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Planar graphs with girth at least 5 are (3, 5)-colorable

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ABSTRACT

A graph is $(d_1, \ldots, d_r)$-colorable if its vertex set can be partitioned into r sets $V_1, \ldots, V_r$ where the maximum degree of the graph induced by V_i is at most d_i for each $i \in \{1, \ldots, r\}$. Let $\mathcal{G}_g$ denote the class of planar graphs with minimum cycle length at least g. We focus on graphs in $\mathcal{G}_5$ since for any d_1 and d_2, Montassier and Ochem constructed graphs in $\mathcal{G}_4$ that are not (d_1, d_2)-colorable. It is known that graphs in $\mathcal{G}_5$ are (2, 6)-colorable and (4, 4)-colorable, but not all of them are (3, 1)-colorable. We prove that graphs in $\mathcal{G}_5$ are (3, 5)-colorable, leaving two interesting questions open: (1) are graphs in $\mathcal{G}_5$ also $(3, d_2)$-colorable for some $d_2 \in \{2, 3, 4\}$? (2) are graphs in $\mathcal{G}_5$ indeed (d_1, d_2)-colorable for all $d_1 + d_2 \geq 8$ where $d_2 \geq d_1 \geq 1$?

1. Introduction

Let $[n] = \{1, \ldots, n\}$. Only finite, simple graphs are considered. Given a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. A *neighbor* of a vertex v is a vertex adjacent to v, and let $N(v)$ denote the set of neighbors of v. The *degree* of v, denoted by $d(v)$, is $|N(v)|$. The *degree* of a face f, denoted by $d(f)$, is the length of a shortest boundary walk of f. A *k-vertex*, *k^+-vertex*, and *k^--vertex* are vertices of degree k, at least k, and at most k, respectively. A *k-face*, *k^+-face* is a face of degree k, at least k, respectively. The *girth* of a graph is the length of a shortest cycle.

A graph is $(d_1, \ldots, d_r)$-colorable if its vertex set can be partitioned into r sets $V_1, \ldots, V_r$ where the maximum degree of the graph induced by V_i is at most d_i for each $i \in [r]$; in other words, there exists a function $f : V(G) \to [r]$ where the graph induced by vertices of color i has maximum degree at most d_i for $i \in [r]$.

There are many papers that study $(d_1, \ldots, d_r)$-colorings of sparse graphs resulting in corollaries regarding planar graphs, sometimes with restrictions on the length of a smallest cycle. The well-known four color theorem [1,2] is exactly the statement that planar graphs are (0, 0, 0, 0)-colorable. Cowen, Cowen, and Woodall [7] proved that planar graphs are (2, 2, 2)-colorable, and Eaton and Hull [8] and Škrekovski [11] proved that this is sharp by exhibiting non-$(1, k, k)$-colorable planar graphs for each k. Thus, the problem is completely solved when $r \geq 3$.

Let $\mathcal{G}_g$ denote the class of planar graphs with girth at least g. Given any d_1 and d_2, consider the following graph constructed by Montassier and Ochem [10]. Let $X_i(d_1, d_2)$ be a copy of K_{2,d_1+d_2+1} where one part is $\{x_i, y_i\}$. Obtain $Y(d_1, d_2)$ in the following way: start with $X_1(d_1, d_2), \ldots, X_{d_1+2}(d_1, d_2)$ and identify $x_1, \ldots, x_{d_1+2}$ into x, and add the edges $y_1y_2, \ldots, y_1y_{d_1+2}$. It is easy to verify that $Y(d_1, d_2)$ is in $\mathcal{G}_4$ but it is not (d_1, d_2)-colorable.

Therefore, we focus on graphs in $\mathcal{G}_5$. There are also many papers [3,5,9,6,4,10] that investigate (d_1, d_2)-colorability for graphs in $\mathcal{G}_g$ for $g \geq 6$; see [10] for the rich history. For example, Borodin, Ivanova, Montassier, Ochem, and Raspaud [3]

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constructed a graph in $\mathcal{G}_6$ (and thus also in $\mathcal{G}_5$) that is not $(0, k)$-colorable for any k. The question of determining if there exists a finite k where all graphs in $\mathcal{G}_5$ are $(1, k)$-colorable is not yet known and was explicitly asked in [10]. On the other hand, Borodin and Kostochka [5] and Havet and Sereni [9], respectively, proved results that imply graphs in $\mathcal{G}_5$ are $(2, 6)$-colorable and $(4, 4)$-colorable.

In this paper, we prove the following theorem, which is not implied by the aforementioned results.

Theorem 1.1. *Planar graphs with girth at least 5 are $(3, 5)$-colorable.*

This solves one of the previously unknown cases of the following question.

Question 1.2. *Are planar graphs with girth at least 5 indeed (d_1, d_2)-colorable for all $d_1 + d_2 \geq 8$ where $d_2 \geq d_1 \geq 1$?*

The only remaining case of Question 1.2 is when $d_1 = 1$ and $d_2 = 7$. As mentioned before, interestingly enough, we do not know even if there is a finite k where graphs in $\mathcal{G}_5$ are $(1, k)$-colorable.

Since there are non-$(3, 1)$-colorable graphs in $\mathcal{G}_5$ [10], Theorem 1.1 implies that the minimum d where graphs in $\mathcal{G}_5$ are $(3, d)$-colorable is in $\{2, 3, 4, 5\}$; determining this d would be interesting.

In the figures throughout this paper, the white vertices do not have incident edges besides the ones drawn, and the black vertices may have other incident edges.

In Section 2, we prove structural lemmas for non-(d_1, d_2)-colorable graphs with minimum order. In Section 3, we reveal some more structures of minimum counterexamples to Theorem 1.1 by focusing on the case when $d_1 = 3$ and $d_2 = 5$. Finally, we prove Theorem 1.1 by using a discharging procedure in Section 4.

2. Non-(d_1, d_2)-colorable graphs with minimum order

In this section, we prove structural lemmas regarding non-(d_1, d_2)-colorable graphs with minimum order; let $H(d_1, d_2)$ be such a graph. It is easy to see that the minimum degree of (a vertex of) $H(d_1, d_2)$ is at least 2 and $H(d_1, d_2)$ is connected.

Given a (partial) coloring f of $H(d_1, d_2)$ and $i \in [2]$, a vertex v with $f(v) = i$ is *i-saturated* if v is adjacent to d_i neighbors colored i. By definition, an i-saturated vertex has at least d_i neighbors.

Lemma 2.1. *Let $H = H(d_1, d_2)$ where $d_1 \leq d_2$. If v is a 2-vertex of H, then v is adjacent to two $(d_1+2)^+$-vertices, one of which is a $(d_2 + 2)^+$-vertex.*

Proof. Let $N(v) = \{v_1, v_2\}$ and let f be a coloring of $H - v$ obtained by the minimality of H. If $f(v_1) = f(v_2)$, then letting $f(v) \in [2] \setminus \{f(v_1)\}$ gives a coloring of H, which is a contradiction. Without loss of generality, assume that $f(v_1) = 1$ and $f(v_2) = 2$. Since setting $f(v) = 1$ must not give a coloring of H, we know v_1 is 1-saturated. Since setting $f(v_1) = 2$ and $f(v) = 1$ must not give a coloring of H, we know v_1 has a neighbor colored 2. This implies $d(v_1) \geq d_1 + 2$. Similar logic implies that $d(v_2) \geq d_2 + 2$. □

Lemma 2.2. *Let $H = H(d_1, d_2)$ where $2 \leq d_1 \leq d_2$. If v is a 3-vertex of H, then v is adjacent to at least two $(d_1+2)^+$-vertices, one of which is a $(d_2 + 2)^+$-vertex.*

Proof. Let $N(v) = \{v_0, v_1, v_2\}$ and let f be a coloring of $H - v$ obtained by the minimality of H. If $f(v_0) = f(v_1) = f(v_2)$, then letting $f(v) \in [2] \setminus \{f(v_0)\}$ gives a coloring of H, which is a contradiction. Without loss of generality, assume that $f(v_1) = 1$ and $f(v_2) = 2$. Further assume that $f(v_0) = i$ for some $i \in [2]$ and let $j \in [2] \setminus \{i\}$.

Since setting $f(v) = j$ must not give a coloring of H, we know that v_j is j-saturated. Since setting $f(v) = j$ and $f(v_j) = i$ must not give a coloring of H, we know that v_j has a neighbor colored i. This implies $d(v_j) \geq d_j + 2$. Since setting $f(v) = i$ must not give a coloring of H, we know either v_0 or v_i is i-saturated. If both $d(v_0), d(v_i) \leq d_i+1$, then recolor each i-saturated vertex in $\{v_0, v_i\}$ with color j, and set $f(v) = i$ to obtain a coloring of H, which is a contradiction. Therefore either v_0 or v_i has degree at least $d_i + 2$. □

Lemma 2.3. *Let $H = H(d_1, d_2)$ where $d_1 + 1 \leq d_2$. If v is a $(d_1 + d_2 + 1)^-$-vertex of H, then v is adjacent to at least one $(d_1 + 2)^+$-vertex.*

Proof. Suppose that no neighbor of v is a $(d_1 + 2)^+$-vertex and let f be a coloring of $H - v$ obtained by the minimality of H. Both colors 1 and 2 must appear on $N(v)$; otherwise, we can easily obtain a coloring of H, which is a contradiction. Since setting $f(v) = 2$ must not give a coloring of H and v cannot be adjacent to a 2-saturated vertex (since a 2-saturated neighbor of v has degree at least $d_2 + 1 \geq d_1 + 2$), we know that v has at least $d_2 + 1$ neighbors colored 2. Since setting $f(v) = 1$ must not give a coloring of H, we know that either v has at least $d_1 + 1$ neighbors colored 1 or v has a 1-saturated neighbor. The former case is impossible because $d(v) \leq d_1 + d_2 + 1$. Since each neighbor of v is a $(d_1 + 1)^-$-vertex, each 1-saturated neighbor of v can be recolored with 2. Now we can let $f(v) = 1$ to obtain a coloring of H, which is a contradiction. □

Lemma 2.4. *Let $H = H(d_1, d_2)$ and let v be a 2-vertex of H where $N(v) = \{v_1, v_2\}$ and $d(v_1) \leq d_2 + 1$. If f is a coloring of $H - v$, then $f(v_1) = 1$ and $f(v_2) = 2$.*

Proof. If $f(v_1) = f(v_2)$, then letting $f(v) \in [2] \setminus \{f(v_1)\}$ gives a coloring of H, which is a contradiction. If $f(v_1) = 2$ and $f(v_2) = 1$, then let $f(v) = 2$ to obtain a coloring of H, unless v_1 is 2-saturated. This implies that $d(v_1) = d_2 + 1$ and $f(z) = 2$ for $z \in N(v_1) \setminus \{v\}$, so we can let $f(v_1) = 1$ to obtain a coloring of H, which is a contradiction. □

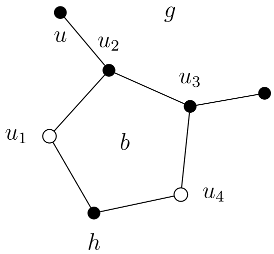

Fig. 1. The head h of a bad face b and a 2-vertex u_1 that is close to a good face g.

3. Non-(3, 5)-colorable planar graphs with minimum order

From now on, let G be a counterexample to Theorem 1.1 with the fewest number of vertices and fix some embedding of G on the plane. It is easy to see that the minimum degree of (a vertex of) G is at least 2 and G is connected.

A 5^+-vertex is *high*, and a 3^--vertex is *low*. Recall that given a (partial) coloring f of G, a vertex v with $f(v) = i$ is *i-saturated* if v is adjacent to $2i + 1$ neighbors colored i.

Lemma 3.1. *Let $u_1u_2u_3u_4$ be a path in G where $d(u_2) = d(u_4) = 2$. If f is a coloring of $G - u_2$ where $f(u_1) = f(u_4) = 1$ and $f(u_3) = 2$, then $d(u_3) \geq 8$.*

Proof. Since setting $f(u_2) = 2$ must not give a coloring of G, it must be that u_3 is 2-saturated. Moreover, since setting $f(u_2) = 2$ and $f(u_3) = 1$ must not give a coloring of G, we know that u_3 has a neighbor colored with 1 that is not u_4. This implies $d(u_3) \geq 8$. □

Lemma 3.2. *Let $u_1u_2u_3$ be a path in G where $d(u_2) = 2$ and $d(u_3) \leq 9$. If f is a coloring of $G - u_2$ where $f(u_1) = 1$ and $f(u_3) = 2$, then u_3 has a high neighbor colored 1.*

Proof. Since setting $f(u_2) = 2$ must not give a coloring of G, it must be that u_3 is 2-saturated. Moreover, since setting $f(u_2) = 2$ and $f(u_3) = 1$ must not give a coloring of G, either u_3 has 4 neighbors colored 1 or at least one 1-saturated neighbor. The former is impossible since $d(u_3) \leq 9$, so u_3 has some 1-saturated neighbors. If all such neighbors are not high, then we can recolor each one with 2 and let $f(u_3) = 1$ and $f(u_2) = 2$ to complete a coloring of G, which is a contradiction. □

A *bad face* is a 5-face incident to two 2-vertices; a face is *good* if it is not bad.

Lemma 3.3. *A 3-vertex cannot be incident to a bad face.*

Proof. Follows immediately from Lemma 2.1 and the observation that a vertex on a bad face must be either a 2-vertex or a neighbor of a 2-vertex. □

A vertex h is the *head* of a bad face $b = hu_1u_2u_3u_4$ if $d(u_1) = d(u_4) = 2$. Note that each bad face has exactly one head. A 2-vertex u_1 incident to a bad face b is *close* to a good face g if u_2u_3 is a common edge of b and g and u, u_2, u_3 are high vertices and u, u_2, u_3 are consecutive vertices of g and u_1, u_2, u_3 are consecutive vertices of b. See Fig. 1.

A vertex v is *chubby* if either $d(v) \in \{7, 8, 9\}$ and v has at least two high neighbors or $d(v) \geq 10$. A vertex v is *fat* if either $d(v) \in \{8, 9\}$ and v has at least two high neighbors or $d(v) \geq 10$ and v has at least one high neighbor. By definition, a fat vertex is also chubby.

Lemma 3.4. *Let $f_0 = x_1v_1vv_2x_2$ be a bad face where $d(v) = 2$, $d(v_1) = 5$, $d(v_2) \geq 7$. If f is a coloring of $G - v$, then $f(v_1) = 1$ and $f(v_2) = 2$, and one of the following holds:*

(i) *If v_1 is the head of f_0, then $f(x_1) = 1$ and $f(x_2) = 2$ and x_2 and v_2 are chubby vertices.*
(ii) *If v_2 is the head of f_0 and $f(x_1) = 2$, then $f(x_2) = 1$ and x_1 is a fat vertex.*
(iii) *If v_2 is the head of f_0 and $f(x_1) = 1$, then v_1 has a 2-saturated neighbor.*

Proof. By Lemma 2.4, $f(v_1) = 1$ and $f(v_2) = 2$. For $i \in [2]$, since setting $f(v) = i$ must not give a coloring of G, we know v_i is i-saturated.

(i) Since v_1 is the head of f_0, we know $d(x_1) = 2$. If $f(x_1) = 2$ so that $f(z) = 1$ for each $z \in N(v_1)\setminus\{v, x_1\}$, then setting $f(v_1) = 2$ and $f(v) = 1$ is a coloring of G, which is a contradiction. Thus $f(x_1) = 1$. If $f(x_2) = 1$, then letting $f(v) = 1$ and $f(x_1) = 2$ is a coloring of G, which is a contradiction. Thus, $f(x_2) = 2$. If $d(v_2) \in \{7, 8, 9\}$, then by applying Lemma 3.2 to v_1vv_2, we know that v_2 must have two high neighbors; namely, x_2 and a neighbor colored 1. Therefore, v_2 is a chubby vertex.

Since $d(x_1) = 2$ and $d(v_1) = 5$, we know $d(x_2) \geq 7$ by Lemma 2.1. Now by letting $f(v) = 1$ and removing the coloring on x_1, the situation is symmetric for x_2; this implies that x_2 is a chubby vertex.

(ii) Since $f(x_1) = 2$, we know $f(z) = 1$ for $z \in N(v_1) \setminus \{v, x_1\}$. If $f(x_2) = 2$, then letting $f(v) = 2$ and $f(x_2) = 1$ is a coloring of G, which is a contradiction. Thus, $f(x_2) = 1$. Since setting $f(v) = 1$ and $f(v_1) = 2$ must not give a coloring of G, we know x_1 is 2-saturated. Note that x_1 has a high neighbor v_1. Since setting $f(v_1) = 2$ and $f(v) = f(x_1) = 1$ must not give a coloring of G, we know that x_1 has a neighbor colored 1 that is neither x_2 nor v_1. This implies $d(x_1) \geq 8$. If $d(x_1) \in \{8, 9\}$ and every 1-saturated neighbor of x_1 except v_1 is a 4-vertex, then recolor each such neighbor with 2 (and let $f(v_1) = 2$ and $f(v) = (x_1) = 1$) to obtain a coloring of G. Thus, if $d(x_1) \in \{8, 9\}$, then x_1 has two high neighbors; namely, v_1 and another neighbor colored 1. Thus, x_1 is a fat vertex.

(iii) Since setting $f(v) = 1$ and $f(v_1) = 2$ must not give a coloring of G, we know v_1 has either a 2-saturated neighbor or 6 neighbors colored 2. Since a 5-vertex v_1 cannot have 6 neighbors of color 2, we know v_1 has a 2-saturated neighbor. □

4. Discharging

Since the embedding of G is fixed, we can let $F(G)$ denote the set of faces of this embedding. In this section, we will prove that G cannot exist by assigning an *initial charge* $\mu(z)$ to each $z \in V(G) \cup F(G)$, and then applying a discharging procedure to end up with *final charge* $\mu^*(z)$ at z. We prove that the final charge has nonnegative total sum, whereas the initial charge sum is negative. The discharging procedure will preserve the total charge sum, and hence we find a contradiction to conclude that the counterexample G does not exist.

For each $z \in V(G) \cup F(G)$, let $\mu(z) = d(z) - 4$. The total initial charge is negative since

$$\sum_{z \in V(G)\cup F(G)} \mu(z) = \sum_{z \in V(G)\cup F(G)} (d(z) - 4) = -4|V(G)| + 4|E(G)| - 4|F(G)| = -8 < 0.$$

The last equality holds by Euler's formula.

The rest of this section will prove that $\mu^*(z)$ is nonnegative for each $z \in V(G) \cup F(G)$.

Recall that a 5^+-vertex is high, and a 3^--vertex is low. A bad face is a 5-face incident to two 2-vertices; a face is good if it is not bad. A vertex h is the *head* of a bad face $hu_1u_2u_3u_4$ if $d(u_1) = d(u_4) = 2$. Note that each bad face has exactly one head. A 2-vertex u_1 incident to a bad face b is *close* to a good face g if u_2u_3 is a common edge of b and g and u, u_2, u_3 are consecutive vertices of g and u_1, u_2, u_3 are consecutive vertices of b, and u, u_2, u_3 are high vertices. See Fig. 1.

The discharging rules (R1)–(R5) are designed so that the faces and high vertices send their excess charge to low vertices. (R6) and (R7) are different from (R1)–(R5) in that 2-vertices with enough charge send excess charge to other 2-vertices that need more charge. (R7) is basically the same as (R6), except we make sure that there is no charge being bounced back and forth between 2-vertices.

Here are the discharging rules:

(R1) Each bad face sends charge $\frac{1}{2}$ to each incident 2-vertex.
(R2) Each good face sends charge $\frac{2}{3}$ to each incident 2-vertex.
(R3) Each good face sends charge $\frac{1}{12}$ to each incident 3-vertex.
(R4) Each good face sends charge $\frac{1}{12}$ to each of its close 2-vertices.
(R5) Each high vertex distributes its initial charge uniformly to each adjacent low vertex.
(R6) Each 2-vertex v distributes its excess charge uniformly to each 2-vertex u where u and v are incident to the same bad face.
(R7) Each 2-vertex v distributes its excess charge uniformly to each 2-vertex u where u and v are incident to the same bad face and u did not send charge to v by (R6).

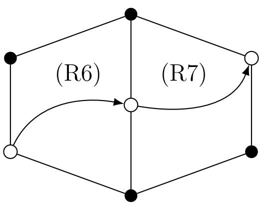

Fig. 2. Discharging rule (R6) and (R7).

See Fig. 2 for an illustration of (R7). We will first show that each face has nonnegative final charge. Then, we will show that each vertex has nonnegative final charge.

Claim 4.1. *Each bad face f has nonnegative final charge.*

Proof. By definition, f is incident to two 2-vertices and has length 5. Since (R1) is the only rule that involves a bad face, it follows that $\mu^*(f) = 1 - 2 \cdot \frac{1}{2} = 0$. □

Claim 4.2. *Each good 5-face f has nonnegative final charge.*

Proof. By definition, f is incident to at most one 2-vertex. Assume that f is incident to one 2-vertex v, which implies that the two neighbors of v (which are both incident to f) are high by Lemma 2.1. If f is incident to at least one 3-vertex, then f has no close vertices, and thus, $\mu^*(f) \geq 1 - \frac{2}{3} - 2 \cdot \frac{1}{12} = \frac{1}{6} > 0$. If f is incident to no 3-vertices, then f has at most four

Table 1
Charge guaranteed from a high vertex.

$d(v)$	5	6	7	8	9	≥ 10
Charge sent to an adjacent low vertex	$\frac{1}{4}$	$\frac{2}{5}$	$\frac{3}{6}$	$\frac{4}{7}$	$\frac{5}{8}$	$\frac{6}{10}$

close vertices, and thus, $\mu^*(f) \geq 1 - \frac{2}{3} - 4 \cdot \frac{1}{12} = 0$.

Now assume f is incident to no 2-vertices. If f is incident to i 3-vertices where $i \in \{3, 4, 5\}$, then f has no close vertices, and thus, $\mu^*(f) = 1 - i \cdot \frac{1}{12} \geq \frac{7}{12} > 0$. If f is incident to two 3-vertices, then f has at most two close vertices, and thus $\mu^*(f) \geq 1 - 4 \cdot \frac{1}{12} = \frac{2}{3} > 0$. If f is incident to one 3-vertex, then f has at most four close vertices, and thus, $\mu^*(f) \geq 1 - 5 \cdot \frac{1}{12} = \frac{7}{12} > 0$. If f is incident to no 3-vertices, then f has at most ten close vertices, and thus, $\mu^*(f) \geq 1 - 10 \cdot \frac{1}{12} = \frac{1}{6} > 0$. □

Claim 4.3. *Each* 6^+*-face* f *has nonnegative final charge.*

Proof. Note that by definition, f is a good face. We will first assign weights on each edge incident to f, and then shift some of these weights to the low vertices incident to f. The initial charge of f will be distributed to incident low vertices and its close vertices according to these weights. Since $d(f) \geq 6$, it follows that $\frac{\mu(f)}{d(f)} = \frac{d(f)-4}{d(f)} \geq \frac{1}{3}$, and thus, we can assign an initial weight of at least $\frac{1}{3}$ to each edge incident to f so that the sum of the weights is $\mu(f)$.

Consider an edge e incident to f. If e is incident to exactly one low vertex v, then shift all of its weight to v. Now, each 2-vertex incident to f has weight at least $2 \cdot \frac{1}{3}$ since a 2-vertex cannot be adjacent to another low vertex by Lemma 2.1. Also, each 3-vertex incident to f has weight at least $\frac{1}{3}$ since a 3-vertex cannot be adjacent to two low vertices by Lemma 2.2. Note that each close vertex of f corresponds to an edge (with weight at least $\frac{1}{3}$) incident to f, and an edge corresponds to at most two close vertices.

This shows that f has enough initial charge to send charge $\frac{2}{3}$ to each incident 2-vertex, $\frac{1}{3} > \frac{1}{12}$ to each incident 3-vertex, and $\frac{1}{3} \cdot \frac{1}{2} > \frac{1}{12}$ to each of its close vertices. □

Claim 4.4. *Each high vertex* v *has nonnegative final charge.*

Proof. Follows immediately since each high vertex has positive initial charge. □

Note that by Lemma 2.3, each high vertex with degree at most 9 is adjacent to at least one high vertex. Table 1 summarizes a lower bound on the amount of charge each high vertex is guaranteed to send to an adjacent low vertex.

Recall that a vertex v is *chubby* if either $d(v) \in \{7, 8, 9\}$ and v has at least two high neighbors or $d(v) \geq 10$. A vertex v is *fat* if either $d(v) \in \{8, 9\}$ and v has at least two high neighbors or $d(v) \geq 10$ and v has at least one high neighbor.

A chubby vertex will send charge at least $\frac{3}{5}$ to each low neighbor, and a fat vertex will send charge at least $\frac{2}{3}$ to each low neighbor.

Claim 4.5. *Each* 4*-vertex* v *has nonnegative final charge.*

Proof. Follows immediately since 4-vertices are not involved in the discharging rules. □

Claim 4.6. *Each* 3*-vertex* v *has nonnegative final charge.*

Proof. By Lemma 3.3, v is incident to three good faces. By Lemma 2.2, v is adjacent to at least two high vertices, one of which is a 7^+-vertex. Thus, $\mu^*(v) \geq -1 + 3 \cdot \frac{1}{12} + \frac{1}{4} + \frac{3}{6} = 0$. □

We split the argument that each 2-vertex has nonnegative final charge into two claims to improve the readability. Note that any 2-vertex receives charge at least $2 \cdot \frac{1}{2} = 1$ from the two incident faces.

Claim 4.7. *Each* 2*-vertex* v *that is not incident to two bad faces has nonnegative final charge.*

Proof. Let $N(v) = \{v_1, v_2\}$. By Lemma 2.1, we may assume $d(v_1) \geq 5$ and $d(v_2) \geq 7$. If v is not incident to a bad face, then each face incident to v sends charge at least $\frac{2}{3}$. Thus, $\mu^*(v) \geq -2 + 2 \cdot \frac{2}{3} + \frac{1}{4} + \frac{3}{6} = \frac{1}{12} > 0$.

Assume v is incident to exactly one bad face $f_0 = x_1v_1vv_2x_2$ so that v receives charge $\frac{2}{3} + \frac{1}{2} = \frac{7}{6}$ from its incident faces. If $d(v_1) \geq 6$, then $\mu^*(v) \geq -2 + \frac{7}{6} + \frac{2}{5} + \frac{3}{6} = \frac{1}{15} > 0$, so assume that $d(v_1) = 5$. If v_1 is the head of f_0, then by Lemma 3.4, v_2 is a chubby vertex. Thus, $\mu^*(v) \geq -2 + \frac{7}{6} + \frac{1}{4} + \frac{3}{5} = \frac{1}{60} > 0$.

So assume that v_2 is the head of f_0. Let f be a coloring of $G - v$ obtained by the minimality of G. By Lemma 3.4 we know $f(v_1) = 1$ and $f(v_2) = 2$. If $f(x_1) = 1$, then Lemma 3.4 tells us that v_1 has a 2-saturated neighbor, which is a high neighbor (of v_1) other than x_1. Thus, $\mu^*(v) \geq -2 + \frac{7}{6} + \frac{1}{3} + \frac{3}{6} = 0$. If $f(x_1) = 2$, then, by Lemma 3.4, x_1 is a fat vertex and $f(x_2) = 1$. By Lemma 3.1, applied to $v_1vv_2x_2$, we know $d(v_2) \geq 8$. Now, x_2 gets charge at least 1 from its incident faces, at least $\frac{4}{7}$ from v_2, and at least $\frac{2}{3}$ from x_1 since it is fat. Thus, the charge at x_2 after (R5) will be at least $-2 + 2 \cdot \frac{1}{2} + \frac{2}{3} + \frac{4}{7} = \frac{5}{21}$. By (R6), x_2 will send charge at least $\frac{5}{42}$ to v. Now, $\mu^*(v) \geq -2 + \frac{7}{6} + \frac{1}{4} + \frac{4}{7} + \frac{5}{42} = \frac{3}{28} > 0$. □

Claim 4.8. *Each* 2*-vertex* v *that is incident to two bad faces has nonnegative final charge.*

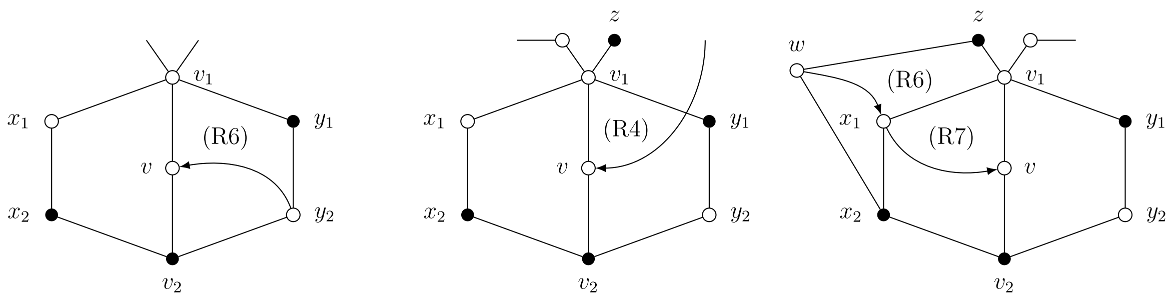

Fig. 3. Three subcases when $d(v_1) = 5$.

Proof. Let $N(v) = \{v_1, v_2\}$. By Lemma 2.1, we may assume $d(v_1) \geq 5$ and $d(v_2) \geq 7$. If $d(v_1) \geq 7$, then $\mu^*(v) \geq -2 + 1 + 2 \cdot \frac{1}{2} = 0$; so assume $d(v_1) \leq 6$.

Let $f_1 = x_1v_1vv_2x_2$ and $f_2 = y_1v_1vv_2y_2$ be the two bad faces incident to v. Let f be a coloring of $G - v$ obtained by the minimality of G. By Lemma 2.4, $f(v_1) = 1$ and $f(v_2) = 2$. For $i \in [2]$, setting $f(v) = i$ must not give a coloring of G, so we know v_i is i-saturated.

Case 1: the faces f_1 and f_2 have the same head.

(i) Assume v_1 is the head of both f_1 and f_2. Note that v_2 has two high neighbors. If $d(v_1) = 6$, then $\mu^*(v) \geq -2+1+\frac{2}{5}+\frac{3}{5} = 0$; so assume $d(v_1) = 5$. If $d(v_2) \geq 10$, then $\mu^*(v) \geq -2 + 1 + \frac{1}{4} + \frac{6}{8} = 0$. By Lemma 3.4, it must be the case that $f(x_i) = f(v_i) = f(y_i) = i$ for $i \in [2]$. If $d(v_2) \in \{7, 8, 9\}$, then Lemma 3.2 applied to v_1vv_2 tells us that v_2 has three high neighbors, which are x_2, y_2, and a high neighbor colored 1. Thus, $\mu^*(v) \geq -2 + 1 + \frac{1}{4} + \frac{3}{4} = 0$.

(ii) Assume that v_2 is the head of both f_1 and f_2. Note that v_1 has two high neighbors x_1 and y_1. If $d(v_1) = 6$, then $\mu^*(v) \geq -2+1+\frac{2}{4}+\frac{3}{6} = 0$, so assume $d(v_1) = 5$. If $f(x_1) = f(y_1) = 1$, then v_1 has a high neighbor that is neither x_1 nor y_1 by Lemma 3.4. Thus, $\mu^*(v) \geq -2+1+\frac{1}{2}+\frac{3}{6} = 0$. Without loss of generality assume that $f(x_1) = 2$. By Lemma 3.4, x_1 is a fat vertex and $f(x_2) = 1$. Therefore, by Lemma 3.1 applied to $v_1vv_2x_2$, $d(v_2) \geq 8$. Now x_2 gets charge at least 1 from its incident faces, at least $\frac{4}{7}$ from v_2, and at least $\frac{2}{3}$ from x_1 since it is fat. Thus, the charge at x_2 after (R5) will be at least $-2 + 1 + \frac{2}{3} + \frac{4}{7} = \frac{5}{21}$. By (R6), v will receive charge at least $\frac{5}{42}$ from x_2. Thus, $\mu^*(v) \geq -2 + 1 + \frac{1}{3} + \frac{4}{7} + \frac{5}{42} = \frac{1}{42} > 0$.

Case 2: the faces f_1 and f_2 have different heads. Without loss of generality, assume that v_i is the head of f_i for $i \in [2]$. Note that each vertex in $\{v_1, v_2\}$ has at least one high neighbor.

(i) Assume that $d(v_1) = 6$. Since letting $f(v) = 1$ and $f(v_1) = 2$ must not give a coloring of G, we know v_1 has a 2-saturated neighbor (a 6-vertex v_1 cannot have six neighbors of color 2 since v has color 1). If $f(y_1) = 1$, then v_1 has two high neighbors, which means v_1 gives charge at least $\frac{2}{4}$ to v, so we are done since $\mu^*(v) \geq -2 + 1 + \frac{2}{4} + \frac{3}{6} = 0$. So $f(y_1) = 2$ and v_1 has only one high neighbor y_1. It must be that $f(y_2) = 1$, since otherwise set $f(v) = 2$ and $f(y_2) = 1$ to obtain a coloring of G. By Lemma 3.1 applied to $v_1vv_2y_2$, we know $d(v_2) \geq 8$. Since setting $f(v) = 1$ and $f(v_1) = 2$ must not give a coloring of G, we know that y_1 is 2-saturated. Also, since setting $f(v) = f(y_1) = 1$ and $f(v_1) = 2$ must not give a coloring of G, we know that y_1 has a neighbor colored 1 that is neither y_2 nor v_1. Thus, $d(y_1) \geq 8$. Now y_2 gets charge at least 1 from its incident faces and at least $\frac{4}{7}$ from each of v_2 and y_1. Thus, the charge at y_2 after (R5) will be at least $-2 + 1 + 2 \cdot \frac{4}{7} = \frac{1}{7}$. By (R6), y_2 will send charge at least $\frac{1}{14}$ to v. Thus, $\mu^*(v) \geq -2 + 1 + \frac{2}{5} + \frac{4}{7} + \frac{1}{14} = \frac{3}{70} > 0$.

(ii) Assume $d(v_1) = 5$. By Lemma 3.4, we know that $f(x_1) = 1$ and $f(x_2) = 2$ and x_2, v_2 are chubby vertices.

(a) If $f(y_1) = 2$, then we know y_1 is a fat vertex and $f(y_2) = 1$ by Lemma 3.4. By Lemma 3.1 applied to $v_1vv_2y_2$, we know $d(v_2) \geq 8$. Since (a chubby vertex) v_2 has a high neighbor x_2 and $d(v_2) \geq 8$, it follows that v_2 is a fat vertex. Thus, v_2 and y_1 each sends charge at least $\frac{2}{3}$ to each bad neighbor. Thus, the charge at y_2 after (R5) will be at least $-2 + 1 + \frac{2}{3} + \frac{2}{3} = \frac{1}{3}$. By (R6), y_2 sends charge at least $\frac{1}{6}$ to v. Thus, $\mu^*(v) \geq -2 + 1 + \frac{1}{4} + \frac{2}{3} + \frac{1}{6} = \frac{1}{12} > 0$.

(b) If $f(y_1) = 1$, we know, v_1 has a 2-saturated (high) neighbor z by Lemma 3.4. If z, v_1, y_1 are consecutive vertices of a face f_3, then v is a close 2-vertex of f_3, which is clearly a good face. By (R4), f_3 will give charge $\frac{1}{12}$ to v. Thus, $\mu^*(v) \geq -2 + 1 + \frac{1}{12} + \frac{1}{3} + \frac{3}{5} = \frac{1}{60} > 0$.

Now consider the case where z, v_1, x_1, x_2 are consecutive vertices of a face f_0. If f_0 is not a bad face, then the charge at x_1 after (R5) will be at least $-2+\frac{1}{2}+\frac{2}{3}+\frac{1}{3}+\frac{3}{5} = \frac{1}{10}$. By (R6), x_1 will send charge at least $\frac{1}{10}$ to v; thus, $\mu^*(v) \geq -2+1+\frac{1}{3}+\frac{3}{5}+\frac{1}{10} = \frac{1}{30} > 0$. So f_0 is a bad face $wzv_1x_1x_2$, which further implies that x_2 is the head of f_0. By letting $f(v) = 1$ and erasing the color on x_1, we can apply Lemma 3.4 to f_0 to conclude that z is a fat vertex and $f(w) = 1$. By Lemma 3.1 applied to $v_1x_1x_2w$, we know $d(x_2) \geq 8$. Since (a chubby vertex) x_2 has a high neighbor v_2 and $d(x_2) \geq 8$, it follows that x_2 is a fat vertex.

Now, after (R5), w will have charge at least $-2 + 1 + 2 \cdot \frac{2}{3} = \frac{1}{3}$ and x_1 will have charge at least $-2 + 1 + \frac{1}{3} + \frac{2}{3} = 0$ and v will have charge at least $-2 + 1 + \frac{1}{3} + \frac{3}{5} = -\frac{1}{15}$. By (R6), w sends charge at least $\frac{1}{6}$ to x_1, so the charge at x_1 is at least $\frac{1}{6}$ after (R6). If the charge at v is still negative after (R6), then v could not have sent charge to w by (R6). Since w sent charge to x_1, by (R7), x_1 will send all of its excess charge to v. Thus, after (R7), v will have charge at least $-\frac{1}{15} + \frac{1}{6} = \frac{1}{10} > 0$ (see Fig. 3). □

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