R6) applied to the face f_y. By symmetry, the contribution to the diamond is also at least $2 + \frac{1}{7}$ from the vertex rules applied to x and (R6) applied to the face f_x. By the vertex rules, z_2 sends charge at least 1 to f_2. Rule (R5), the four faces adjacent to the diamond send $\frac{1}{7}$ each. Rule (R6) applies to the face incident to z_1 that is not f_1, f_x, or f_y, giving $\frac{1}{7}$ to the diamond. Thus $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 2(2 + \frac{1}{7}) + 1 + 4 \cdot \frac{1}{7} + \frac{1}{7} = 0$.

Case 2: Both 3-faces f_1 and f_2 are bad. We consider the degree of x and order z_1 and z_2 such that z_1, y, and z_2 appear consecutively in the clockwise ordering of the neighbors of x. Observe that when $d(x) > 3$, we have $d(z_1) = d(z_2) = 3$ since f_1 and f_2 are bad.

(*i*) Assume $d(x) \geq 5$. By (R3) and (R4), both x and y each send at least $2 \cdot \frac{9}{7}$. By (R5), the four incident faces contribute charge $4 \cdot \frac{1}{7}$ to the diamond and by (R6), two of the faces incident to z_1 or z_2 contribute at least $2 \cdot \frac{1}{7}$. Thus the final charge on the diamond is $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 4 \cdot \frac{9}{7} + 4 \cdot \frac{1}{7} + 2 \cdot \frac{1}{7} = 0$.

(*ii*) Assume $d(x) = 4$. By (R1), the vertex x sends charge 1 to each face f_i. Since G contains no (C8), if $d(y) = 5$ then y is not incident to a bad 3-face other than f_1 and f_2. Thus, y sends charge $\frac{3}{2}$ to each face f_i, and after the vertex rules the charge on the diamond is $-6 + 2 \cdot 1 + 2 \cdot \frac{3}{2} = -1$, and the faces f_i are hungry. By (R5), the four faces adjacent to f_1 and f_2 each send charge $\frac{1}{7}$ to the diamond. By (R6), three of the four faces adjacent to f_1 and f_2 each send charge $\frac{1}{7}$ to the diamond. Thus, $\mu^*(T_1) + \mu^*(T_2) = -6 + 2 \cdot 1 + 2 \cdot \frac{3}{2} + 4 \cdot \frac{1}{7} + 3 \cdot \frac{1}{7} = 0$.

(*iii*) Assume $d(x) = 3$. Since G contains no (C5), we have $d(y) \geq 5$. Let f be the face incident to z_1, x, z_2. If f is a 3-face, then $z_1 x z_2 y$ is a facial K_4, handled in Claim 4.4.7. If f is a 4-face, then G contains a 5-cycle, a contradiction. Thus, $d(f) \geq 7$, and let

$s_0, s_1, s_2, \ldots$ be the faces adjacent to f in cyclic clockwise order where $s_1 = f_1$ and $s_2 = f_2$. Since G contains no (C9), $d(y) = 5$ implies that the vertex y is not adjacent to a bad 3-face other than f_1 and f_2. By (R3C), (R3D), and (R4), y sends charge $\frac{3}{2}$ to each face f_i. By (R5), four faces adjacent to the diamond each contribute at least $\frac{1}{7}$. Since G contains no (C3), it follows that $d(z_i) \geq 4$ for each $i \in \{1, 2\}$.

If $d(z_i) \geq 5$ for some $i \in \{1, 2\}$, then by the vertex rules, z_1, z_2 together will contribute at least $1 + \frac{10}{7}$ to the diamond. Thus, $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 1 + \frac{10}{7} + 3 + 4 \cdot \frac{1}{7} = 0$. We can now assume $d(z_1) = d(z_2) = 4$.

(a) If $d(s_3) \geq 7$, then z_2 sends charge 2 to f_2 by (R1) and z_1 sends charge at least 1 to f_1 by the vertex rules. Thus $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 2 \cdot \frac{3}{2} + 2 + 1 = 0$. By symmetry this also solves the case when $d(s_0) \geq 7$.

(b) If $d(s_3) = 4$, then since s_3 and s_2 are not a copy of (C11), the face s_3 must be incident to at least two high vertices. By (R2B), z_3 sends charge $\frac{10}{7}$ to f_2 and by the vertex rules, z_1 sends charge at least 1 to f_2. Thus, $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 2 \cdot \frac{3}{2} + \frac{10}{7} + 1 + 4 \cdot \frac{1}{7} = 0$. By symmetry this also solves the case when $d(s_0) = 4$.

(c) If $d(s_0) = d(s_3) = 3$, then since the faces s_0 and f_1 (or the faces f_2 and s_3) do not form a copy of (C12), the 3-faces s_0 and s_3 are not bad 3-faces.

Let z_0 be the vertex such that $z_0 z_1$ is the edge between f and s_0; similarly let z_3 be the vertex such that $z_3 z_2$ is the edge between f and s_3. Let w_i be the other vertex of s_i different from z_i for $i \in \{0, 3\}$. For $i \in \{0, 3\}$, we have $d(z_i) \geq 4$ and $d(w_i) \geq 4$ since s_i is not a bad 3-face. See Figure 4.5 for a sketch of the situation.

Let i be in $\{0, 3\}$. If s_i is an isolated 3-face, then it receives charge at least 1 from each of its incident vertices and is not hungry after the vertex rules. If s_i is in a diamond with a bad 3-face t_i then since $t_i, s_i, f_{|i-1|}$ is not a copy of (C10), at least one of z_i and w_i has degree at least 5. By symmetry, assume z_i has degree at least

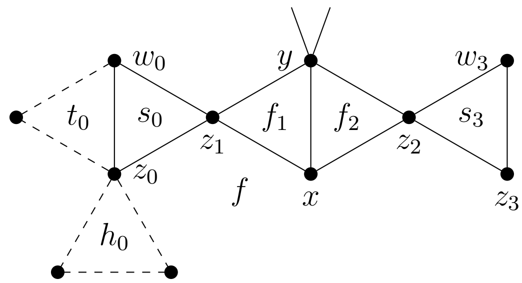

Figure 4.5: Situation in Case 3.iii(c).

5. If (R3D) does not apply to z_i, then the diamond formed by s_i, t_i receives charge at least 6 from its vertices and is not hungry after the vertex rules. Hence (R3D) applies to z_i and there must be a bad 3-face h_i incident to z_i that is not t_i. If w_i has degree at least 5, by symmetry, (R3D) applies and t_i, s_i are not hungry. If w_i has degree 4 then the faces s_i, t_i, h_i, and $f_{|i-1|}$ form a copy of (C14). Therefore the diamond s_i, t_i is not hungry after the vertex rules. Therefore s_0 and s_3 are not hungry after the vertex rules.

By (R5), the three faces adjacent to f_1 and f_2 send charge $4 \cdot \frac{1}{7}$ to the diamond. The rule (R6) applied on f and edge z_2z_3 sends charge $\frac{1}{7}$ to f_2 and (R6) applied on edge z_1w_0 and face containing z_1w_0 sendscharge $\frac{1}{7}$ to f_1. Finally, we show that (R7) applies on z_1z_0 which gives additional $\frac{1}{7}$. If $d(z_0) \geq 5$ then (R7) applies. Suppose that $d(z_0) = 4$. If s_0 is in a diamond with t_0, then (R6) cannot apply on z_1z_0. If z_0 is in another bad 3-face h_0, then h_0, s_0, f_1 form reducible configuration (C13). Hence (R7) indeed applies. Thus, $\mu^*(f_1) + \mu^*(f_2) \geq -6 + 2 \cdot \frac{3}{2} + 1 + 1 + 4 \cdot \frac{1}{7} + 2 \cdot \frac{1}{7} + \frac{1}{7} = 0$.

In all cases, our diamond has nonnegative total charge. □

Claim 4.4.9. *For each isolated 3-face t, the final charge $\mu^*(t)$ is nonnegative.*

Proof. Note that $\mu(t) = -3$. If t is good, then each incident vertex sends charge at least 1 by the vertex rules and $\mu^*(t) \geq -3 + 3 = 0$. We now assume t is incident to at least one low vertex. Moreover, we assume that t is hungry.

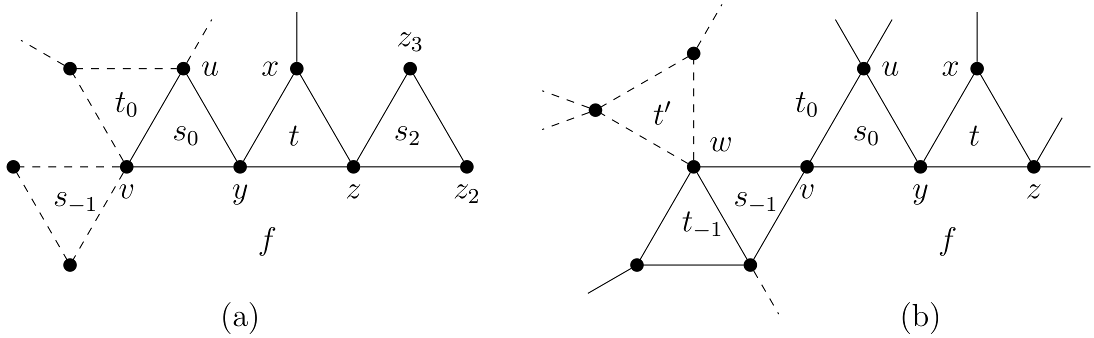

Figure 4.6: Claim 4.4.9 where $d(s_0) = d(s_2) = 3$.

Since G contains no (C3), t is incident to exactly one low vertex. Let x, y, and z be the vertices incident to t in counter-clockwise order where x is low, and let f be the face sharing the edge yz with t. The three faces adjacent to t each send charge $\frac{1}{7}$ to t by (R5). The faces having zx in common with t sends charge $\frac{1}{7}$ to t by (R6).

If one of y and z has degree at least 5, then y and z together send charge at least $1 + \frac{10}{7}$ to t by the vertex rules. Thus, $\mu^*(t) \geq -3 + 1 + \frac{10}{7} + 3 \cdot \frac{1}{7} + \frac{1}{7} = 0$.

We now assume $d(y) = d(z) = 4$. Let $s_0, s_1, s_2, \ldots$ be the faces adjacent to f in clockwise order so that $s_1 = t$. Observe y is incident to s_0 and s_1, while z is incident to s_1 and s_2.

If $d(s_2) \geq 7$, then z sends charge 2 to t by (R1) and y sends charge at least 1 to t by the vertex rules, so $\mu^*(t) \geq -3 + 2 + 1 = 0$. Hence $d(s_2) \leq 4$ and by symmetry $d(s_1) \leq 4$.

If $d(s_2) = 4$, then since G contains no (C11), s_2 must be incident to at least two high vertices. Thus, z sends charge $\frac{10}{7}$ to t by (R2B), and y sends charge at least 1 to t by the vertex rules. Therefore, $\mu^*(t) \geq -3 + \frac{10}{7} + 1 + 3 \cdot \frac{1}{7} + \frac{1}{7} = 0$. Hence $d(s_2) = 3$ and by symmetry, also $d(s_0) = 3$.

Since neither the pair s_0 and s_1, nor the pair s_1 and s_2 form a copy of (C12), the faces s_0 and s_2 are good 3-faces. Let $V(s_0) = \{u, v, y\}$ so that vy is the edge between s_0 and f. See Figure 4.6, for an example of this situation.

Suppose that s_0 is hungry after vertex rules. Since s_0 is not a bad face, it must be in a diamond with a bad face t_0. Since (C10) is reducible, at least one of u, v has degree at least

5. If both u and v have degree at least 5 or one has degree at least 6, then t_0 and s_0 receive enough charge after the vertex rules and s_0 is not hungry. Hence without loss of generality, assume $d(v) = 5$ and $d(u) = 4$. If v is incident to another bad face s_{-1}, then s_0, s_{-1}, t_0, s_t form a reducible configuration (C14). Hence s_{-1} is not a bad 3-face, so (R3C) applies and the faces s_0 and t_0 are not hungry.

By a symmetric argument, s_2 is also not hungry after the vertex rules.

Recall that $d(x) = 3$, $d(y) = d(z) = 4$, and s_0 and s_2 are good 3-faces. Thus, y and z send charge 1 to t by (R1). By (R6), the three faces adjacent to t gives charge $\frac{3}{7}$ to t. If $d(v) \geq 5$, then (R7) applies to f and f contributes $\frac{1}{7}$ to t. Hence $d(v) = 4$. If s_{-1} is not hungry, then again (R7) is applied and f contributes $\frac{1}{7}$. Hence s_{-1} is hungry. Therefore, s_{-1} is a 3-face and s_0 is an isolated 3-face. If s_{-1} is a bad triangle, then s_{-1}, s_0, t form (C13) which is reducible. Hence s_{-1} is not bad and it forms a diamond with a bad 3-face t_{-1}. See Figure 4.6(b). If both vertices shared by s_{-1} and t_{-1} have degree four, faces s_{-1}, t_{-1}, s_0, t form configuration (C15). If both shared vertices have degree at least 5, then the diamond has nonnegative charge after the vertex rules so s_{-1} cannot be hungry.

Hence one shared vertex w has degree 5 and the other is of degree four. If w is not incident to any other bad 3-face, then the faces s_{-1} and t_{-1} are not hungry. If w is in a bad 3-face t', then t', s_{-1}, t_{-1}, s_0, and t form a copy of (C16). Therefore, (R7) applies and contributes $\frac{1}{7}$.

Thus $\mu^*(t) \geq -3 + 1 + 1 + 3 \cdot \frac{1}{7} + 3 \cdot \frac{1}{7} + \frac{1}{7} = 0$. □

4.4.3 Reducibility

We now show that any minimum counterexample (G, L) to Theorem 4.4.1 is prime. Since we already proved that no pair (G, L) is prime, this shows that no counterexample exists.

We start by proving basic properties of G. If G is not connected, there exists an L-coloring for every connected component of G which together give an L-coloring of G. Hence G is connected.

Suppose that G has a cut-vertex v. Let G_1 and G_2 be proper subgraphs of G such that $G_1 \cap G_2 = v$, $G = G_1 \cup G_2$ and $p \in G_1$. By the minimality of G, there exists an L-coloring φ of G_1. Let L' be lists on G_2 where $L'(v) = \varphi(v)$ and $L'(u) = L(u)$ for $u \in V(G_2) - v$. By the minimality of G, there exists an L'-coloring ψ of G_2. Colorings φ and ψ together give an L-coloring of G, a contradiction. Hence G is 2-connected.

Suppose $v \in V(G) - p$ has degree at most two. An L-coloring φ of $G - v$ can be extended to v since $|L(v)| \geq 3$. Hence $d(v) \geq 3$ for every $v \in V(G) - p$.

Suppose $d(p) = 1$. Let v be the neighbor of p. Since G is 2-connected, v is not a cut-vertex. So $V(G) = \{p, v\}$ and G is L-colorable. Hence $d(p) \geq 2$.

By the minimality of G, lists of endpoints of every edge $e \in E(G)$ have a color in common. If not, e can be removed from G without changing possible L-colorings. We denote the color shared by the endpoints of e by $c(e)$.

Lemma 4.4.10. *There is no vertex v with a color $c \in L(v)$ not appearing on the edges incident to v.*

Proof. Suppose $v \in V(G)$ has a color $c \in L(v)$ not appearing in the lists of the adjacent vertices. Let φ be an L-coloring of $G - v$. An L-coloring of G can be obtained by assigning assigning $\varphi(v) = c$. □

Lemma 4.4.11. *G does not contain a trail of three edges $e_1e_2e_3$ where $c(e_1) = c(e_3) \neq c(e_2)$.*

Proof. Suppose G contains a trail of three edges $e_1e_2e_3$ where $c(e_1) = c(e_3) \neq c(e_2)$. The lists of the two endpoints of e_2 both contain the colors $c(e_1)$ and $c(e_2)$, which is a contradiction to the L-list assignment if $c(e_1) \neq c(e_2)$. □

Lemma 4.4.12. *G does not contain a 3-face $e_1e_2e_3$ incident to a low vertex with $c(e_1) = c(e_2)$.*

Proof. If v is a low vertex in a 3-face bounded by $e_1e_2e_3$, then by Lemma 4.4.10 the edges incident to v have distinct colors. Thus, if $c(e_1) = c(e_2)$, they are not both incident to v and $c(e_3) \neq c(e_1) = c(e_2)$, and thus a trail from Lemma 4.4.11 is contained in G. □

Lemma 4.4.13. *G contains no copy of (C1).*

Proof. Suppose puv is a 3-face. Let $c = L(p) = c(pu) = c(pv)$. By Lemma 4.4.11, also $c(uv) = c$. Let φ be and L-coloring of $G - uv$. Since $\varphi(p) = c$, $\varphi(u) \neq c$ and $\varphi(v) \neq c$. Hence $\varphi(u) \neq \varphi(v)$ and φ is an L-coloring of G. □

We will show that a minimum counterexample (G, L) contains no copy of the configurations (C2)–(C16) by using a concrete form of reducibility.

Definition 4.4.14. A configuration C is *reducible* if there exist disjoint sets $X, R \subseteq V(C)$ where X is nonempty, $p \notin X \cup R$, the set $X \cup R$ contains exactly the vertices of C with at most one neighbor outside C, and for every L-coloring φ of $G - X$, there exists an L-coloring ψ of G satisfying the following properties:

- $\varphi(v) = \psi(v)$ for all $v \notin X \cup R$,
- if $\varphi(x) \neq \psi(x)$ for some $x \in R$ with one neighbor outside of C, then $|L(x) \cap \{\psi(y) : y \in N(x) \cap V(C)\}| \leq 1$, and
- if $\varphi(x) \neq \psi(x)$ for some $x \in R$ with no neighbor outside of C, then $|L(x) \cap \{\psi(y) : y \in N(x) \cap V(C)\}| \leq 2$.

If a graph G contains a copy of a reducible configuration, then it is not a minimum counterexample since L-colorings of proper subgraphs extend to L-colorings of G. We now use this definition to prove our simple configurations (C2)–(C6) are reducible.

Lemma 4.4.15. *(C2) is reducible.*

Proof. Let $F = v_1v_2v_3v_4$ be a normal 4-face where $d(v_i) = 3$ for $i \in \{1, 2, 3, 4\}$; see Figure 4.7. Let $X = F$ and $R = \varnothing$ and let φ be an L-coloring of $G - X$. Each vertex v_i has at most one color forbidden by φ, which implies that $|L_\varphi(v_i)| \geq 2$. The only case when a cycle is not 2-choosable is when the cycle has odd length and each vertex has the same list of size 2. Hence φ can be extended to an L-coloring ψ of G. □

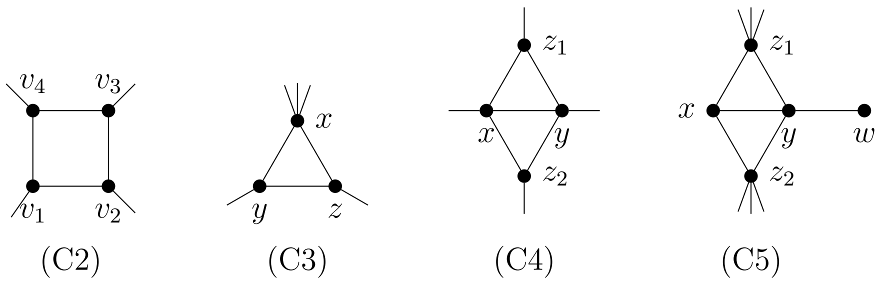

Figure 4.7: Situations in Lemmas 4.4.15, 4.4.16, 4.4.17, and 4.4.18.

Lemma 4.4.16. *(C3) is reducible.*

Proof. Let xyz be a 3-face where y and z have degree 3; see Figure 4.7. Let $X = \{y, z\}$ and $R = \varnothing$, and let φ be an L-coloring of $G - X$. By Lemmas 4.4.10 or 4.4.11, the color $\varphi(x)$ is not equal to both $c(xy)$ and $c(xz)$. Thus, without loss of generality, we assume $\varphi(x) \neq c(xz)$. Observe $|L_\varphi(y)| \geq 1$ and $|L_\varphi(z)| \geq 2$, and thus we can color y and then z to find an L-coloring ψ of G. □

Lemma 4.4.17. *(C4) is reducible.*

Proof. Let z_1xyz_2 be a diamond as in (C4); see Figure 4.7. Let $X = \{z_1, x, y, z_2\}$ and $R = \varnothing$, and let φ be an L-coloring of $G - X$. Observe $|L_\varphi(v)| \geq 2$ for all $v \in \{z_1, x, y, z_2\}$. If the color $c(xy)$ no longer appears in both $L_\varphi(x)$ and $L_\varphi(y)$, then we can remove the edge xy and extend the coloring to an L-coloring ψ of G since z_1xz_2y is a 4-cycle, which is 2-choosable.

Suppose that $c(xy) \in L_\varphi(x)$. By Lemma 4.4.12, $c(xy) \neq c(xz_1)$ and $c(xy) \neq c(xz_2)$. Set $\varphi(x) = c(xy)$. Now φ can be extended to an L-coloring of G by coloring y and then z_1, z_2 in a greedy way. □

Lemma 4.4.18. *(C5) is reducible.*

Proof. Let z_1xyz_2 be a diamond as in (C5); see Figure 4.7. Let $X = \{x\}$ and $R = \{y\}$, and let φ be an L-coloring of $G - X$. By Lemma 4.4.10, the colors $c(xv)$ are distinct for $v \in \{z_1, y, z_2\}$. If $\varphi(v) \neq c(xv)$ for some $v \in \{z_1, y, z_2\}$, then we color $\varphi(x) = c(xv)$ to find an L-coloring of G without recoloring $y \in R$. Thus, $\varphi(v) = c(xv)$ for all $v \in \{z_1, y, z_2\}$.

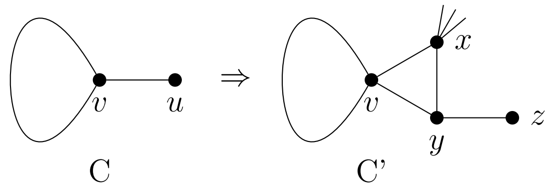

Figure 4.8: Creating compound reducible configurations in Lemma 4.4.20.

By Lemma 4.4.11, the color $c(z_i x)$ is distinct from $c(z_i y)$ for each $i \in \{1,2\}$. Thus, the colors $\varphi(z_i)$ are not in $L(y)$, so $|L(y) \cap \{\varphi(v) : v \in \{x, z_1, z_2\}\}| = 1$. Thus there is at least one color $a \in L(y)$ other than $c(xy)$ and $c(yw)$, where w is the neighbor of y outside the diamond. We color $\psi(x) = c(xy)$ and recolor $\psi(y) = a$ to find an L-coloring of G. □

Lemma 4.4.19. *(C6) is reducible.*

Proof. Observe that (C6) contains (C5) as a subgraph and the proof for (C5) works also for (C6). □

To complete the list of reducible configurations, we describe a way to build compound reducible configurations from simple reducible configurations by adding a bad face.

Lemma 4.4.20 (Iterative Construction)**.** *Let C be a reducible configuration and let $v \in V(C)$ have a unique neighbor $u \in N(v) \setminus V(C)$. Let C' be obtained from C by removing the edge vu and adding two new vertices x, y such that vxy is a 3-face, y has exactly one neighbor z in $N(y) \setminus V(C')$ and x has at least one neighbor in $N(x) \setminus V(C')$. If C is reducible, then C' is reducible.*

See Figure 4.8 for a visualization of this construction.

Proof. Let G contain C'. Observe that since y has degree three, Lemmas 4.4.10 and 4.4.11 guarantee that $c(vy)$, $c(yx)$, and $c(vx)$ are distinct.

Let $X, R \subseteq V(C)$ be given by the definition of C being reducible. Let $X' = X$ and $R' = R \cup \{y\}$. We consider cases based on whether v is in X or R.

Case 1: $v \in X$. For an L-coloring φ of $G - X$, we use the method of extending a coloring to the vertices in C given by its proof of reducibility. When coloring v, the method expects only one color from $L(v)$ appearing in its neighbors outside of C. If at most one of $\varphi(x)$ or $\varphi(y)$ appears in $L(v)$, then the method to color C completes with an L-coloring ψ of G. Otherwise, $\varphi(y) = c(vy)$ and $\varphi(x) = c(vx) \neq c(yx)$. Thus, we recolor $\psi(y) = c(yx)$ and assign $\psi(v) = \varphi(y)$. This recolors y with a color that does not appear in its neighbors, and y has exactly one color restricted within C'.

Case 2: $v \in R$. For an L-coloring φ of $G - X'$, we use the method of extending a coloring to the vertices in C given by its proof of reducibility. If it can be colored without recoloring v, then the resulting coloring is an L-coloring on G. However, if v must be recolored, then v has at most one color restricted from within C. Since $c(vx) \neq c(vy)$, if v has no available colors for this recoloring, we have $\varphi(y) = c(vy)$ and $\varphi(x) = c(vx) \neq c(xy)$. Thus, we recolor $\psi(v) = \varphi(y) = c(vy)$ and $\psi(y) = c(yx)$. Observe that v has at most two colors restricted by its neighbors in C', and y has at most one color restricted by its neighbors in C'.

In either case, we have modified the coloring algorithm for C to apply for C'. □

Lemma 4.4.21. *(C7)–(C16) are reducible.*

Proof. Each configuration is built using Lemma 4.4.20 from a known reducible configuration. We use the notation "(Ci) $\longrightarrow$ (Cj)" to denote "Applying Lemma 4.4.20 to (Ci) results in (Cj)," in the following pairs:

$$\begin{array}{ccc} \text{(C6)} \longrightarrow \text{(C7)} & \text{(C4)} \longrightarrow \text{(C8)} & \text{(C5)} \longrightarrow \text{(C9)} \\ \text{(C4)} \longrightarrow \text{(C10)} & \text{(C2)} \longrightarrow \text{(C11)} & \text{(C3)} \longrightarrow \text{(C12)} \\ \text{(C12)} \longrightarrow \text{(C13)} & \text{(C8)} \longrightarrow \text{(C14)} & \text{(C8)} \longrightarrow \text{(C15)} \\ & \text{(C15)} \longrightarrow \text{(C16)} & \end{array}$$

Thus, by Lemma 4.4.20 and previous lemmas, these configurations are reducible. □

4.5 Conclusion

The main problem if planar graphs are $(3,1)$-choosable remains open. We hope that this result could serve as an inspiration of possible approaches to the problem. Unfortunately, the conditions of Theorem 4.2.1 and Theorem 4.3.1 are not valid for all planar graphs; see Figure 4.9. Let us note that we do not have an example where P has length one and the endpoints are not in a triangle.

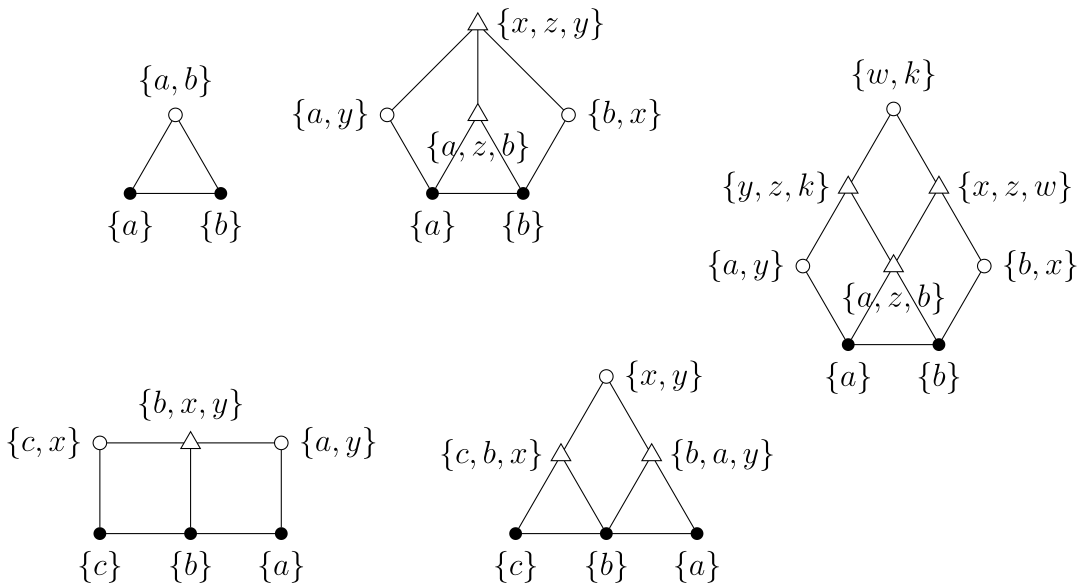

Figure 4.9: Some examples where conditions of Theorem 4.2.1 and Theorem 4.3.1 do not generalize to all planar graphs.

The last thing we promised is a planar graph G without 4-cycles and 5-cycles that is not $(3,2)$-choosable. It is a modification of a construction of Wang, Wen, and Wang [51]. The main building gadget is the graph H depicted in Figure 4.10. It has two vertices with lists of size one. The graph G is created by taking 9 copies of H and identifying vertices with lists $\{a\}$ into one vertex v and vertices with lists $\{b\}$ into one. Vertices u and v get disjoint lists and we assign to every 9 possible colorings of u and v one gadget, where the coloring of u and v cannot be extended. By inspecting the gadget, the reader can check that G cannot be colored and that G has no cycles of length 4 or 5.

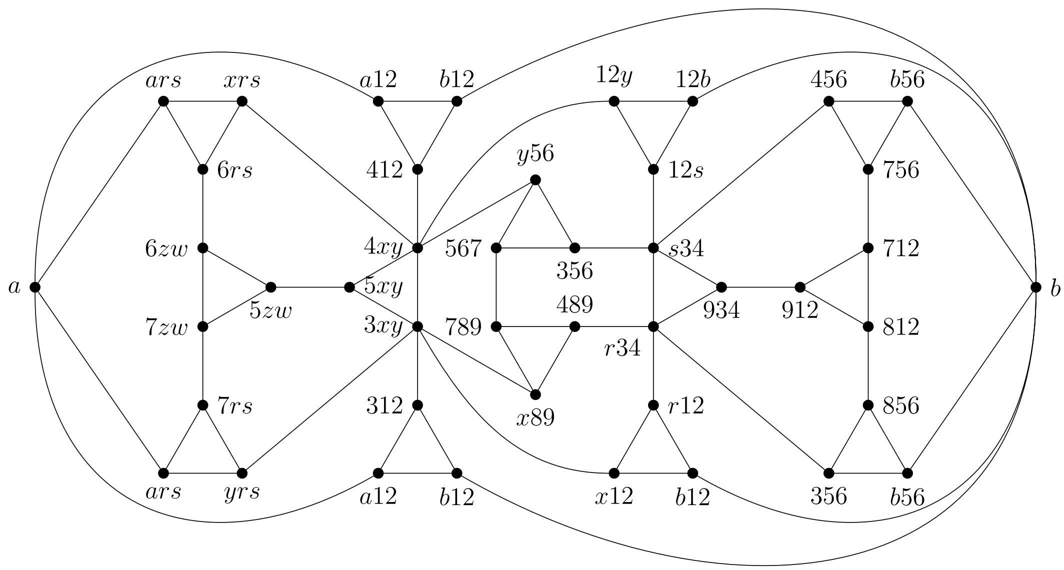

Figure 4.10: Building block of a non $(3,2)$-choosable planar graph without cycles of length 4 and 5.

We thank Mohit Kumbhat for introducing them to the problem during 3rd Emléktábla Workhops and thank Kyle F. Jao for fruitful discussions and encouragement in the early stage of the project.

Chapter 5

Vertex Arboricity of Toroidal Graphs

5.1 Introduction

The *vertex arboricity* of a graph G, denoted $a(G)$, is the minimum k such that $V(G)$ can be partitioned into k sets $V_1, \ldots, V_k$ where $G[V_i]$ is a forest for each $i \in [k]$. This can be viewed as a vertex coloring f with k colors where each color class V_i induces a forest; namely, $G[f^{-1}(i)]$ is an acyclic graph for each $i \in [k]$. Note that a graph with no cycles is a forest, and it has vertex arboricity 1.

Vertex arboricity, also known as point arboricity, was first introduced by Chartrand, Kronk, and Wall [13] in 1968. Among other things, they proved Theorem 5.1.1. Shortly after, Chartrand and Kronk [12] showed that Theorem 5.1.1 is sharp by constructing a planar graph with vertex arboricity 3, and they also proved Theorem 5.1.2.

Theorem 5.1.1 ([13])**.** *If G is a planar graph, then $a(G) \leq 3$.*

Theorem 5.1.2 ([12])**.** *If G is an outerplanar graph, then $a(G) \leq 2$.*

We direct the readers to the work of Stein [44] and Hakimi and Schmeichel [27] for a complete characterization of maximal plane graphs with vertex arboricity 2.

In 2008, Raspaud and Wang [40] not only determined the order of the smallest planar graph G with $a(G) = 3$, but also found several sufficient conditions for a planar graph to have vertex arboricity at most 2 in terms of forbidden small structures; namely, they proved that a planar graph with either no triangles at distance less than 2 or no k-cycles for some fixed $k \in \{3, 4, 5, 6\}$ has vertex arboricity at most 2. Chen, Raspaud, and Wang [14] showed that forbidding intersecting triangles is also sufficient for planar graphs. In [40], Raspaud and Wang asked the following question:

Question 5.1.3 ([40])**.** *What is the maximum integer μ where for all $k \in \{3, \ldots, \mu\}$, a planar graph G with no k-cycles has $a(G) \leq 2$?*

Raspaud and Wang's results imply $6 \leq \mu \leq 21$. The lower bound was increased to 7 by Huang, Shiu, and Wang [30] since they proved planar graphs without 7-cycles have vertex arboricity at most 2.

We completely answer the question for toroidal graphs, which are graphs that are embeddable on a torus with no crossings.

Kronk [34] and Cook [20] investigated vertex arboricity on higher surfaces in 1969 and 1974, respectively.

Theorem 5.1.4 ([34])**.** *If G is a graph embeddable on a surface of positive genus g, then $a(G) \leq \lfloor \frac{9+\sqrt{1+48g}}{4} \rfloor$.*

Theorem 5.1.5 ([20])**.** *If G is a graph embeddable on a surface of genus g with no 3-cycles, then $a(G) \leq 2 + \sqrt{g}$.*

Theorem 5.1.6 ([20])**.** *If G is a graph embeddable on a surface of positive genus g with girth at least $5 + 4\log_3 g$, then $a(G) \leq 2$.*

Theorem 5.1.4 says every toroidal graph G has $a(G) \leq 4$. Theorem 5.1.5 says a toroidal graph with no 3-cycles has vertex arboricity at most 3, and Theorem 5.1.6 only guarantees that toroidal graphs with girth at least 5 have vertex arboricity at most 2. Both of these cases were improved by Kronk and Mitchem [35] who showed Theorem 5.1.7. Recently, Zhang [55] showed Theorem 5.1.8, which says that forbidding 5-cycles in toroidal graphs is sufficient to guarantee vertex arboricity at most 2.

Theorem 5.1.7 ([35])**.** *If G is a toroidal graph with no 3-cycles, then $a(G) \leq 2$.*

Theorem 5.1.8 ([55])**.** *If G is a toroidal graph with no 5-cycles, then $a(G) \leq 2$.*

Since the complete graph on 5 vertices is a toroidal graph with no cycles of length at least 6 and has vertex arboricity 3, the only remaining case is when 4-cycles are forbidden in toroidal graphs; this is our main result.

Theorem 5.1.9. *If G is a toroidal graph with no 4-cycles, then $a(G) \leq 2$.*

In Section 5.2, we will prove some structural lemmas needed in Section 5.3, where we prove Theorem 5.1.9 using (simple) discharging rules. Note that Theorem 5.1.9 implies that every planar graph without 4-cycles have vertex arboricity at most 2, which is a result in [40].

5.2 Lemmas

From now on, let G be a counterexample to Theorem 5.1.9 with the fewest number of vertices. It is easy to see that G must be 2-connected and the minimum degree of a vertex of G is at least 4.

A graph is k-regular if every vertex in the graph has degree k. A set $S \subseteq V(G)$ of vertices is k-regular if every vertex in S has degree k in G. A *triangular cycle* is a cycle adjacent to a triangle. A (partial) 2-coloring f of G is *good* if each color class induces a forest.

Lemma 5.2.1. *If $V(G)$ contains a 4-regular set S where $G[S]$ is a cycle C, then every good coloring f of $G[V(G) \setminus S]$ that does not extend to all of G has either*

Case 1: $f(v)$ the same for every vertex $v \notin S$ that has a neighbor in S, or

Case 2: $f(x) \neq f(y)$ for all $v \in S$ such that $N(v) \setminus S = \{x, y\}$ and C is an odd cycle.

Proof. Let $S = \{v_1, \ldots, v_s\}$ where $v_1, \ldots, v_s$ are the vertices of C in this order. For each $i \in [s]$, let $\{x_i, y_i\} = N(v_i) \setminus S$. Obtain a good coloring f of $G[V(G) \setminus S]$ by the minimality of G. We will show that if f does not satisfy one of the two conditions in the statement, then f can be extended to all of G.

If s is even and $\{f(x_i), f(y_i)\} = \{1, 2\}$ for each $i \in [s]$, then let $f(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$ to extend f to all of G.

We know that there exists at least one index $j \in [s]$ where $f(x_j) = f(y_j)$ since we are not in Case 2. For each $i \in [s]$ where $f(x_i) = f(y_i)$, let $f(v_i) = \begin{cases} 1 & \text{if } f(x_i) = f(y_i) = 2 \\ 2 & \text{if } f(x_i) = f(y_i) = 1 \end{cases}$.
Now, consider the vertices of C in cyclic order starting with $i = j$, and for $f(v_i)$ that is not defined yet, let $f(v_i) = \begin{cases} 1 & \text{if } f(v_{i-1}) = 2 \\ 2 & \text{if } f(v_{i-1}) = 1 \end{cases}$ for all i. We claim that this coloring f is now a good coloring of all of G, which is a contradiction.

Note that f cannot have a monochromatic cycle that only uses vertices of $V(G) \setminus S$. Also, f cannot have a monochromatic cycle where x_i, v_i, y_i are consecutive vertices on this cycle since $f(x_i) = f(v_i) = f(y_i)$ never happens. Moreover, f cannot have a monochromatic cycle where v_i, v_{i+1}, x_i are consecutive vertices on this cycle since $f(v_i) = f(v_{i+1})$ implies that $f(x_{i+1}) = f(y_{i+1}) \neq f(v_{i+1})$. Thus, a monochromatic cycle in f must be C itself, which is possible only in Case 1. □

Lemma 5.2.2. *$V(G)$ does not contain a 4-regular set S where $G[S]$ is a triangular cycle.*

Proof. Let $S = \{v_1, \ldots, v_s, u\}$, so that u, v_1, v_2 are the vertices of a triangle and let $C = S \setminus \{u\}$. Let $v_1, \ldots, v_s$ be the vertices of C in this order. For $i \in [2]$, let v'_i be the neighbor of v_i that is not in S. We will obtain a good coloring of all of G to show that S does not exist. Obtain a good coloring f of $G[V(G) \setminus C]$ by the minimality of G.

Assume that the first case of Lemma 5.2.1 happens and without loss of generality, assume $f(v) = 1$ for every vertex $v \notin C$ that has a neighbor in C. For $i \in [s] \setminus \{1\}$, let $f(v_i) = 2$ and let $f(v_1) = 1$. If f is not a good coloring, then in the graph induced by $f^{-1}(1)$, there must exist a cycle where v'_1, v_1, u, z are consecutive vertices on the cycle for some $z \in N(u) \setminus \{v_1, v_2\}$. Now, alter f by letting $f(u) = 2$ to obtain a good coloring of all of G.

Assume that the second case of Lemma 5.2.1 happens and without loss of generality, assume $f(v_1') = f(v_2') = 1$ and $f(u) = 2$. Note that s must be odd. For $i \in [s] \setminus \{1\}$, let $f(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 2 & \text{if } i \text{ is even} \end{cases}$, let $f(v_1) = 2$, and change $f(u)$ from 2 to 1. If f is not a good coloring, then in the subgraph induced by $f^{-1}(1)$, there must exist a cycle where u and two of its neighbors that are not v_1, v_2 are consecutive vertices on the cycle. Now for $i \in [s] \setminus \{1\}$, alter f by letting $f(v_i) = \begin{cases} 2 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even} \end{cases}$ (but keep $f(u) = 2$) to obtain a good coloring of all of G. $\square$

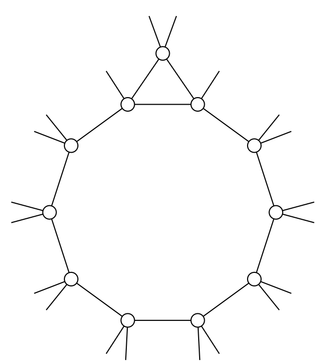

Figure 5.1: Forbidden Configuration. The white vertices do not have incident edges besides the ones drawn.

Let a vertex v be *bad* if $d(v) = 4$ and v is incident to two triangles; a vertex is *good* if it is not bad. Let $H = H(G)$ be the graph where $V(H)$ is the set of triangles of G incident to at least one bad vertex and let $uv \in E(H)$ if and only if there is a bad vertex of G that is incident to both triangles that correspond to u, v.

Claim 5.2.3. *Each component of H is either a cycle or a tree.*

Proof. Assume for the sake of contradiction that H has a component D with a cycle C where C is not the entire component. Let $v \in V(D) \setminus V(C)$ be a vertex that has a neighbor in $V(C)$. The graph in G that corresponds to this structure is forbidden by Lemma 5.2.2, which is a contradiction. See Figure 5.2. $\square$

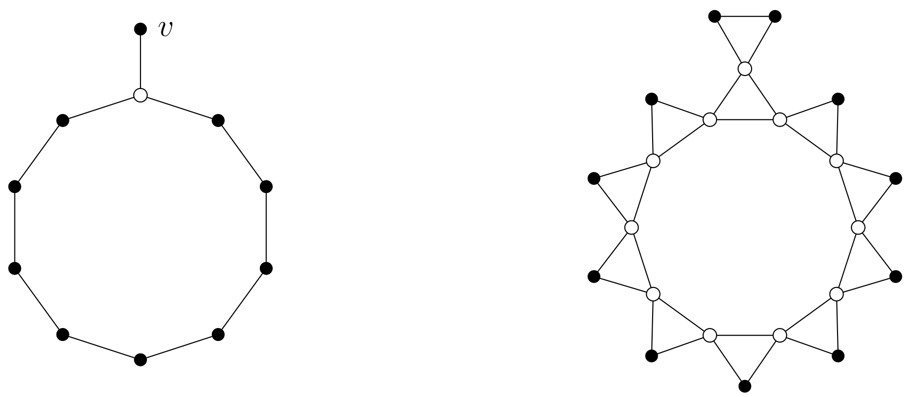

Figure 5.2: The cycle C in the proof of Claim 5.2.3 (left) and the corresponding graph in G (right). The white vertices do not have incident edges besides the ones drawn. The black vertices may have other incident edges.

Here is a lemma that will help later on.

Lemma 5.2.4. *Every n-vertex tree where with maximum degree 3 has exactly 2 more vertices of degree 1 than vertices of degree 3.*

Proof. Let z_i be the number of vertices of degree i. An n-vertex tree has $n-1$ edges and the sum of the degrees is twice the number of edges. Thus we have $n = z_1 + z_2 + z_3$ and $2(n-1) = z_1 + 2z_2 + 3z_3$. By eliminating z_2, we get $z_1 = z_3 + 2$. □

5.3 Discharging

In this section, we will prove that G cannot exist. Fix an embedding of G and let $F(G)$ be the set of faces. We assign an *initial charge* $\mu(z)$ to each $z \in V(G) \cup F(G)$, and then we will apply a discharging procedure to end up with *final charge* $\mu^*(z)$ at z. We prove that the final charge has positive total sum, whereas the initial charge sum is at most zero. The discharging process will preserve the total charge sum, and hence we find a contradiction to conclude that G does not exist.

For every vertex $v \in V(G)$, let $\mu(v) = d(v) - 6$, and for every face $f \in F(G)$, let

$\mu(f) = 2d(f) - 6$. The total initial charge is zero since

$$\sum_{z \in V(G) \cup F(G)} \mu(z) = \sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in V(F)} (2d(f) - 6) = 6|E(G)| - 6|V(G)| - 6|F(G)| \leq 0.$$

The final equality holds by Euler's formula.

For the discharging procedure we introduce the notion of a *bank*, which serves as a placeholder for charges. For each component D of the auxiliary graph $H(G)$, we will define a separate bank; let $b(D)$ denote the bank. We give each bank an initial charge of zero and we will show that either some vertex or some bank has positive final charge. The rest of this section will prove that the sum of the final charge after the discharging phase is positive.

Recall that a vertex v is *bad* if $d(v) = 4$ and v belongs to two triangles and a vertex is *good* if it is not bad. A good vertex v is incident to a bank $b(D)$ if there is a vertex u of D where v is incident to the triangle in G that corresponds to u. Note that each bad vertex of G is an edge of $H(G)$.

Here are the discharging rules:

(R1) Each face distributes its initial charge uniformly to each incident vertex.

(R2) Each good vertex v sends charge $\frac{2}{5}$ to each bank $b(D)$ each time v is incident to $b(D)$.

(R3) For each component D of $H(G)$, the bank $b(D)$ sends charge $\frac{2}{5}$ to each bad vertex in G that corresponds to an edge in D.

It is trivial that each face has nonnegative final charge. Moreover, each face f with $d(f) \geq 5$ sends charge $\frac{\mu(f)}{d(f)} = \frac{2d(f)-6}{d(f)} \geq \frac{4}{5}$ to each incident vertex. We will first show that each vertex has nonnegative final charge. Then we will show that either some bank or some vertex has positive final charge.

Note that since G has no 4-cycles, each vertex v is incident to at most $\lfloor \frac{d(v)}{2} \rfloor$ triangles, and therefore at most $\lfloor \frac{d(v)}{2} \rfloor$ banks.

Claim 5.3.1. *Each vertex has nonnegative final charge. Moreover, each vertex of degree at least 5 has positive final charge.*

Proof. A vertex v with $d(v) \geq 6$ has nonnegative initial charge and receives at least $\frac{4}{5} \cdot \frac{d(v)}{2}$ after (R1). Since v is incident to at most $\lfloor \frac{d(v)}{2} \rfloor$ incident banks, $\mu^*(v) \geq \frac{4}{5} \cdot \frac{d(v)}{2} - \frac{2}{5} \cdot \lfloor \frac{d(v)}{2} \rfloor > 0$. A vertex v with $d(v) = 5$ will receive charge from at least 3 incident faces and will give charge to at most 2 incident banks. Therefore, $\mu^*(v) \geq -1 + 3 \cdot \frac{4}{5} - 2 \cdot \frac{2}{5} > 0$.

A good vertex v with $d(v) = 4$ will receive charge from at least 3 faces and will give charge to at most 1 incident bank. Therefore, $\mu^*(v) \geq -2 + 3 \cdot \frac{4}{5} - \frac{2}{5} = 0$. A bad vertex v will receive charge at least $\frac{4}{5}$ from two faces and $\frac{2}{5}$ from exactly one bank. Therefore, $\mu^*(v) \geq -2 + 2 \cdot \frac{4}{5} + \frac{2}{5} = 0$. □

Given a component D of $H(G)$, since an edge of D corresponds to a bad vertex of G, we need to check that $b(D)$ has enough charge for each edge of D.

Claim 5.3.2. *Each bank $b(D)$ where D is a cycle has nonnegative final charge.*

Proof. Assume D is a cycle C with n vertices. Since D is a cycle, each triangle in G that corresponds to a vertex in D must be incident to one good vertex; each good vertex will send charge $\frac{2}{5}$ to $b(D)$. Thus, $b(D)$ receives charge $\frac{2}{5}n$ and there are n edges in D so $b(D)$ has nonnegative final charge. □

Claim 5.3.3. *Each bank $b(D)$ where D is a tree has positive final charge.*

Proof. Assume T has n vertices. T has maximum degree at most 3 since a triangle in G cannot be incident to more than 3 bad vertices. For $i \in [3]$, let z_i be the number of vertices of degree i in T.

Each triangle in G that corresponds to a degree 1 vertex in T is incident to 2 good vertices, and each triangle in G that corresponds to a degree 2 vertex in T is incident to 1 good vertex. Thus $b(T)$ gets charge $\frac{4}{5}z_1 + \frac{2}{5}z_2$, and must spend $\frac{2}{5}|E(T)| = \frac{2}{5}(n-1)$. Since $z_1 = z_3 + 2$ by Lemma 5.2.4, it follows that $\frac{4}{5}z_1 + \frac{2}{5}z_2 = \frac{2}{5}n + \frac{4}{5} > \frac{2}{5}n - \frac{2}{5}$. Thus, $b(T)$ has positive final charge. □

If $H(G)$ has a component that is a tree T, then $b(T)$ has positive final charge. If $H(G)$ has a component that is a cycle, then there exists a vertex of degree at least 5 in G, and by Claim 5.3.1, this vertex has positive final charge. If $H(G)$ has no components, then there are no bad vertices, and we are done since either some bank or some vertex will have positive final charge.

Chapter 6

Improper Coloring of Planar Graphs

6.1 Introduction

A graph is $(d_1, \ldots, d_r)$-colorable if its vertex set can be partitioned into r sets $V_1, \ldots, V_r$ where the maximum degree of the graph induced by V_i is at most d_i for each $i \in [r]$; in other words, there exists a function $f : V(G) \to [r]$ where the graph induced by vertices of color i has maximum degree at most d_i for $i \in [r]$.

There are many papers that study $(d_1, \ldots, d_r)$-colorings of sparse graphs resulting in corollaries regarding planar graphs, sometimes with restrictions on the length of a smallest cycle. The well-known Four Color Theorem [1, 2] is exactly the statement that planar graphs are $(0, 0, 0, 0)$-colorable. Cowen, Cowen, and Woodall [21] proved that planar graphs are $(2, 2, 2)$-colorable, and Eaton and Hull [23] and Škrekovski [42] proved that this is sharp by exhibiting non-$(k, k, 1)$-colorable planar graphs for each k. Thus, the problem is completely solved when $r \geq 3$.

Let $\mathcal{G}_g$ denote the class of planar graphs with minimum cycle length at least g. Given any d_1 and d_2, consider the following graph constructed by Montassier and Ochem [38]. Let $X_i(d_1, d_2)$ be a copy of K_{2,d_1+d_2+1} where one part is $\{x_i, y_i\}$. Obtain $Y(d_1, d_2)$ in the following way: start with $X_1(d_1, d_2), \ldots, X_{d_1+2}(d_1, d_2)$ and identify $x_1, \ldots, x_{d_1+2}$ into x, and add the edges $y_1y_2, \ldots, y_1y_{d_1+2}$. It is easy to verify that $Y(d_1, d_2)$ is not (d_1, d_2)-colorable.

Therefore, we focus on graphs in $\mathcal{G}_5$. There are also many papers [5, 7, 29, 6, 9] that investigate (d_1, d_2)-colorability for graphs in $\mathcal{G}_g$ for $g \geq 6$. Regarding graphs in $\mathcal{G}_5$, we know that they are $(2, 6)$-colorable [7] and $(4, 4)$-colorable [29].

In this section, we prove the following theorem:

Theorem 6.1.1. *Planar graphs with girth at least* 5 *are* $(3,5)$*-colorable.*

Since there are non-$(3,1)$-colorable graphs in $\mathcal{G}_5$ [38], Theorem 6.1.1 implies that the minimum d where graphs in $\mathcal{G}_5$ are $(3,d)$-colorable is in $\{2,3,4,5\}$; determining this d would be interesting.

Also, this solves one of the previously unknown cases of the following question:

Question 6.1.2. *Are planar graphs with girth at least* 5 *indeed* (d_1,d_2)*-colorable for all* $d_1+d_2=8$ *where* $d_2 \geq d_1 \geq 1$*?*

The only remaining case of Question 6.1.2 is when $d_1 = 1$ and $d_2 = 7$. Interestingly enough, we do not know even if there is a finite k where graphs in $\mathcal{G}_5$ are $(1,k)$-colorable.

In the figures throughout this section, the white vertices do not have incident edges besides the ones drawn, and the black vertices may have other incident edges.

In Section 6.2, we prove structural lemmas for non-(d_1,d_2)-colorable graphs with minimum order. In Section 6.3, we reveal some more structure of minimum counterexamples to Theorem 6.1.1 by focusing on the case when $d_1 = 3$ and $d_2 = 5$. Finally, we prove Theorem 6.1.1 by using a discharging procedure in Section 6.4.

6.2 Non-(d_1,d_2)-colorable graphs with minimum order

In this section, we prove structural lemmas regarding non-(d_1,d_2)-colorable graphs with minimum order; let $H(d_1,d_2)$ be such a graph. It is easy to see that the minimum degree of (a vertex of) $H(d_1,d_2)$ is at least 2 and $H(d_1,d_2)$ is connected.

Given a (partial) coloring f of $H(d_1,d_2)$ and $i \in [2]$, a vertex v with $f(v)=i$ is *i-saturated* if v is adjacent to d_i neighbors colored i. By definition, an i-saturated vertex has at least d_i neighbors.

Lemma 6.2.1. *Let $H = H(d_1, d_2)$ where $d_1 \leq d_2$. If v is a 2-vertex of H, then v is adjacent to two $(d_1 + 2)^+$-vertices, one of which is a $(d_2 + 2)^+$-vertex.*

Proof. Let $N(v) = \{v_1, v_2\}$ and let f be a coloring of $H - v$ obtained by the minimality of H. If $f(v_1) = f(v_2)$, then letting $f(v) \in [2] \setminus \{f(v_1)\}$ gives a coloring of H, which is a contradiction. Without loss of generality, assume $f(v_1) = 1$ and $f(v_2) = 2$. Since setting $f(v) = 1$ must not give a coloring of H, we know v_1 is 1-saturated. Since setting $f(v_1) = 2$ and $f(v) = 1$ must not give a coloring of H, we know v_1 has a neighbor colored 2. This implies $d(v_1) \geq d_1 + 2$. Similar logic implies that $d(v_2) \geq d_2 + 2$. □

Lemma 6.2.2. *Let $H = H(d_1, d_2)$ where $d_1 \leq d_2$. If v is a 3-vertex of H, then v is adjacent to at least two $(d_1 + 2)^+$-vertices, one of which is a $(d_2 + 2)^+$-vertex.*

Proof. Let $N(v) = \{v_0, v_1, v_2\}$ and let f be a coloring of $H - v$ obtained by the minimality of H. If $f(v_0) = f(v_1) = f(v_2)$, then letting $f(v) \in [2] \setminus \{f(v_0)\}$ gives a coloring of H, which is a contradiction. Without loss of generality, assume $f(v_1) = 1$ and $f(v_2) = 2$. Further assume $f(v_0) = i$ for some $i \in [2]$ and let $j \in [2] \setminus \{i\}$.

Since setting $f(v) = j$ must not give a coloring of H, we know v_j is j-saturated. Since setting $f(v) = j$ and $f(v_j) = i$ must not give a coloring of H, we know v_j has a neighbor colored i. This implies $d(v_j) \geq d_j + 2$. Since setting $f(v) = i$ must not give a coloring of H, we know either v_0 or v_i is i-saturated. If both $d(v_0), d(v_i) \leq d_i + 1$, then recolor each i-saturated vertex in $\{v_0, v_i\}$ with color j, and set $f(v) = i$ to obtain a coloring of H, which is a contradiction. Therefore either v_0 or v_i has degree at least $d_i + 2$. □

Lemma 6.2.3. *Let $H = H(d_1, d_2)$ where $d_1 + 1 \leq d_2$. If v is a $(d_1 + d_2 + 1)^-$-vertex of H, then v is adjacent at least one $(d_1 + 2)^+$-vertex.*

Proof. Suppose no neighbor of v is a $(d_1 + 2)^+$-vertex and let f be a coloring of $H - v$ obtained by the minimality of H. Both colors 1 and 2 must appear on $N(v)$, otherwise, we can easily obtain a coloring of H, which is a contradiction. Since setting $f(v) = 2$ must not give a coloring of H and v cannot be adjacent to a 2-saturated vertex (since a 2-saturated neighbor of v has degree at least $d_2 + 1 \geq d_1 + 2$), we know that v has at least $d_2 + 1$ neighbors colored 2. Since setting $f(v) = 1$ must not give a coloring of H, we know that either v has at least $d_1 + 1$ neighbors colored 1 or v has a 1-saturated neighbor. The former case is impossible since $d(v) \leq d_1 + d_2 + 1$. Since each neighbor of v is a $(d_1 + 1)^-$-vertex, each 1-saturated neighbor of v can be recolored with 2. Now we can let $f(v) = 1$ to obtain a coloring of H, which is a contradiction. $\square$

Lemma 6.2.4. *Let $H = H(d_1, d_2)$ where $d(v_1) \leq d_2 + 1$ and let $N(v) = \{v_1, v_2\}$ for a 2-vertex v of H. If f is a coloring of $H - v$, then $f(v_1) = 1$ and $f(v_2) = 2$.*

Proof. If $f(v_1) = f(v_2)$, then letting $f(v) \in [2] \setminus \{f(v_1)\}$ gives a coloring of H, which is a contradiction. If $f(v_1) = 2$ and $f(v_2) = 1$, then let $f(v) = 2$ to obtain a coloring of H, unless v_1 is 2-saturated. This implies that $d(v_1) = d_2 + 1$ and $f(z) = 2$ for $z \in N(v_1) \setminus \{v\}$, so we can let $f(v_1) = 1$ to obtain a coloring of H, which is a contradiction. $\square$

6.3 Non-$(3,5)$-colorable planar graphs with minimum order

From now on, let G be a counterexample to Theorem 6.1.1 with the fewest number of vertices, and fix some embedding of G on the plane. It is easy to see that the minimum degree of (a vertex of) G is at least 2 and G is connected.

A 5^+-vertex is *high*, and a 3^--vertex is *low*. Recall that given a (partial) coloring f of G, a vertex v with $f(v) = i$ is *i-saturated* if v is adjacent to $2i + 1$ neighbors colored i.

Lemma 6.3.1. *Let $u_1u_2u_3u_4$ be a path in G where $d(u_2) = d(u_4) = 2$. If f is a coloring of $G - u_2$ where $f(u_1) = f(u_4) = 1$ and $f(u_3) = 2$, then $d(u_3) \geq 8$.*

Proof. Since setting $f(u_2) = 2$ must not give a coloring of G, it must be that u_3 is 2-saturated. Moreover, since setting $f(u_2) = 2$ and $f(u_3) = 1$ must not give a coloring of G, we know that u_3 has a neighbor colored with 1 that is not u_4. This implies $d(u_3) \geq 8$. $\square$

Lemma 6.3.2. *Let $u_1u_2u_3$ be a path in G where $d(u_2) = 2$ and $d(u_3) \leq 9$. If f is a coloring of $G - u_2$ where $f(u_1) = 1$ and $f(u_3) = 2$, then u_3 has a high neighbor colored* 1.

Proof. Since setting $f(u_2) = 2$ must not give a coloring of G, it must be that u_3 is 2-saturated. Moreover, since setting $f(u_2) = 2$ and $f(u_3) = 1$ must not give a coloring of G, either u_3 has 4 neighbors colored 1 or at least one 1-saturated neighbor. The former is impossible since $d(u_3) \leq 9$, so u_3 has some 1-saturated neighbors. If all such neighbors are not high, then we can recolor each one with 2 and let $f(u_3) = 1$ and $f(u_2) = 2$ to complete a coloring of G, which is a contradiction. $\square$

A *bad face* is a 5-face incident to two 2-vertices; a face is *good* if it is not bad.

Lemma 6.3.3. *A* 3*-vertex cannot be incident to a bad face.*

Proof. Follows immediately from Lemma 6.2.1 and the observation that a vertex on a bad face must be either a 2-vertex or a neighbor of a 2-vertex. $\square$

A vertex h is the *head* of a bad face $hu_1u_2u_3u_4$ if $d(u_1) = d(u_4) = 2$. Note that each bad face has exactly one head. A 2-vertex u_1 incident to a bad face b is *close* to a good face g if u_2u_3 is a common edge of b and g and u, u_2, u_3 are high vertices and u, u_2, u_3 are consecutive vertices of g and u_1, u_2, u_3 are consecutive vertices of b. See Figure 6.1.

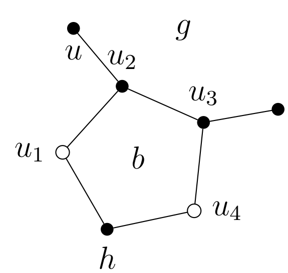

Figure 6.1: The head h of a bad face b and a 2-vertex u_1 that is close to a good face g.

A vertex v is *chubby* if either $d(v) \in \{7, 8, 9\}$ and v has at least two high neighbors or $d(v) \geq 10$. A vertex v is *fat* if either $d(v) \in \{8, 9\}$ and v has at least two high neighbors or $d(v) \geq 10$ and v has at least one high neighbor. By definition, a fat vertex is also chubby.

Lemma 6.3.4. *Let $f_0 = x_1v_1vv_2x_2$ be a bad face where $d(v) = 2$, $d(v_1) = 5$, $d(v_2) \geq 7$. If f is a coloring of $G - v$, then $f(v_1) = 1$ and $f(v_2) = 2$, and one of the following holds:*

(i) If v_1 is the head of f_0, then $f(x_1) = 1$ and $f(x_2) = 2$ and x_2 and v_2 are chubby vertices.

(ii) If v_2 is the head of f_0 and $f(x_1) = 2$, then $f(x_2) = 1$ and x_1 is a fat vertex.

(iii) If v_2 is the head of f_0 and $f(x_1) = 1$, then v_1 has a 2-saturated neighbor.

Proof. By Lemma 6.2.4, $f(v_1) = 1$ and $f(v_2) = 2$. For $i \in [2]$, since setting $f(v) = i$ must not give a coloring of G, we know v_i is i-saturated.

(i) : Since v_1 is the head of f_0, we know $d(x_1) = 2$. If $f(x_1) = 2$ so that $f(z) = 1$ for each $z \in N(v_1) \setminus \{v, x_1\}$, then setting $f(v_1) = 2$ and $f(v) = 1$ is a coloring of G, which is a contradiction. Thus $f(x_1) = 1$. If $f(x_2) = 1$, then letting $f(v) = 1$ and $f(x_1) = 2$ is a coloring of G, which is a contradiction. Thus, $f(x_2) = 2$. If $d(v_2) \in \{7, 8, 9\}$, then by applying Lemma 6.3.2 to v_1vv_2, we know that v_2 must have two high neighbors; namely, x_2 and a neighbor colored 1. Therefore, v_2 is a chubby vertex.

Since $d(x_1) = 2$ and $d(v_1) = 5$, we know $d(x_2) \geq 7$ by Lemma 6.2.1. Now by letting $f(v) = 1$ and removing the coloring on x_1, the situation is symmetric for x_2; this implies that x_2 is a chubby vertex.

(ii) : Since $f(x_1) = 2$, we know $f(z) = 1$ for $z \in N(v_1) \setminus \{v, x_1\}$. If $f(x_2) = 2$, then letting $f(v) = 2$ and $f(x_2) = 1$ is a coloring of G, which is a contradiction. Thus, $f(x_2) = 1$. Since setting $f(v) = 1$ and $f(v_1) = 2$ must not give a coloring of G, we know x_1 is 2-saturated. Note that x_1 has a high neighbor v_1. Since setting $f(v_1) = 2$ and $f(v) = f(x_1) = 1$ must not give a coloring of G, we know that x_1 has a neighbor colored 1 that is neither x_2 nor v_1. This implies $d(x_1) \geq 8$. If $d(x_1) \in \{8, 9\}$ and every 1-saturated neighbor of x_1 except v_1 is a

4-vertex, then recolor each such neighbor with 2 (and let $f(v_1) = 2$ and $f(v) = (x_1) = 1$) to obtain a coloring of G. Thus, if $d(x_1) \in \{8, 9\}$, then x_1 has two high neighbors; namely, v_1 and another neighbor colored 1. Thus, x_1 is a fat vertex.

(iii) : Since setting $f(v) = 1$ and $f(v_1) = 2$ must not give a coloring of G, we know v_1 has either a 2-saturated neighbor or 6 neighbors colored 2. Since a 5-vertex v_1 cannot have 6 neighbors of color 2, we know v_1 has a a 2-saturated neighbor. $\square$

6.4 Discharging

Since the embedding of G fixed, we can let $F(G)$ denote the set of faces of this embedding. In this section, we will prove that G cannot exist by assigning an *initial charge* $\mu(z)$ to each $z \in V(G) \cup F(G)$, and then applying a discharging procedure to end up with *final charge* $\mu^*(z)$ at z. We prove that the final charge has nonnegative total sum, whereas the initial charge sum is negative. The discharging procedure will preserve the total charge sum, and hence we find a contradiction to conclude that the counterexample G does not exist.

For each $z \in V(G) \cup F(G)$, let $\mu(z) = d(z) - 4$. The total initial charge is negative since

$$\sum_{z \in V(G) \cup F(G)} \mu(z) = \sum_{z \in V(G) \cup F(G)} (d(z) - 4) = -4|V(G)| + 4|E(G)| - 4|F(G)| = -8 < 0.$$

The last equality holds by Euler's formula.

The rest of this section will prove that $\mu^*(z)$ is nonnegative for each $z \in V(G) \cup F(G)$.

Recall that a 5^+-vertex is high, and a 3^--vertex is low. A bad face is a 5-face incident to two 2-vertices; a face is good if it is not bad. A vertex h is the *head* of a bad face $hu_1u_2u_3u_4$ if $d(u_1) = d(u_4) = 2$. Note that each bad face has exactly one head. A 2-vertex u_1 incident to a bad face b is *close* to a good face g if u_2u_3 is a common edge of b and g and u, u_2, u_3 are consecutive vertices of g and u_1, u_2, u_3 are consecutive vertices of b, and u, u_2, u_3 are high vertices. See Figure 6.1.

The discharging rules (R1)–(R5) are designed so that the faces and high vertices send their excess charge to low vertices. (R6) and (R7) different from (R1)–(R5) in that 2-vertices with enough charge send excess charge to other 2-vertices that need more charge. (R7) is basically the same as (R6), except we make sure that there is no charge being bounced back and forth between 2-vertices.

Here are the discharging rules:

(R1) Each bad face sends charge $\frac{1}{2}$ to each incident 2-vertex.

(R2) Each good face sends charge $\frac{2}{3}$ to each incident 2-vertex.

(R3) Each good face sends charge $\frac{1}{12}$ to each incident 3-vertex.

(R4) Each good face sends charge $\frac{1}{12}$ to each of its close 2-vertices.

(R5) Each high vertex distributes its initial charge uniformly to each adjacent low vertex.

(R6) Each 2-vertex v distributes its excess charge uniformly to each 2-vertex u where u and v are incident to the same bad face.

(R7) Each 2-vertex v distributes its excess charge uniformly to each 2-vertex u where u and v are incident to the same bad face and u did not send charge to v by (R6).

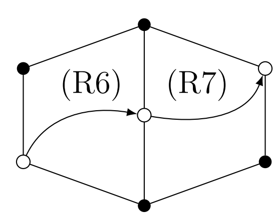

Figure 6.2: Discharging Rule (R6) and (R7).

We will first show that each face has nonnegative final charge. Then, we will show that each vertex has nonnegative final charge.

Claim 6.4.1. *Each bad face f has nonnegative final charge.*

Proof. By definition, f is incident to two 2-vertices and has length 5. Since (R1) is the only rule that involves a bad face, it follows that $\mu^*(f) = 1 - 2 \cdot \frac{1}{2} = 0$. $\square$

Claim 6.4.2. *Each good* 5*-face* f *has nonnegative final charge.*

Proof. By definition, f is incident to at most one 2-vertex. Assume f is incident to one 2-vertex v, which implies that the two neighbors of v (which are both incident to f) are high by Lemma 6.2.1. If f is incident to at least one 3-vertex, then f has no close vertices, and thus, $\mu^*(f) \geq 1 - \frac{2}{3} - 2 \cdot \frac{1}{12} = \frac{1}{6} > 0$. If f is incident to no 3-vertices, then f has at most four close vertices, and thus, $\mu^*(f) \geq 1 - \frac{2}{3} - 4 \cdot \frac{1}{12} = 0$.

Now assume f is incident to no 2-vertices. If f is incident to i 3-vertices where $i \in \{3, 4, 5\}$, then f has no close vertices, and thus, $\mu^*(f) = 1 - i \cdot \frac{1}{12} \geq \frac{7}{12} > 0$. If f is incident to two 3-vertices, then f has at most two close vertices, and thus $\mu^*(f) \geq 1 - 4 \cdot \frac{1}{12} = \frac{2}{3} > 0$. If f is incident to one 3-vertex, then f has at most four close vertices, and thus, $\mu^*(f) \geq 1 - 5 \cdot \frac{1}{12} = \frac{7}{12} > 0$. If f is incident to no 3-vertices, then f has at most ten close vertices, and thus, $\mu^*(f) \geq 1 - 10 \cdot \frac{1}{12} = \frac{1}{6} > 0$. $\square$

Claim 6.4.3. *Each* 6^+*-face* f *has nonnegative final charge.*

Proof. Note that by definition, f is a good face. We will first assign weights on each edge incident to f, and then shift some of these weights to the low vertices incident to f. The initial charge of f will be distributed to incident low vertices and its close vertices according to these weights. Since $d(f) \geq 6$, it follows that $\frac{\mu(f)}{d(f)} = \frac{d(f)-4}{d(f)} \geq \frac{1}{3}$, and thus, we can assign an initial weight of at least $\frac{1}{3}$ to each edge incident to f so that the sum of the weights are $\mu(f)$.

Consider an edge e incident to f. If e is incident to exactly one low vertex v, then shift all of its weight to v. Now, each 2-vertex incident to f has weight at least $2 \cdot \frac{1}{3}$ since a 2-vertex cannot be adjacent to another low vertex by Lemma 6.2.1. Also, each 3-vertex incident to f has weight at least $\frac{1}{3}$ since a 3-vertex cannot be adjacent to two low vertices by Lemma 6.2.2.

Note that each close vertex of f corresponds to an edge (with weight at least $\frac{1}{3}$) incident to f, and an edge corresponds to at most two close vertices.

This shows that f has enough initial charge to send charge $\frac{2}{3}$ to each incident 2-vertex, $\frac{1}{3} > \frac{1}{12}$ to each incident 3-vertex, and $\frac{1}{3} \cdot \frac{1}{2} > \frac{1}{12}$ to each of its close vertices. $\square$

Claim 6.4.4. *Each high vertex v has nonnegative final charge.*

Proof. Follows immediately since each high vertex has positive initial charge. $\square$

Note that by Lemma 6.2.3, each high vertex with degree at most 9 is adjacent to at least one high vertex. Table 6.1 summarizes a lower bound on the amount of charge each high vertex is guaranteed to send to an adjacent low vertex.

$d(v)$	5	6	7	8	9	≥ 10
charge sent to an adjacent low vertex	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{3}{6}$	$\frac{4}{7}$	$\frac{5}{8}$	$\frac{6}{10}$

Table 6.1: Charge guaranteed from a high vertex.

Recall that a vertex v is *chubby* if either $d(v) \in \{7, 8, 9\}$ and v has at least two high neighbors or $d(v) \geq 10$. A vertex v is *fat* if either $d(v) \in \{8, 9\}$ and v has at least two high neighbors or $d(v) \geq 10$ and v has at least one high neighbor.

A chubby vertex will send charge at least $\frac{3}{5}$ to each low neighbor, and a fat vertex will send charge at least $\frac{2}{3}$ to each low neighbor.

Claim 6.4.5. *Each 4-vertex v has nonnegative final charge.*

Proof. Follows immediately since 4-vertices are not involved in the discharging rules. $\square$

Claim 6.4.6. *Each 3-vertex v has nonnegative final charge.*

Proof. By Lemma 6.3.3, v is incident to three good faces. By Lemma 6.2.2, v is adjacent to at least two high vertices, one of which is a 7^+-vertex. Thus, $\mu^*(v) \geq -1 + 3 \cdot \frac{1}{12} + \frac{1}{4} + \frac{3}{6} = 0$. $\square$

We split the argument that each 2-vertex has nonnegative final charge into two claims to improve the readability. Note that any 2-vertex receives charge at least $2 \cdot \frac{1}{2} = 1$ from the two incident faces.

Claim 6.4.7. *Each* 2*-vertex* v *that is not incident to two bad faces has nonnegative final charge.*

Proof. Let $N(v) = \{v_1, v_2\}$. By Lemma 6.2.1, we may assume $d(v_1) \geq 5$ and $d(v_2) \geq 7$. If v is not incident to a bad face, then each face incident to v sends charge at least $\frac{2}{3}$. Thus, $\mu^*(v) \geq -2 + 2 \cdot \frac{2}{3} + \frac{1}{4} + \frac{3}{6} = \frac{1}{12} > 0$.

Assume v is incident to exactly one bad face $f_0 = x_1 v_1 v v_2 x_2$ so that v receives charge $\frac{2}{3} + \frac{1}{2} = \frac{7}{6}$ from its incident faces. If $d(v_1) \geq 6$, then $\mu^*(v) \geq -2 + \frac{7}{6} + \frac{2}{5} + \frac{3}{6} = \frac{1}{15} > 0$, so assume $d(v_1) = 5$. If v_1 is the head of f_0, then by Lemma 6.3.4, v_2 is a chubby vertex. Thus, $\mu^*(v) \geq -2 + \frac{7}{6} + \frac{1}{4} + \frac{3}{5} = \frac{1}{60} > 0$.

So assume v_2 is the head of f_0. Let f be a coloring of $G - v$ obtained by the minimality of G. By Lemma 6.3.4 we know $f(v_1) = 1$ and $f(v_2) = 2$. If $f(x_1) = 1$, then Lemma 6.3.4 tells us that v_1 has a 2-saturated neighbor, which is a high neighbor (of v_1) other than x_1. Thus, $\mu^*(v) \geq -2 + \frac{7}{6} + \frac{1}{3} + \frac{3}{6} = 0$. If $f(x_1) = 2$, then, by Lemma 6.3.4, x_1 is a fat vertex and $f(x_2) = 1$. By Lemma 6.3.1, applied to $v_1 v v_2 x_2$, we know $d(v_2) \geq 8$. Now, x_2 gets charge at least 1 from its incident faces, at least $\frac{4}{7}$ from v_2, and at least $\frac{2}{3}$ from x_1 since it is fat. Thus, the charge at x_2 after (R5) will be at least $-2 + 2 \cdot \frac{1}{2} + \frac{2}{3} + \frac{4}{7} = \frac{5}{21}$. By (R6), x_2 will send charge at least $\frac{5}{42}$ to v. Now, $\mu^*(v) \geq -2 + \frac{7}{6} + \frac{1}{4} + \frac{4}{7} + \frac{5}{42} = \frac{3}{28} > 0$. □

Claim 6.4.8. *Each* 2*-vertex* v *that is incident to two bad faces has nonnegative final charge.*

Proof. Let $N(v) = \{v_1, v_2\}$. By Lemma 6.2.1, we may assume $d(v_1) \geq 5$ and $d(v_2) \geq 7$. If $d(v_1) \geq 7$, then $\mu^*(v) \geq -2 + 1 + 2 \cdot \frac{1}{2} = 0$, so assume $d(v_1) \leq 6$.

Let $f_1 = x_1 v_1 v v_2 x_2$ and $f_2 = y_1 v_1 v v_2 y_2$ be the two bad faces incident to v. Let f be a coloring of $G - v$ obtained by the minimality of G. By Lemma 6.2.4, $f(v_1) = 1$ and $f(v_2) = 2$. For $i \in [2]$, setting $f(v) = i$ must not give a coloring of G, so we know v_i is i-saturated.

Case 1: The faces f_1 and f_2 have the same head.

(i) Assume v_1 is the head of both f_1 and f_2 so that v_2 has two high neighbors. If $d(v_1) = 6$, then $\mu^*(v) \geq -2+1+\frac{2}{5}+\frac{3}{5}=0$, so assume $d(v_1)=5$. If $d(v_2) \geq 10$, then $\mu^*(v) \geq -2+1+\frac{1}{4}+\frac{6}{8}=0$. By Lemma 6.3.4, it must be the case that $f(x_i)=f(v_i)=f(y_i)=i$ for $i \in [2]$. If $d(v_2) \in \{7,8,9\}$, then Lemma 6.3.2 applied to v_1vv_2 tells us that v_2 has three high neighbors, which are x_2, y_2, and a high neighbor colored 1. Thus, $\mu^*(v) \geq -2+1+\frac{1}{4}+\frac{3}{4}=0$.

(*ii*) Assume v_2 is the head of both f_1 and f_2 so that v_1 has two high neighbors x_1 and y_1. If $d(v_1)=6$, then $\mu^*(v) \geq -2+1+\frac{2}{4}+\frac{3}{6}=0$, so assume $d(v_1)=5$. If $f(x_1)=f(y_1)=1$, then v_1 has a high neighbor that is neither x_1 nor y_1 by Lemma 6.3.4. Thus, $\mu^*(v) \geq -2+1+\frac{1}{2}+\frac{3}{6}=0$. Without loss of generality assume $f(x_1)=2$. By Lemma 6.3.4, x_1 is a fat vertex and $f(x_2)=1$. Therefore, by Lemma 6.3.1 applied to $v_1vv_2x_2$, $d(v_2) \geq 8$. Now x_2 gets charge at least 1 from its incident faces, at least $\frac{4}{7}$ from v_2, and at least $\frac{2}{3}$ from x_1 since it is fat. Thus, the charge at x_2 after (R5) will be at least $-2+1+\frac{2}{3}+\frac{4}{7}=\frac{5}{21}$. By (R6), v will receive charge at least $\frac{5}{42}$ from x_2. Thus, $\mu^*(v) \geq -2+1+\frac{1}{3}+\frac{4}{7}+\frac{5}{42}=\frac{1}{42}>0$.

Case 2: The faces f_1 and f_2 have different heads. Without loss of generality, assume v_i is the head of f_i for $i \in [2]$. Note that each vertex in $\{v_1, v_2\}$ has at least one high neighbor.

(*i*) Assume $d(v_1)=6$. Since letting $f(v)=1$ and $f(v_1)=2$ must not give a coloring of G, we know v_1 has a 2-saturated neighbor (a 6-vertex v_1 cannot have six neighbors of color 2 since v has color 1). If $f(y_1)=1$, then v_1 has two high neighbors, which means v_1 gives charge at least $\frac{2}{4}$ to v, so we are done since $\mu^*(v) \geq -2+1+\frac{2}{4}+\frac{3}{6}=0$. So $f(y_1)=2$ and v_1 has only one high neighbor y_1. It must be that $f(y_2)=1$, since otherwise set $f(v)=2$ and $f(y_2)=1$ to obtain a coloring of G. By Lemma 6.3.1 applied to $v_1vv_2y_2$, we know $d(v_2) \geq 8$. Since setting $f(v)=1$ and $f(v_1)=2$ must not give a coloring of G, we know that y_1 is 2-saturated. Also, since setting $f(v)=f(y_1)=1$ and $f(v_1)=2$ must not give a coloring of G, we know that y_1 has a neighbor colored 1 that is neither y_2 nor v_1. Thus, $d(y_1) \geq 8$. Now y_2 gets charge at least 1 from its incident faces and at least $\frac{4}{7}$ from each of v_2 and y_1. Thus, the charge at y_2 after (R5) will be at least $-2+1+2\cdot\frac{4}{7}=\frac{1}{7}$. By (R6), y_2 will send charge at least $\frac{1}{14}$ to v. Thus, $\mu^*(v)\geq -2+1+\frac{2}{5}+\frac{4}{7}+\frac{1}{14}=\frac{3}{70}>0$.

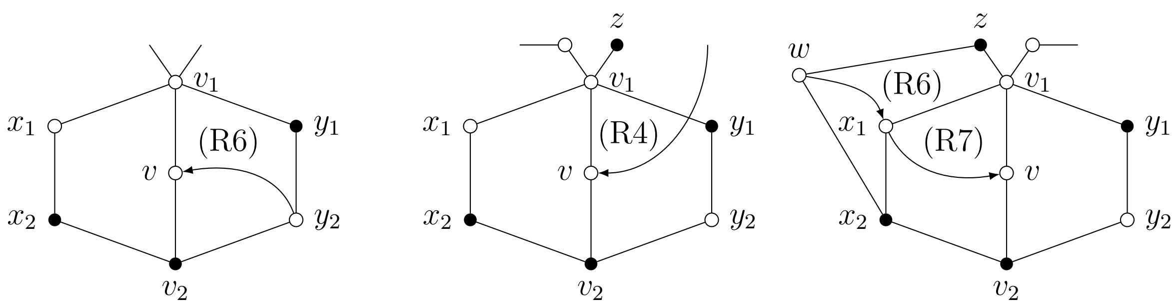

Figure 6.3: Three subcases when $d(v_1)=5$.

(ii) Assume $d(v_1)=5$. By Lemma 6.3.4, we know that $f(x_1)=1$ and $f(x_2)=2$ and x_2,v_2 are chubby vertices.

(a) If $f(y_1)=2$, then we know y_1 is a fat vertex and $f(y_2)=1$ by Lemma 6.3.4. By Lemma 6.3.1 applied to $v_1vv_2y_2$, we know $d(v_2)\geq 8$. Since (a chubby vertex) v_2 has a high neighbor x_2 and $d(v_2)\geq 8$, it follows that v_2 is a fat vertex. Thus, v_2 and y_1 each sends charge at least $\frac{2}{3}$ to each bad neighbor. Thus, the charge at y_2 after (R5) will be at least $-2+1+\frac{2}{3}+\frac{2}{3}=\frac{1}{3}$. By (R6), y_2 sends charge at least $\frac{1}{6}$ to v. Thus, $\mu^*(v)\geq -2+1+\frac{1}{4}+\frac{2}{3}+\frac{1}{6}=\frac{1}{12}>0$.

(b) If $f(y_1)=1$, we know, v_1 has a 2-saturated (high) neighbor z by Lemma 6.3.4.

If z,v_1,y_1 are consecutive vertices of a face f_3, then v is a close 2-vertex of f_3, which is clearly a good face. By (R4), f_3 will give charge $\frac{1}{12}$ to v. Thus, $\mu^*(v)\geq -2+1+\frac{1}{12}+\frac{1}{3}+\frac{3}{5}=\frac{1}{60}>0$.

Now consider the case where z,v_1,x_1,x_2 are consecutive vertices of a face f_0. If f_0 is not a bad face, then the charge at x_1 after (R5) will be at least $-2+\frac{1}{2}+\frac{2}{3}+\frac{1}{3}+\frac{3}{5}=\frac{1}{10}$. By (R6), x_1 will send charge at least $\frac{1}{10}$ to v, thus, $\mu^*(v)\geq -2+1+\frac{1}{3}+\frac{3}{5}+\frac{1}{10}=$

$\frac{1}{30} > 0$. So f_0 is a bad face $wzv_1x_1x_2$, which further implies that x_2 is the head of f_0. By letting $f(v) = 1$ and erasing the color on x_1, we can apply Lemma 6.3.4 to f_0 to conclude that z is a fat vertex and $f(w) = 1$. By Lemma 6.3.1 applied to $v_1x_1x_2w$, we know $d(x_2) \geq 8$. Since (a chubby vertex) x_2 has a high neighbor v_2 and $d(x_2) \geq 8$, it follows that x_2 is a fat vertex.

Now, after (R5), w will have charge at least $-2+1+2\cdot\frac{2}{3} = \frac{1}{3}$ and x_1 will have charge at least $-2+1+\frac{1}{3}+\frac{2}{3} = 0$ and v will have charge at least $-2+1+\frac{1}{3}+\frac{3}{5} = -\frac{1}{15}$. By (R6), w sends charge at least $\frac{1}{6}$ to x_1, so the charge at w is at least $\frac{1}{6}$ after (R6). If the charge at v is still negative after (R6), then v could not have sent charge to w by (R6). Since w sent charge to x_1, by (R7), x_1 will send all of its excess charge to v. Thus, after (R7), v will have charge at least $-\frac{1}{15}+\frac{1}{6} = \frac{1}{10} > 0$.

$\square$

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