Boolean Algebras, Boolean Rings and Stone's Representation Theorem

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Abstract

This is a part of a supplementary note for a Logic and Set Theory course. The main goal is to constitute equivalences between the category of Boolean algebras, Boolean rings and Stone spaces.

Contents

1 Boolean Algebras

1.1 Definition and First Properties

Definition 1.1.1. A boolean algebra B is a set together with complement operation $\neg$ and two binary operations $\vee$ and $\wedge$ subject to

- Commutativity
$$a \vee b = b \vee a, \qquad a \wedge b = b \wedge a$$
- Associativity
$$a \vee (b \vee c) = (a \vee b) \vee c, \qquad a \wedge (b \wedge c) = (a \wedge b) \wedge c$$
- Distributivity
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \qquad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
- Absorption
$$a \vee (a \wedge b) = a, \qquad a \wedge (a \vee b) = a$$

- There are two different elements 0 and 1 in B such that

$$a \vee 0 = a, \qquad a \wedge 1 = a$$

  and

$$a \vee \neg a = 1, \qquad a \wedge \neg a = 0$$

Definition 1.1.2. Given two Boolean algebras B_1 and B_2 the function $f : B_1 \to B_2$ is called a Boolean algebra homomorphism (or simply homomorphism if no confusion occurs) if f satisfies the following properties: $f(a \vee b) = f(a) \vee f(b)$, $f(a \wedge b) = f(a) \wedge f(b)$, $f(0) = 0$ and $f(1) = 1$.

Proposition 1.1.3. *Let $f : B_1 \to B_2$ be an homomorphism between Boolean algebras B_1 and B_2. For any $x \in B_1$, we have $f(\neg x) = \neg f(x)$.*

Proof. Observe:

$$1 = f(1) = f(x \vee \neg x) = f(x) \vee f(\neg x)$$

and

$$0 = f(0) = f(x \wedge \neg x) = f(x) \wedge f(\neg x).$$

Now the proposition is a consequence of the following lemma. □

Lemma 1.1.4. *If $x \wedge y = 0$ and $x \vee y = 1$ then $y = \neg x$.*

Proof. To this end, we first observe that

$$\begin{aligned} \neg x &= \neg x \wedge 1 \\ &= \neg x \wedge (x \vee y) \\ &= (\neg x \wedge x) \vee (\neg x \wedge y) \\ &= 0 \vee (\neg x \wedge y) \\ &= \neg x \wedge y. \end{aligned}$$

Since $0 = x \wedge y$ we finally have

$$\begin{aligned} \neg x &= \neg x \vee 0 \\ &= (\neg x \wedge y) \vee (x \wedge y) \\ &= ((\neg x \wedge y) \vee x) \wedge ((\neg x \wedge y) \vee y) \\ &= ((\neg x \vee x) \wedge (y \vee x)) \wedge y \\ &= (y \vee x) \wedge y \\ &= y. \end{aligned}$$

We used the absorption property several times. □

Example 1.1.5 (Two-element Boolean Algebra)**.** This is the one of motivating examples of Boolean algebras. Let $T = \{0, 1\}$. Interpret 0, 1, $\vee$, $\wedge$, $\neg$ as "false", "true", "or", "and", "not" respectively. Then we get a Boolean algebra which is called a two-element Boolean algebra.

Example 1.1.6 (Power Set Boolean Algebra)**.** Let X be a set and consider its power set 2^X. Let $0 = \emptyset$ and $1 = X$ and interpret $\vee$, $\wedge$, $\neg$ as $\cup$, $\cap$, $X \setminus -$ respectively. Then $(2^X, \vee, \wedge, \neg)$ becomes a Boolean algebra.

Example 1.1.7. Let X be a topological space. Let $\mathcal{B}(X)$ be the set of clopen subsets of X. With the same interpretations of $\vee$, $\wedge$, $\neg$ as in the power set algebra, we get a Boolean algebra $(\mathcal{B}(X), \vee, \wedge, \neg)$. For a connected space X, $\mathcal{B}(X)$ is isomorphic to the two-element Boolean algebra. To get a non-trivial $\mathcal{B}(X)$, the space X should have many connected components. $\mathcal{B}(X)$ is particularly interesting when X is totally disconnected, Hausdorff and compact. In that case, we will see that the Boolean algebra $\mathcal{B}(X)$ is universal.

Example 1.1.8 (Duality)**.** Let $(B, \vee, \wedge, \neg)$ be a given Boolean algebra. We can construct another Boolean algebra B^* with is dual to B as follows. As a set, B^* is the same as B. The operations $\vee^*$, $\wedge^*$, $\neg^*$ of B^* are defined by $a \vee^* b = a \wedge b$, $a \wedge^* b = a \vee b$ and $\neg^* a = \neg a$. 0 in B^* is 1 in B and 1 in B^* is 0 in B. Then $B^* = (B, \vee^*, \wedge^*, \neg^*)$ is a Boolean algebra.

We can define a filter for a Boolean algebra.

Definition 1.1.9. Let B be a Boolean algebra. A filter F of B is a subset of B such that for all $x, y \in F$, $x \wedge y \in F$ and for all $x \in F$ and all $z \in B$, $x \vee z \in F$. A subset F is a prime filter if F is a filter and for all $x, y \in B$, $x \vee y \in F$ implies either $x \in F$ or $y \in F$. A ultrafilter is a proper filter which is maximal with respect to the inclusion.

The definition of a filter for a Boolean algebra is consistent with that of a set if we regard 2^X as a Boolean algebra in the sense of example 1.1.6.

There is a dual notion of a filter which is called an ideal.

Definition 1.1.10. An ideal of a Boolean algebra B is a filter of its dual B^*. Prime and maximal ideal is prime, ultrafilter of B^* respectively.

Proposition 1.1.11. *Let M be an maximal ideal of a Boolean algebra B. Then there is a homomorphism $f : B \to T$ into the two-element Boolean algebra T such that $M = f^{-1}(0)$. Conversely, given a homomorphism $f : B \to T$ into the two-element Boolean algebra T, $f^{-1}(0)$ is a maximal ideal.*

Proof. Assume that $f : B \to T$ is given. It is not difficult to show that $f^{-1}(0)$ is an ideal. To show that $f^{-1}(0)$ is maximal, suppose that there is another ideal I containing $f^{-1}(0)$. Then there is $a \in I$ such that $a \notin f^{-1}(0)$, that is, $f(a) = 1$. Since $f(\neg a) = \neg f(a) = 0$ we see that $\neg a \in f^{-1}(0)$. Then since I is an ideal, $a \wedge \neg a = 1 \in I$. But if $1 \in I$ we see that for all $x \in B$, $1 \wedge x = x \in B$. Namely, $I = B$. Therefore, $f^{-1}(0)$ is a maximal ideal.

Conversely, given a maximal ideal M we construct a homomorphism $f : B \to T$ into the two-elements Boolean algebra. For this, we construct a quotient Boolean algebra B/M and prove that B/M is two-element Boolean algebra. We do not prove this in detail. $\square$

Corollary 1.1.12. *A filter U of a Boolean algebra B is a ultrafilter if and only if for each $x \in B$, either $x \in U$ or $\neg x \in U$.*

1.2 Stone's Representation theorem

Let B be a Boolean algebra. Let $\mathcal{S}(B)$ be the set of all ultrafilters. We give a topology on $\mathcal{S}(B)$ be declaring the open sets are unions of sets of the form $U_a = \{\xi \in \mathcal{S}(B) \,|\, a \in \xi\}$.

Let us investigate properties of the topological space $\mathcal{S}(B)$.

Proposition 1.2.1. *Let B be a Boolean algebra and $a, b \in B$.*

1. $U_{a\wedge b} = U_a \cap U_b$ *and* $U_{a\vee b} = U_a \cup U_b$.
2. U_a *is closed. Consequently* U_a *are all clopen subsets.*
3. *For each point* $\xi \in \mathcal{S}(B)$, *the singleton* $\{\xi\}$ *is closed.*

Proof. (1) follows from the absorption property.
(2) is because $U_a = \mathcal{S}(B) \setminus U_{\neg a}$.
(3) since $\{\xi\} = \bigcap_{a\in\xi} U_a$ is closed. □

Definition 1.2.2. A topological space X is totally disconnected if no subspace other than singleton is connected.

Definition 1.2.3. A nonempty totally disconnected compact Hausdorff space is called a Stone space.

A Cantor set is an example of a Stone space.

Theorem 1.2.4. *For a Boolean algebra B, $\mathcal{S}(B)$ is a Stone space.*

Proof. Let C be any subset of $\mathcal{S}(B)$ containing at least two elements. Let ξ and η be distinct elements in C. Then we can choose $z \in \xi \setminus \eta$. One can observe that $\{U_z \cap C, U_{\neg z} \cap C\}$ is a separation of C. Therefore, C is not connected proving that $\mathcal{S}(B)$ is totally disconnected.

We prove the compactness of $\mathcal{S}(B)$. Suppose that we are given a set of open sets $\{O_\alpha\}_{\alpha\in\Lambda}$. We have to find a finite subcover, that is, finite subset Λ' of Λ such that $\{O_\alpha\}_{\alpha\in\lambda'}$ covers $\mathcal{S}(B)$. Since each O_α is a union of base open sets U_a, $a \in B$, it is enough to show the following statement

> For any subset B' of B such that $\bigcup_{a\in B'} U_a = \mathcal{S}(B)$, one can find a finite subset B'' of B' such that $\bigcup_{a\in B''} U_a = \mathcal{S}(B)$.

Since $\mathcal{S} \setminus U_a = U_{\neg a}$, the above statement is equivalent to

> For any subset B' of B such that $\bigcap_{a\in B'} U_a = \emptyset$, one can find a finite subset B'' of B' such that $\bigcap_{a\in B''} U_a = \emptyset$.

We will prove the last statement. Given a subset A of B, denote by $\langle A\rangle$ the smallest filter of B containing A. Then one can prove that

$$\langle A\rangle = \{(a_1 \vee b_1) \wedge (a_2 \vee b_2) \cdots \wedge (a_n \vee b_n) \,|\, a_i \in A, b_i \in B, n \in \mathbb{N}\}.$$

Moreover, we have $U_a \cap U_b = U_{a\wedge b} = U_{\langle\{a,b\}\rangle}$. Now since $\bigcap_{a\in B'} U_a = U_{\langle B'\rangle} = \emptyset$ we see that $\langle B'\rangle$ contains 0. Therefore, 0 can be written as $0 = (a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge \cdots \wedge (a_n \vee b_n)$ for $a_i \in B'$ and $b_i \in B$. Define a finite subset $B'' = \{a_1, a_2, \cdots, a_n\}$ of B'. Then we have that 0 is in $\langle B''\rangle$ so that $\bigcap_{b\in B''} U_b = \emptyset$. Hence $\mathcal{S}(B)$ is compact.

To show that $\mathcal{S}(B)$ is Hausdorff, choose two points ξ and η in $\mathcal{S}(B)$. Then there is $a \in \xi \setminus \eta$. Recall that U_a and $U_{\neg a}$ are disjoint open sets. Since $a \notin \eta$, $\neg a \in \eta$. Therefore, U_a and $U_{\neg a}$ separate ξ and η. So, $\mathcal{S}(B)$ is Hausdorff. □

Corollary 1.2.5. *All clopen subset of $\mathcal{S}(B)$ is of the form U_a for some $a \in B$.*

Proof. We proved that subsets U_a are all clopen in proposition 1.2.1. It U is an open set then U can be written as a union $\bigcup_{x\in B'} U_b$ for some subset B' of B. Suppose that U is closed as well. Then since $\mathcal{S}(B)$ is compact, U must be compact. Therefore, there is a finite subset $\{x_1, \cdots, x_n\}$ of B' such that $U = \bigcup_{i=1}^n U_{x_i}$. By proposition 1.2.1, we see that $U = \bigcup_{i=1}^n U_{x_i} = U_{x_1\vee\cdots\vee x_n}$. □

Let $f : B_1 \to B_2$ be a homomorphism of Boolean algebras B_1 and B_2. We have seen that a ultrafilter ξ of B_2 is the kernel of a homomorphism $g : B_2^* \to T$ into the two-element Boolean algebra T. Since $g \circ f^*$ is a homomorphism from B_1^* to T, its kernel, denoted by $\mathcal{S}(f)(\xi)$, is an ultrafilter of B_1. One can prove that $\mathcal{S}(f)(\xi) = f^{-1}(\xi)$. Hence given a homomorphism $f : B_1 \to B_2$ we get a map $\mathcal{S}(f) : \mathcal{S}(B_2) \to \mathcal{S}(B_1)$. Since $\mathcal{S}(B_1)$ and $\mathcal{S}(B_2)$ are given a topology, we expect that $\mathcal{S}(f)$ is continuous.

Theorem 1.2.6. *If $f : B_1 \to B_2$ is a homomorphism of Boolean algebras B_1 and B_2 then $\mathcal{S}(f) : \mathcal{S}(B_2) \to \mathcal{S}(B_1)$ is continuous.*

Proof. For a basic open set U_1 of $\mathcal{S}(B_1)$, $a \in B_1$, its inverse image $\mathcal{S}(f)^{-1}(U_a)$ is a basic open set $U_{f(a)}$ of $\mathcal{S}(B_2)$. □

Now we do a reverse engineering. Given a Stone space X, we could associate a Boolean algebra $\mathcal{B}(X)$. If $f : X_1 \to X_2$ is a continuous function between Stone spaces, there is a map $\mathcal{B}(f) : \mathcal{B}(X_2) \to \mathcal{B}(X_1)$ sending a clopen subset U of X_2 to its inverse image $f^{-1}(U)$. This map is a homomorphism of Boolean algebras.

As we mentioned before Boolean algebras of the type $\mathcal{B}(X)$ is universal in the following sense.

Theorem 1.2.7 (Stone's Representation Theorem)**.** *Let B be a Boolean algebra. Then B is isomorphic to a Boolean algebra $\mathcal{B}(\mathcal{S}(B))$.*

Proof. We have seen that U_x is clopen for each $x \in B$. Hence define a map $F : B \to \mathcal{B}(\mathcal{S}(B))$ by $F(x) = U_x$. This map is bijection by corollary 1.2.5. Proving that F is homomorphism is routine. □

Remark 1.2.8. The meaning of the Stone's representation theorem is that, there is essentially only one type of Boolean algebra, that is $\mathcal{B}(X)$. We have seen a similar theorem in linear algebra course: Every finite dimensional vector space over a field F is isomorphic to F^n. However, there is a big difference between these two theorems. When we construct an isomorphism between a given vector space and F^n, we have to choose some basis. Such a choice of a basis is very non-canonical so your isomorphism is not natural. On the other hand, Stone's representation theorem does not involve any choice in the construction of the isomorphism. In this sense, the Stone's isomorphism is a god-given nature of Boolean algebras and Stone's representation theorem reveals that intrinsic property.

2 Boolean Rings and its Spectrum

2.1 Boolean Rings

We consider the the functors $\mathcal{S}$ and $\mathcal{B}$ defined in the previous section. The categories we are interested are the category BA of Boolean algebras and the category Stone of Stone spaces. We proved that $\mathcal{S} : \mathsf{BA} \to \mathsf{Stone}$ is a contravariant functor. Moreover, there is another contravariant functor $\mathcal{B} : \mathsf{Stone} \to \mathsf{BA}$. What the Stone's representation theorem 1.2.7 tells us is that these two functors $\mathcal{S}$ and $\mathcal{B}$ are inverse of one another. In other words, the category BA and Stone are categorically equivalent. Since they are isomorphic one can say that

> Stone spaces can be classified up to homeomorphisms by the invariant $\mathcal{B}$

or

> $\mathcal{B}$ is a complete invariant of the category of Stone spaces.

Moreover, every categorical term in one category can be translated into the other category. Let us give one example.

Definition 2.1.1. The object 0 of a category C is called initial if, for each object X of C, there is unique morphism from 0 to X. Its dual notion is the final object 1 which is defined by the property that for each object X, there is unique morphism from X to 1.

Not every category has an initial/final object but if exists, it must be unique up to isomorphism. Note that the initial and final object may coincide.

Example 2.1.2. In the category BA, the initial object of BA is the two-element Boolean algebra T and the corresponding final (recall that the functor $\mathcal{S}$ is contravariant) object in Stone is a singleton $\{pt\}$.

Now, we introduce another category which is naturally arisen from BA.

Definition 2.1.3. A Boolean ring is a ring R with unity such that $x^2 = x \cdot x = x$ for all $x \in R$.

Proposition 2.1.4. *Let R be a Boolean ring. For all $x, y \in R$ we have*

1. $x + x = 2x = 0$.
2. $x \cdot y = y \cdot x$.

In other words, R is necessarily commutative and has characteristic 2.

Proof. We have $2x = (2x)^2 = (x + x)^2 = x^2 + 2x + x^2 = 4x$ which proves (1).
Observe that

$$x + y = (x + y)^2 = x^2 + x \cdot y + y \cdot x + y^2 = x + y + x \cdot y + y \cdot x.$$

Hence, $x \cdot y = -y \cdot x$. But by (1), we have $-y \cdot x = y \cdot x$. So, we get (2). $\square$

Proposition 2.1.5. *Let R be a Boolean ring. The following statements are equivalent.*

1. *The cardinality of R is* 2.
2. *R is a field.*
3. *R is an integral domain.*

4. $R = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

Proof. The only nontrivial implication is (3) $\Rightarrow$ (4). For any $x \in R$, we have $x^2 = x$. So, $x(x-1) = 0$. Since R is assumed to be an integral domain, we see that either $x = 0$ or $x = 1$. $\square$

Let BR be the category of Boolean rings. Define a functor $\mathcal{R}$ from the category of Boolean algebras BA into the category of Boolean rings BR as follows. For a given Boolean algebra $B = (B, \vee, \wedge, \neg)$, $\mathcal{R}(B)$ is, as a set, the same as B. The operations $+$, $\cdot$ are defined respectively by $a + b = (a \vee b) \wedge \neg(a \wedge b)$ and $a \cdot b = a \wedge b$. One can show that $\mathcal{R}(B) = (B, +, \cdot)$ is a Boolean ring. For instance, given any $x \in B$, we have

$$x^2 = x \cdot x = x \wedge x = x \wedge (x \vee 0) = x.$$

Finally, $\mathcal{R}$ sends a morphism $f : B_1 \to B_2$ to itself (as a set function), i.e. $\mathcal{R}(f)(x) = f(x) \in \mathcal{R}(B_2)$, $x \in \mathcal{R}(B_1)$.

Example 2.1.6. Consider a power set algebra 2^X. The $+$ operation of associated Boolean ring is nothing but the symmetric difference: $A + B = (A \cup B) \setminus (A \cap B)$.

Now we construct a functor $\mathcal{A}$ from BR to BA. As before, $\mathcal{A}$ sends a Boolean ring $R = (R, +, \cdot)$ to itself as a set. The operations $\vee$, $\wedge$, and $\neg$ are defined by $a \vee b = a + b + a \cdot b$, $a \wedge b = a \cdot b$, and $\neg a = 1 + a$ respectively. We have to check that $\mathcal{A}(R) = (R, \vee, \wedge, \neg)$ so defined satisfies all the axioms of a Boolean algebra. In particular, we have, by proposition 2.1.4, that

$$a \vee \neg a = a \vee (1 + a) = a + (1 + a) + a \cdot (a + 1) = 1 + 3a + a^2 = 1 + 4a = 1$$

and

$$a \wedge \neg a = a \wedge (1 + a) = a \cdot (1 + a) = a + a^2 = a + a = 0.$$

A proof of the following statement is straightforward.

Theorem 2.1.7. *The functors $\mathcal{R}$ and $\mathcal{A}$ are inverse of each other.*

In other words, two categories BA and BR are equivalent.

2.2 Spec of Boolean Rings

We have the notion of ideals, prime ideals and maximal ideals in a Boolean algebra. They are in one to one correspondence with ideals, prime ideals and maximal ideals in the associated Boolean ring.

Proposition 2.2.1. *A subset I of a Boolean algebra B is an ideal (prime, maximal) if and only if I is an ideal (prime, maximal respectively) in $\mathcal{R}(B)$.*

Proof. Proving that I is an ideal (prime ideal) in B if and only if I is an ideal (prime ideal respectively) in $\mathcal{R}(B)$ is straightforward.

Let M be a maximal ideal in B. Then we have a morphism $f : B \to T$ into a two-element Boolean algebra T such that $M = f^{-1}(0)$. Then, $M = \ker \mathcal{R}(f)$. Observe that $\mathcal{R}(T)$ is a two-element Boolean ring, which is a field $\mathbb{F}_2$. It is general fact that the kernel of ring homomorphism into a field is a maximal ideal. Therefore, M is maximal ideal in $\mathcal{R}(B)$.

For the converse, it suffices to show that if a Boolean ring R is a field, then $R = \mathbb{F}_2$. But it is the case by proposition 2.1.5. $\square$

Proposition 2.2.2. *Let I be an ideal of a Boolean ring. Then I is prime if and only if I is maximal.*

Proof. I is prime (maximal) if and only if I is a kernel of a ring homomorphism $f : M \to D$ into an integral domain (field, respectively) D. Due to proposition 2.1.5, D is a field if and only if D is an integral domain. So, I is prime if and only if I is maximal. □

We have seen that BA and Stone are equivalent categories. Since BA and BR are equivalent, BR and Stone must be equivalent. Can we find an explicit functor that gives the equivalence between BR and Stone?

Definition 2.2.3. Let R be a commutative ring with unity. Define a topological space $\operatorname{Spec} R$, the (prime) spectrum of R, as follows:

- As a set, $\operatorname{Spec} R$ is the set of prime ideals.
- We give a topology on $\operatorname{Spec} R$ by declaring that the closed sets are subsets of the type $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R \,|\, I \subset \mathfrak{p}\}$ where I is an ideal of R. The topology defined in this way is called the Zariski topology.

Remark 2.2.4. A spectrum of ring is a central object in algebraic geometry.

In fact the functor Spec from BR to Stone can be written as $\operatorname{Spec} = \mathcal{S} \circ \mathcal{A}$.

$$\begin{tikzcd} \mathsf{BA} \arrow[rr, "\mathcal{S}"] & & \mathsf{Stone} \\ & \mathsf{BR} \arrow[ul, "\mathcal{A}"] \arrow[ur, "\mathrm{Spec}"'] & \end{tikzcd}$$