The very basics of the NLS

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Abstract

I will focus on the semilinear Schrödinger equation of the form $i\partial_t u + \Delta u = \mu|u|^p u$ whose initial datum lies in some L^2-based Sobolev space. At first, I will briefly explain the linear Schrödinger equation and its dispersive phenomenon, and then quantify it in terms of Lebesgue norms to obtain the Strichartz estimates. With the contraction mapping principle, this allows us to have the local well-posedness of the equation for a certain range of p. On the other hand, as a Hamiltonian equation, symmetries of the energy functional give rise to some conserved quantities (via Noether's theorem). Using these, one can upgrade the local well-posedness to the global well-posedness. Finally, if time permits, I will briefly introduce the scattering theory. Most of the materials are brought from [Tao].

1 Notations and preliminaries

Fourier transform: the Fourier transform on $\mathbb{R}^d$ is defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} d\xi.$$

Fourier multiplier: let $m : \mathbb{R} \to \mathbb{R}$ be a measurable function whose growth is at most polynomial. We define $m(\frac{\nabla}{i})$ as a linear operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ satisfying

$$[\widehat{m(D)f}](\xi) = m(\xi)\widehat{f}(\xi)$$

where $D = \frac{\nabla}{i}$.

Sobolev norms: we define the Sobolev norms for $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ as

$$\|f\|_{W^{s,p}} := \|\langle\nabla\rangle^s f\|_{L^p}, \qquad \|f\|_{\dot{W}^{s,p}} := \||\nabla|^s f\|_{L^p}.$$

Mixed Lebesgue norms: for $1 \leq q, r \leq \infty$, we define

$$\|F(t,x)\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} := \Big\|\|F(t)\|_{L_x^r(\mathbb{R}^d)}\Big\|_{L_t^q(I)}.$$

Nonlinear Schrödinger equation (NLS): for $p \geq 0$, $s \in \mathbb{R}$ and $\mu \in \{-1, 0, 1\}$,

$$\begin{aligned} i\partial_t u + \Delta u &= \mu|u|^p u = \mu F(u) \\ u(0,x) &= u_0(x) \in H_x^s(\mathbb{R}^d). \end{aligned}$$

If $\mu = 0$, this is called the *linear* (or *free*) Schrödinger equation. If $\mu = 1$ (resp., -1), it is called the *defocusing* (resp., *focusing*).

Duhamel's formula: If there exists a solution $u \in C_{t,loc}^\infty \mathcal{S}_x(I\times\mathbb{R}^d)$ with initial datum u_0, the spatial Fourier transform yields that

$$i\partial_t \widehat{u(t)} - |\xi|^2\widehat{u(t)} = \mu\widehat{F(u(t))}.$$

From Duhamel's formula in ODE theory, we obtain

$$\widehat{u(t)} = e^{-it|\xi|^2}\widehat{u_0} - i\mu\int_0^t e^{-i(t-s)|\xi|^2}\widehat{F(u(s))}ds.$$

Finally, we take the inverse Fourier transform to obtain

$$u(t) = e^{it\Delta}u_0 - i\mu\int_0^t e^{i(t-s)\Delta}F(u(s))ds.$$

Hamiltonian formulation: consider the Hamiltonian H and the symplectic form w on $L^2(\mathbb{R}^d)$ as follows

$$H(u) := \frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2 + \frac{\mu}{p+2}\int|u|^{p+2}, \qquad w(f,g) := \operatorname{Im}\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx.$$

The corresponding Hamiltonian equation becomes

$$\partial_t u = \nabla_w H(u) = i\Delta u - i\mu|u|^p u.$$

This H enjoys lots of *symmetries.* From Noether's theorem, they correspond to some *conserved quantities.* We list them as follows. Also, the equation

Symmetries	Conserved quantities
time translation	Energy: $E = \frac{1}{2}\int \|\nabla u\|^2 + \frac{\mu}{p+2}\int \|u\|^{p+2}$
space translation	Momentum: $P = 2\int \mathrm{Im}(\overline{u}\nabla u)$
space rotation	Angular momentum: $L_{jk} = i\int \overline{u}[x_j\partial_k u - x_k\partial_j u]$
phase rotations	Mass: $M = \int \|u\|^2$
Galilean invariance	Normalized center of mass: $\vec{x} - t\vec{p}$

Table 1: Symmetries and Conserved quantities

itself enjoys some symmetries: let u be a solution. We have the scaling symmetry

$$u(t,x) \mapsto \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x).$$

In particular, if $p = \frac{4}{d}$ (called *mass-critical*), then the above transformation preserves the L^2 norm. If $p < \frac{4}{d}$ (resp. $p > \frac{4}{d}$), we call that case *mass-subcritical* (resp. *mass-supercritical*).

Also, we have the Galilean invariance

$$u(t,x) \mapsto e^{ix\cdot\xi_0 - it|\xi_0|^2} u(t, x - 2\xi_0 t).$$

There are discrete symmetries: the time-reversal symmetry

$$u(t,x) \mapsto \overline{u}(-t,x)$$

and the pseudo-conformal symmetry (only holds when $p = \frac{4}{d}$ or $\mu = 0$)

$$u(t,x) \mapsto t^{-\frac{d}{2}} e^{i|x|^2/4t} u\Big(-\frac{1}{t}, \frac{x}{t}\Big).$$

2 The linear Schrödinger equation

In order to study a nonlinear equation, it is customary to regard the nonlinear equation as a perturbation of the linear equation at first time. Indeed, the Duhamel formula exactly shows that one can obtain u by adding some nonlinear feedback to the linear solution.

Let us first start with a linear dispersive equation of the form

$$\partial_t u = L(D)u,$$

where $L(z) = ih(\frac{z}{i})$ with a real polynomial h. Here, h is called a dispersion relation. For example, the linear Schrödinger equation $i\partial_t u + \Delta u = 0$ has the dispersion relation $h(\xi) = -|\xi|^2$, the Airy equation $u_t + u_{xxx} = 0$ has the dispersion relation $h(\xi) = \xi^3$, and the transport equation $\partial_t u = -v \cdot \nabla u$ with $v \in \mathbb{R}^d$ has the dispersion relation $h(\xi) = -v \cdot \xi$.

We first observe that

$$u(t,x) = e^{i\xi_0 \cdot x + ith(\xi_0)}, \qquad \forall \xi_0 \in \mathbb{R}^d$$

is a linear solution. This says that u has the velocity $-\frac{\xi_0}{|\xi_0|}\frac{h(\xi_0)}{|\xi_0|}$. This velocity is called the *phase velocity*, as one shall see soon that *group velocity* is somewhat different. Suppose that $\widehat{\phi}$ is supported near the frequency ξ_0 and $u(0) = \phi$. Then,

$$u(t) = \sum_{\xi} \widehat{\phi}(\xi) e^{ith(\xi)} e^{i\xi \cdot x} \approx \sum_{\xi} \widehat{\phi}(\xi) e^{it\nabla h(\xi_0)\cdot(\xi-\xi_0)} e^{ith(\xi_0)} e^{i\xi\cdot x}.$$

The term $\widehat{\phi}(\xi)e^{-it(2\xi_0\cdot\xi)}$ makes the translation in the physical space by $2t\xi_0$. Therefore, u has the group velocity $2\xi_0$. With the superposition principle, we see that each plane waves of different frequency moves in different velocity. So the dispersive effect is apparent. However on $\mathbb{T}^d$, one cannot have an easy dispersive effect as in $\mathbb{R}^d$ because of periodicity.

Roughly, as the linear solution disperses, its support becomes larger but its height becomes lower. A common way to quantify this effect is to estimate the L_x^p norm for large p because it is less sensitive to the width of the support of the function as p becomes larger.

Lemma 2.1 (Fundamental solution). *We have* $e^{it\Delta}u_0 = K_t * u_0$ *where*

$$K_t(x) = (4\pi it)^{-\frac{d}{2}} e^{i|x-y|^2/4t}$$

for all $t \neq 0$.

Proof. Formally, we have

$$\begin{aligned} e^{it\Delta}u_0(y) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-it|\xi|^2} e^{-i(x-y)\cdot\xi} u_0(x)\,dx\,d\xi \\ &= \int_{\mathbb{R}^d} \underbrace{\int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} e^{-it|\xi|^2} e^{i(y-x)\cdot\xi}\,d\xi}_{K_t(y-x)}\, u_0(x)\,dx. \end{aligned}$$

The formula of K_t can be obtained using analytic continuation. $\square$

Proposition 2.2 (Dispersive estimate)**.** *We have*

$$\|e^{it\Delta}u_0\|_{L_x^p(\mathbb{R}^d)} \lesssim_{d,p} |t|^{-d(\frac{1}{2}-\frac{1}{p})}\|u_0\|_{L_x^{p'}(\mathbb{R}^d)}$$

for all $t \neq 0$ and $2 \leq p \leq \infty$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. From the above lemma, $\|e^{it\Delta}u_0\|_{L_x^\infty(\mathbb{R}^d)} \lesssim_d |t|^{-\frac{d}{2}}\|u_0\|_{L_x^1(\mathbb{R}^d)}$. On the other hand, the Plancherel theorem tells us that $e^{it\Delta}$ is a unitary operator on $L_x^2(\mathbb{R}^d)$. Therefore, the claim follows from the Riesz-Thorin interpolation theorem. □

Definition (Schrödinger admissible pairs)**.** For $2 \leq q, r \leq \infty$, (q, r) is called a Schrödinger admissible pair if

$$\frac{2}{q}+\frac{d}{r}=\frac{d}{2}, \qquad \text{and} \qquad (q,r,d) \neq (2,\infty,2).$$

Theorem 2.3 (Strichartz estimate)**.** *Let (q, r), $(\widetilde{q}, \widetilde{r})$ be Schrödinger admissible pairs. Then,*

$$\|e^{it\Delta}u_0\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{d,q,r} \|u_0\|_{L_x^2(\mathbb{R}^d)},$$
$$\Big\|\int_{\mathbb{R}} e^{-is\Delta}F(s,x)ds\Big\|_{L_x^2(\mathbb{R}^d)} \lesssim_{d,\widetilde{q},\widetilde{r}} \|F\|_{L_t^{\widetilde{q}'}L_x^{\widetilde{r}'}(\mathbb{R}\times\mathbb{R}^d)},$$
$$\Big\|\int_{s<t} e^{i(t-s)\Delta}F(s,x)ds\Big\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim_{d,q,r,\widetilde{q},\widetilde{r}} \|F\|_{L_t^{\widetilde{q}'}L_x^{\widetilde{r}'}(\mathbb{R}\times\mathbb{R}^d)}.$$

Proof. We will only show the non-endpoint cases. The endpoint estimates are proved in [KeelTao]. Let

$$[Tu_0](t,x) \coloneqq [e^{it\Delta}u_0](x).$$

Then, the dual operator becomes

$$[T^*F](x) \coloneqq \int_{\mathbb{R}} e^{-is\Delta}F(s,x)ds.$$

We use the TT^*-argument; i.e., we will show that TT^* is bounded. Observe that

$$[TT^*F](t,x) \coloneqq \int_{\mathbb{R}} e^{i(t-s)\Delta}F(s,x)ds.$$

Now, we apply Minkowski's inequality, dispersive estimate and Hardy-Littlewood-Sobolev inequality to obtain

$$\begin{aligned}\|TT^*F\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)} &\lesssim \Big\| \int_{\mathbb{R}} \|e^{i(t-s)\Delta}F(s,x)\|_{L^r_x(\mathbb{R}^d)} ds\Big\|_{L^q_t(\mathbb{R})} \\ &\lesssim_{d,r} \Big\| \int_{\mathbb{R}} |t-s|^{-d(\frac{1}{2}-\frac{1}{r})} \|F(s)\|_{L^{r'}_x(\mathbb{R}^d)} ds\Big\|_{L^q_t(\mathbb{R})} \\ &\lesssim_{d,r} \Big\| \|F(s)\|_{L^{r'}_x(\mathbb{R}^d)} \Big\|_{L^{q'}_t(\mathbb{R})}\end{aligned}$$

where we used $0 < \frac{2}{q} = \frac{d}{2} - \frac{d}{r} < 1$ (non-endpoint assumption) in the HLS inequality. This completes the proof of the first two estimates. In order to obtain the last estimate, apply the Christ-Kiselev lemma which is stated below. $\square$

Lemma 2.4 (Christ-Kiselev lemma)**.** *Let X, Y be Banach spaces; let I be a time interval, and let $K \in C^0(I \times I \to \mathcal{B}_{X\to Y})$ be a kernel. Suppose that we have*

$$\Big\| \int_I K(t,s)f(s)ds \Big\|_{L^q_t(I\to Y)} \le A\|f\|_{L^p_t(I\to X)}$$

for some $1 \le p < q \le \infty$ and $A > 0$. Then,

$$\Big\| \int_I K(t,s)f(s)ds \Big\|_{L^q_t(I\to Y)} \lesssim_{p,q} A\|f\|_{L^p_t(I\to X)}.$$

Proof. See [Tao2]. $\square$

3 Well-posedness of the NLS

Definition 3.1. (Strong L^2_x solution) Suppose that $u_0 \in L^2_x(\mathbb{R}^d)$ are given. If $u \in C^0_tL^2_x(I\times\mathbb{R}^d)$ satisfies the integral equation, u is called a strong $L^2_x(\mathbb{R}^d)$ solution of NLS with initial datum u_0.

Definition 3.2. (Strichartz space) The Strichartz space $S^0(I\times\mathbb{R}^d)$ is defined with its norm

$$\begin{aligned}\|u\|_{S^0(I\times\mathbb{R}^d)} &= \sup_{(q,r):\ \text{admissible}} \|u\|_{L^q_tL^r_x(I\times\mathbb{R}^d)} \text{ and} \\ \|u\|_{N^0(I\times\mathbb{R}^d)} &= \inf_{(q,r):\ \text{admissible}} \|u\|_{L^{q'}_tL^{r'}_x(I\times\mathbb{R}^d)}.\end{aligned}$$

Remark. If $d = 2$, the set of admissible pairs is not compact (not containing $(2, \infty)$). Therefore, we may truncate the supremum such as $q \geq 2 + \epsilon$.

Remark. By Riesz-Thorin interpolation on $L_t^q L_x^r$ space, the above Strichartz norm is equivalent to the supremum norm of two end-point admissible pairs. Similarly, this applies to N^0.

The Strichartz estimates can be written as:

$$\|e^{it\Delta/2}u_0\|_{S^0(I\times\mathbb{R}^d)} \lesssim_d \|u_0\|_{L_x^2(\mathbb{R}^d)},$$
$$\left\| \int_0^t e^{i(t-s)\Delta/2}F(s)ds \right\|_{S^0(I\times\mathbb{R}^d)} \lesssim_d \|F\|_{N^0(I\times\mathbb{R}^d)}.$$

Theorem 3.3 (L_x^2-subcritical local well-posedness)**.** *Let $d = 1$ and $p = 2$. For any $R > 0$ and $u_0 \in B_{L_x^2}(0; R)$, there exists $T = T(R) > 0$ such that u_0 admits a unique strong L_x^2 solution in $S^0([-T,T]\times\mathbb{R}) \subset C_t^0 L_x^2([-T,T]\times\mathbb{R})$. Furthermore, the solution map $u_0 \mapsto u$ is Lipschitz map from $B_{L_x^2}(0; R)$ to $S^0([-T,T]\times\mathbb{R})$.*

Remark. The T can be chosen depending only on R. This says that T does not depends on the profile of the initial datum and whether the equation is focusing or defocusing.

Proof. We want to apply the contraction mapping principle to the map $\Phi_{u_0} : B_{S^0}(0; R_1) \to B_{S^0}(0; R_1)$ defined by

$$\Phi_{u_0}(u) = e^{it\Delta/2}u_0 - i\mu \int_0^t e^{i(t-s)\Delta/2}F(u(s))ds.$$

1. (Self-map) For $u \in B_{S^0}(0; R_1)$, the Strichartz estimate and Hölder inequality yields that

$$\begin{aligned}
\|\Phi_{u_0}(u)\|_{S^0} &\leq \|e^{it\Delta}u_0\|_{S^0} + \left\| \int_0^t e^{i(t-s)\Delta}|u(s)|^2 u(s)ds \right\|_{S^0} \\
&\lesssim \|u_0\|_{L_x^2} + \||u|^2u\|_{L_t^1 L_x^2} \\
&\lesssim \|u_0\|_{L_x^2} + T^{\frac{1}{2}}\|u\|_{L_{t,x}^6}^3 \\
&\leq C_1(R + T^{\frac{1}{2}}R_1^3).
\end{aligned}$$

2. (Contraction) For $u, v \in B_{S^0}(0; R_1)$, the Hölder inequality yields

$$\begin{aligned}\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{S^0} &\lesssim \||u|^2u - |v|^2v\|_{L^1_tL^2_x} \\ &\lesssim T^{\frac{1}{2}}\|u - v\|_{L^6_{t,x}}(\|u\|^2_{L^6_{t,x}} + \|v\|^2_{L^6_{t,x}}) \\ &\leq C_2T^{\frac{1}{2}}R_1^2\|u - v\|_{S^0}.\end{aligned}$$

3. Note that $\Phi_{u_0}(u) \in C^0_tL^2_x$ because of the following fact:

$$\text{if } t \to t_0 \text{ in } \mathbb{R} \text{ and } f \to f_0 \text{ in } L^2_x, \text{ then } e^{it\Delta}f \to e^{it_0\Delta}f_0 \text{ in } L^2_x.$$

By 1 and 2, we may choose R_1 large such that $R_1 \geq 2C_1R$. Then, we choose T small such that $T^{\frac{1}{2}}R_1^3 \leq R$ and $C_2T^{\frac{1}{2}}R_1^2 \leq \frac{1}{2}$. By the contraction mapping principle, there exists a unique solution u in $B_{S^0}(0; R_1)$.

Now, we show that u is indeed unique in the whole space S^0. Suppose we are given two solutions u, $\widetilde{u}$ for the initial datum u_0. Since $u, \widetilde{u} \in C^0_tL^2_x$, the set

$$A := \{t \in [-T, T] : u(t) = \widetilde{u}(t)\}$$

is closed. Hence, it suffices to show that A is also open in $[-T, T]$. Let $t_0 \in A$ be arbitrary and choose $R_1 \geq 2C_1 \max\{\|u\|_{S^0}, \|\widetilde{u}\|_{S^0}\}$. Then, by the observation 1 and 2, we can choose $\delta > 0$ sufficiently small such that the map $\Phi_{u(t_0)}$ (whose initial datum is $u(t_0)$ at time t_0 and S^0 is defined on the interval $[t_0 - \delta, t_0 + \delta]$) becomes a contraction. Therefore, $u(t) = \widetilde{u}(t)$ for all $t \in [t_0 - \delta, t_0 + \delta]$ by the uniqueness. Thus, A is open.

Finally, we prove Lipschitz continuous dependence on the initial data. For any $u_0, v_0 \in B_{L^2_x}(0; R)$, let u, v be the solutions, respectively. Then,

$$\begin{aligned}\|u - v\|_{S^0} &\leq \|e^{it\Delta/2}(u_0 - v_0)\|_{S^0} + \|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{S^0} \\ &\leq C_3\|u_0 - v_0\|_{L^2_x} + \frac{1}{2}\|u - v\|_{S^0}.\end{aligned}$$

This shows that u is a contraction. □

Combining with mass conservation, we have global L^2_x well-posedness:

Corollary 3.4 (L^2_x-subcritical global well-posedness)**.** *Let $0 \leq p < \frac{4}{d}$. For any $u_0 \in L^2_x(\mathbb{R}^d)$, there exists a unique strong L^2_x solution u in $S^0_{t,loc}(\mathbb{R}\times\mathbb{R}^d) \subset C^0_{t,loc}L^2_x(\mathbb{R} \times \mathbb{R}^d)$. Moreover, the solution map $u_0 \mapsto u$ from L^2_x to $S^0_{t,loc}$ is continuous.*

When p is mass-critical, we need to change the statement slightly:

Theorem 3.5 (L^2_x-critical local well-posedness)**.** *Let $p = 3$ and $d = 2$. For any $u_* \in L^2_x(\mathbb{R}^2)$, there exists $\epsilon_0 > 0$ such that whenever $u_* \in L^2_x(\mathbb{R}^2)$ and I satisfies $\|e^{it\Delta}u_*\|_{L^4_{t,x}(I\times\mathbb{R}^2)} \le \epsilon_0$, any $u_0 \in B_{L^2_x}(u_*;\epsilon_0)$ admits a strong $L^2_x(\mathbb{R}^2)$ solution in $S^0(I\times\mathbb{R}^2) \subset C^0_t L^2_x(I\times\mathbb{R}^2)$. Moreover, the solution map $u_0 \mapsto u$ from $B_{L^2_x}(u_*;\epsilon_0)$ to $S^0(I\times\mathbb{R}^2)$ is Lipschitz continuous.*

Remark. The lifespan of the solution depends on the profile of the initial datum. So we cannot obtain global well-posedness using mass conservation in the critical case. Indeed, choose a solution of the form $e^{-it\tau}Q(x)$ where $Q \in \mathcal{S}_x(\mathbb{R}^d)$ (which is available in the focusing case). Using the pseudo-conformal transformation, we see that $e^{i\tau/t}e^{i|x|^2/2t}\overline{Q(x/t)}$ is also a solution to the NLS. This solution has constant L^2_x-norm (by mass conservation) but has short existence time interval when $t \to 0$.

Proof. Define an artificial norm on $S^0(I\times\mathbb{R}^2)$ by

$$\|u\|_X = \delta\|u\|_{S^0(I\times\mathbb{R}^2)} + \|u\|_{L^4_{t,x}(I\times\mathbb{R}^2)},$$

where $0 < \delta \le 1$ will be chosen later. Here, choosing δ small means that we will not have a good control on the size of $\|u\|_{S^0}$ (i.e., it might be very large) but we only require $u \in S^0(I\times\mathbb{R}^2)$. Define $\Phi_{u_0} : B_X(0;\eta) \to B_X(0;\eta)$ such that

$$\Phi_{u_0}(u) := e^{it\Delta}u_0 - i\mu\int_0^t e^{i(t-s)\Delta}|u(s)|^2u(s)ds.$$

As before, we obtain

$$\begin{aligned}
\|\Phi_{u_0}(u)\|_X &\le \|e^{it\Delta}(u_0 - u_*)\|_X + \|e^{it\Delta}u_*\|_X + \left\|\int_0^t e^{i(t-s)\Delta}|u(s)|^2u(s)ds\right\|_X \\
&\lesssim (1+\delta)\epsilon_0 + (\epsilon_0 + \delta\|u_*\|_{L^2_x}) + \||u|^2u\|_{L^{4/3}_{t,x}} \\
&\le C_1(\epsilon_0 + \delta\|u_*\|_{L^2_x} + \eta^3)
\end{aligned}$$

and

$$\begin{aligned}
\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_X &\lesssim \|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{S^0} \\
&\lesssim \|F(u) - F(v)\|_{L^{4/3}_{t,x}} \\
&\le C_2\eta^2\|u - v\|_X.
\end{aligned}$$

Note that $\Phi_{u_0}(u) \in C^0_t L^2_x$ follows as before. Therefore, we may choose $\eta, \epsilon_0 > 0$ small such that $\max\{C_1, C_2\}\eta^2 < \frac{1}{2}$ and $0 < C_1\epsilon_0 < \frac{1}{2}(\eta - C_1\eta^3)$. Then, choose $\delta > 0$ small such that $\delta\|u_*\|_{L^2_x} < \eta - C_1(\epsilon_0 + \eta^p)$.

The uniqueness in $S^0(I \times \mathbb{R}^2)$ follows as before. For Lipschitz continuous dependence on the initial data, observe for the two solutions u, v in $B_X(0;\eta)$ whose initial data $u_0, v_0 \in B_{L^2_x}(u_*;\epsilon_0)$ that

$$\begin{aligned}\|u - v\|_X &= \|e^{it\Delta/2}(u_0 - v_0)\|_X + \|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_X \\ &\leq C\|u_0 - v_0\|_{L^2_x} + \frac{1}{2}\|u - v\|_X.\end{aligned}$$

Since X norm depends on δ, the Lipschitz constant for $B_{L^2_x}(u_*;\epsilon_0) \to S^0(I \times \mathbb{R}^d)$ actually depends on u_*. $\square$

Remark. This argument works for all $p = \frac{4}{d}$ in any dimension.

Remark. When u_0 has very small norm, then we can apply the theorem with $u_* = 0$ and $I = \mathbb{R}$. In this case, the solution globally exists and $\|u\|_{S^0(\mathbb{R}\times\mathbb{R}^2)} < \infty$.

Remark. Note that we cannot have such local well-posedness in the supercritical case. Suppose that p is mass-supercritical and recall the scaling invariance

$$u(t, x) \mapsto \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x).$$

When $\lambda > 1$, the lifespan of $\lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x)$ becomes shorter and the profile becomes sharper. When p is supercritical, the lifespan becomes shorter while its L^2_x norm becomes smaller. That means that there could exist a profile u_0 whose L^2_x-norm is arbitrarily small and its lifespan is arbitrarily short. Hence it seems that the mass-supercritical NLS is ill-posed in some sense.

One can also obtain H^1_x-(sub)critical local well-posedness.

Theorem 3.6 (H^1_x-subcritical local well-posedness)**.** *Let $p = 3$ and $d = 3$. For any $R > 0$ and $u_0 \in B_{H^1_x}(0;R)$, there exists $T = T(R) > 0$ such that u_0 admits a unique strong H^1_x solution in $S^1([-T,T] \times \mathbb{R}^3) \subset C^0_t H^1_x([-T,T] \times \mathbb{R}^3)$. Furthermore, the solution map $u_0 \mapsto u$ is Lipschitz map from $B_{H^1_x}(0;R)$ to $S^1([-T,T] \times \mathbb{R}^3)$.*

Proof. Proof strategy is almost same as before. We only note some estimates:

$$\begin{aligned}\|F(u)\|_{L^2_t W^{1,\frac{6}{5}}} &\lesssim T^{\frac{1}{5}} \|F(u)\|_{L_t^{\frac{10}{3}} W^{1,\frac{6}{5}}} && \text{by Hölder} \\ &\lesssim T^{\frac{1}{5}} \|u\|^2_{L^{10}_t L^5_x} \|u\|_{L^{10}_t W^{1,\frac{10}{3}}_x} && \text{by fractional chain rule} \\ &\lesssim T^{\frac{1}{5}} \|u\|^3_{S^1}. && \text{by Sobolev embedding}\end{aligned}$$

Thus, we may choose T small. $\square$

Theorem 3.7 ($\dot{H}^1_x$-critical local well-posedness)**.** *Let $p = 5$ and $d = 3$. For any $u_* \in \dot{H}^1_x(\mathbb{R}^3)$, there exists $\epsilon_0 > 0$ such that whenever $u_* \in \dot{H}^1_x(\mathbb{R}^3)$ and I satisfies $\|e^{it\Delta}u_*\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)} \leq \epsilon_0$, any $u_0 \in B_{\dot{H}^1_x}(u_*;\epsilon_0)$ admits a strong $\dot{H}^1_x(\mathbb{R}^3)$ solution in $\dot{S}^1(I\times\mathbb{R}^3) \subset C^0_t\dot{H}^1_x(I\times\mathbb{R}^3)$. Moreover, the solution map $u_0 \mapsto u$ from $B_{\dot{H}^1_x}(u_*;\epsilon_0)$ to $\dot{S}^1(I\times\mathbb{R}^3)$ is Lipschitz continuous.*

Proof. Observe that

$$\|F(u)\|_{L^2_t\dot{W}^{1,\frac{6}{5}}} \lesssim \|u\|^4_{L^{10}_{t,x}}\|u\|_{L^{10}_t\dot{W}^{1,\frac{30}{13}}_x} \lesssim \|u\|^4_{L^{10}_{t,x}}\|u\|_{\dot{S}^1}.$$

Define X-norm by $\|u\|_X \coloneqq \|u\|_{L^{10}_{t,x}} + \delta\|u\|_{\dot{S}^1}$ and proceed as before. □

Definition. A solution u to the NLS scatters in $\dot{H}^s_x$, forward (resp. backward) in time if there exists $u_\pm$ such that $\|u(t) - e^{it\Delta}u_\pm\|_{\dot{H}^s_x} \to 0$ as $t \to \pm\infty$.

Proposition 3.8 (Finite Strichartz norm implies scattering)**.** *Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan solution to the NLS. If $\|u\|_{S^0(I)} < \infty$, then $I = \mathbb{R}$ and u scatters in L^2_x, both forward and backward in time.*

Proof. $I = \mathbb{R}$ follows from inspecting the proof of local well-posedness. In order to prove that u scatters, the Duhamel formula implies

$$e^{-it\Delta}u(t) = u(0) - i\mu\int_0^t e^{-is\Delta}F(s)ds.$$

Therefore, u scatters if the above integral conditionally converges. Since we assume $\|u\|_{S^0(I)} < \infty$, this is the case. □

Theorem 3.9 (Mass-critical scattering theorem)**.** *Let $u_0 \in L^2_x(\mathbb{R}^d)$ and $u : I\times\mathbb{R}^d \to \mathbb{C}$ be the maximal lifespan solution to the NLS with initial datum u_0. If $\mu = +1$, then $I = \mathbb{R}$ and there exists a continuous function $L : [0,\infty) \to [0,\infty)$ satisfying*

$$\|u\|_{S^0(\mathbb{R}\times\mathbb{R}^d)} \leq L(\|u_0\|_{L^2_x}).$$

If $\mu = -1$, we further assume that $\|u_0\|_{L^2_x} < \|Q\|_{L^2_x}$ and the above statement holds with $L : [0, \|Q\|_{L^2_x}) \to [0,\infty)$.

We conclude this note with a remark. In the mass-critical case, if $u_0 \in H^1_x$, then one can use H^1_x-subcritical local well-posedness. If $\mu = +1$, then energy conservation shows that the solution exists globally. If $\mu = -1$, then we further assume $M(u) < M(Q)$ and use the sharp Gagliardo-Nirenberg inequality to obtain global existence. However, for an initial data only in L_x^2, the global well-posedness and scattering were conjectured for a long time. This conjecture has been completely solved by B. Dodson (for a reference, [Dodson] and will be enough). The proof uses the concentration-compactness technique which originates from the elliptic theory. The major contribution of Dodson is that he proved the longtime Strichartz estimate. Roughly, it says that a (nonlinear) solution which seems to scatter must obey the linear Strichartz estimate plus some amenable error.

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