CONVERGENCE OF MULTIGRID METHODS FOR NONCONFORMING FINITE ELEMENTS WITHOUT REGULARITY ASSUMPTIONS

Zhangxin Chen and Do Y. Kwak

Abstract. In this paper we consider multigrid methods for discretizations of second-order elliptic problems using nonconforming finite elements. These multigrid methods are based on the "Galerkin approach" where the quadratic forms over coarse grids are constructed from the quadratic form on the finest grid and the iterated coarse-to-fine grid operators. In terms of this approach we show that the nonconforming multigrid operator can be written in a product form, which, together with an appropriate approximation hypothesis, implies convergence estimates for both the $\mathcal{V}$-cycle and $\mathcal{W}$-cycle multigrid methods for partial differential problems without regularity assumptions. These estimates can thus be applied to problems with rough coefficients and local grid refinement. Numerical results are also presented to illustrate the present theory.

1. Introduction

Recently, there has been some attempt to prove convergence estimates for nonconforming multigrid methods for partial differential problems with less than full elliptic regularity [16, 42]. Since nonconforming finite element spaces may not be subspaces of fractional order Sobolev spaces, the idea of using conforming "relatives" in the analysis of the nonconforming multigrid methods has been exploited. Namely, for a given nonconforming finite element space, the idea is to find a conforming finite element space which contains it as a subspace, and then to obtain multigrid convergence results for the nonconforming element by applying its connection with the conforming element. In terms of this approach, only the $\mathcal{W}$-cycle nonconforming multigrid method has been shown to converge under the assumption that the number of smoothing iterations on all levels is sufficiently large, which follows the methodology developed for the conforming multigrid method in [4].

In this paper we shall prove new convergence results for both $\mathcal{V}$-cycle and $\mathcal{W}$-cycle nonconforming multigrid methods under regularity-free assumptions. The convergence analysis adopted here generalizes that in [11] for multigrid methods for conforming finite element methods, but does not use any conforming relatives.

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In [11] it was required that the approximate multilevel subspaces be nested and the discretization operators over these spaces be inherited. This is true for the conforming finite elements. However, it is known that the usual multigrid method for the nonconforming finite elements does not satisfy these two requirements. Here we shall follow the approach introduced in [20] where multigrid methods for nonconforming finite element methods for partial differential problems with full elliptic regularity have been developed and analyzed. In this new approach the quadratic forms (i.e., the bilinear forms) over coarse grids are constructed from the quadratic form on the finest grid and the iterated coarse-to-fine grid operators (i.e., the so-called "Galerkin approach" is exploited in the context of finite difference methods [27]), while the usual multigrid method uses discrete equations on all levels which are defined by the same discretization. Exploiting this new approach, we shall prove that the nonconforming multigrid operator can be written in a product form, which, together with an appropriate approximation hypothesis, implies convergence estimates without regularity assumptions. Hence the theory here can be applied to problems with just bounded measurable coefficients, to nonconforming finite element methods with local grid refinement, and even to problems with large jump coefficients. As example, the theory here shows that the convergence estimates for the $\mathcal{V}$-cycle method with any number of smoothing iterations is no worse than $1 - /(CK)$, where K is the number of levels in the multigrid method, for second-order elliptic problems with bounded measurable coefficients, which may have large jumps. In contrast, there is no such theory for the usual nonconforming multigrid method.

The remainder of the paper is presented as follows. In the next section we derive fundamental recurrence relations for the multigrid operator and give a general convergence theory. A key ingredient for the development of the fundamental recurrence relations is the introduction of both iterated coarse-to-fine and fine-to-coarse grid operators. Then we analyze the P_1 and rotated Q_1-nonconforming finite elements for second-order elliptic problems without regularity assumptions as examples of the application of the present theory in §3 and §4, respectively. In §5 we analyze the application to problems with large jump coefficients. Finally, numerical results for illustrating the theory are given in §6. A comparison between the present and usual methods is presented there.

We mention that, due to the equivalence between nonconforming and mixed finite element methods for the problems under consideration [2, 3, 17, 19], all the results derived here carry over directly to the mixed methods. We also mention that there has been an extensive literature on the convergence proof of the $\mathcal{W}$-cycle method with a sufficiently large number of smoothing steps for partial differential problems with full elliptic regularity; see [2, 6, 8, 13, 16, 18, 21, 23, 29, 32, 42, 43] for second-order differential problems, [14, 26, 28, 34, 35, 37, 38, 41, 44] for biharmonic and plate bending problems, and [15, 39] for Stokes problems, for example. For the convergence theory for conforming multigrid methods, see the references in the books [7, 27, 30].

2. Multigrid Methods

In this section we describe multigrid methods, derive fundamental recurrence relations, and present a convergence theory.

Let us assume that we are given a sequence of nonconforming finite element spaces

$$V_0,\, V_1, \dots, V_K,$$

along with the nonsingular coarse-to-fine grid operators $I_k : V_{k-1} \to V_k$ for $k = 1, \dots, K$. In addition, assume that we are given symmetric positive definite quadratic forms $a_K(\cdot,\cdot)$ on $V_K \times V_K$ over the finest grid and $(\cdot,\cdot)_k$ on $V_k \times V_k$ for $k = 0, \dots, K$.

We define the iterates of I_k by

$$H_k^K = I_K \cdots I_{k+1} : V_k \to V_K,$$

and the symmetric positive definite quadratic forms $b_k(\cdot,\cdot)$ on $V_k \times V_k$ by

$$b_k(v,w) = a_K(H_k^K v, H_k^K w), \qquad \forall v,\, w \in V_k,\; k = 0, \dots, K, \tag{2.1}$$

where $H_K^K = I$ is the identity operator on V_K. The norms corresponding to $(\cdot,\cdot)_k$ and $b_k(\cdot,\cdot)$ will be denoted by $||\cdot||_k$ and $||\cdot||_{1,k}$, respectively. Examples of spaces, operators, and quadratic forms will be given in §3–§5 later.

Given $f \in V_K$, we are interested in the solution of the problem: Find $u_K \in V_K$ such that

$$a_K(u_K, v) = (f, v)_K, \qquad \forall v \in V_K. \tag{2.2}$$

To introduce multigrid methods for solving (2.2), we define the discretization operator $A_k : V_k \to V_k$ on level k given by

$$(A_k v, w)_k = b_k(v,w), \qquad \forall\, w \in V_k,\; k = 0, \dots, K. \tag{2.3}$$

Note that the operator A_k is clearly symmetric (in both the $b_k(\cdot,\cdot)$ and $(\cdot,\cdot)_k$ inner products) and positive definite. Also, we define the operators $P_{k-1} : V_k \to V_{k-1}$ and $P_{k-1}^0 : V_k \to V_{k-1}$ by

$$b_{k-1}(P_{k-1}v, w) = b_k(v, I_k w), \qquad \forall\, w \in V_{k-1},\; k = 1, \dots, K, \tag{2.4}$$

and

$$\left(P_{k-1}^0 v, w\right)_{k-1} = (v, I_k w)_k, \qquad \forall\, w \in V_{k-1},\; k = 1, \dots, K.$$

It is obvious that $I_k P_{k-1}$ is a symmetric operator wtih respect to the b_k form. Finally, let $R_k : V_k \to V_k$ for $k = 1, \dots, K$ be linear operators. On V_0, let $R_0 = A_0^{-1}$; i.e., we solve exactly on the coarsest level. We assume that R_k is a symmetric and positive semi-definite operator in the $(\cdot,\cdot)_k$ form and define $J_k = I - R_k A_k$ on V_k. Note that J_k is symmetric in the $b_k(\cdot,\cdot)$ form. For the analysis below we assume that J_k is nonnegative in the sense that $b_k(J_k v, v) \geq 0$ for all $v \in V_k$. These hypotheses yield that the spectrum of J_k is in $[0, 1]$.

Following [12], we now define the following symmetric and nonsymmetric multigrid methods:

Multigrid Method 2.1. Let $1 \leq k \leq K$ and p be a positive integer. Set $B_0^s = A_0^{-1}$. Assume that B_{k-1}^s has been defined and define $B_k^s g$ for $g \in V_k$ by

(1) Set $x^0 = 0$ and $q^0 = 0$.
(2) Define x^l for $l = 1, \ldots, m(k)$ by

$$x^l = x^{l-1} + R_k(g - A_k x^{l-1}).$$

(3) Define $y^{m(k)} = x^{m(k)} + I_k q^p$, where q^i for $i = 1, \ldots, p$ is given by

$$q^i = q^{i-1} + B_{k-1}^s \left[P_{k-1}^0 \left(g - A_k x^{m(k)} \right) - A_{k-1} q^{i-1} \right].$$

(4) Define y^l for $l = m(k)+1, \ldots, 2m(k)$ by

$$y^l = y^{l-1} + R_k \left(g - A_k y^{l-1} \right).$$

(5) Set $B_k^s g = y^{2m(k)}$.

Multigrid Method 2.2. Let $1 \leq k \leq K$ and p be a positive integer. Set $B_0^n = A_0^{-1}$. Assume that B_{k-1}^n has been defined and define $B_k^n g$ for $g \in V_k$ by

(1) Set $x^0 = 0$ and $q^0 = 0$.
(2) Define x^l for $l = 1, \ldots, m(k)$ by

$$x^l = x^{l-1} + R_k(g - A_k x^{l-1}).$$

(3) Define $B_k^n g = x^{m(k)} + I_k q^p$, where q^i for $i = 1, \ldots, p$ is given by

$$q^i = q^{i-1} + B_{k-1}^n \left[P_{k-1}^0 \left(g - A_k x^{m(k)} \right) - A_{k-1} q^{i-1} \right].$$

The definition of the nonsymmetric multigrid method 2.2 differs from the symmetric multigrid method 2.1 in that the smoothing step after the coarse correction is excluded. In these two multigrid methods, $m(k)$ gives the number of pre-smoothing (respectively, post-smoothing) iterations and can vary as a function of k. $p = 1$ and $p = 2$ correspond to $\mathcal{V}$ and $\mathcal{W}$-cycle methods, respectively.

We now write $I - B_k^n A_k$ and $I - B_k^s A_k$ in a product form. For these, we need the following preliminary results. For simplicity, we focus on the $\mathcal{V}$-cycle method since the analysis below does not give an improved rate of convergence for $p > 1$ under regularity-free assumptions. The analysis can be extended to the $\mathcal{W}$-cycle method as in the conforming case [11], though. In contrast, the analysis in [20] with the regularity and approximation assumption does improve the convergence rate.

Lemma 2.1. *It holds that*

$$P^0_{k-1}A_k = A_{k-1}P_{k-1}, \tag{2.5}$$

$$P_{k-1}I_k = I \quad \text{on } V_{k-1}. \tag{2.6}$$

This lemma can be easily seen from the definitions of these operators.

Let $g = A_k x$. It is clear from method 2.2 that

$$q^1 = B^n_{k-1}P^0_{k-1}A_k(x - x^{m(k)}),$$

so that, by (2.5), we see that

$$q^1 = B^n_{k-1}A_{k-1}P_{k-1}(x - x^{m(k)}). \tag{2.7}$$

Also, we have

$$J^{m(k)}_k x = x - x^{m(k)},$$

which combines with (2.7) to yield that

$$I - B^n_k A_k = (I - I_k B^n_{k-1}A_{k-1}P_{k-1})J^{m(k)}_k. \tag{2.8}$$

A similar argument for method 2.1 implies that

$$I - B^s_k A_k = J^{m(k)}_k (I - I_k B^s_{k-1}A_{k-1}P_{k-1})J^{m(k)}_k. \tag{2.9}$$

By Lemma 2.1 and an induction argument we see that

$$I - B^s_k A_k = (I - B^n_k A_k)^*(I - B^n_k A_k), \tag{2.10}$$

where $(I - B^n_k A_k)^*$ denotes the adjoint with respect to the inner product $b_k(\cdot,\cdot)$.

To see that $I - B^n_k A_k$ and thus $I - B^s_k A_k$ can be written as a product of operators, we introduce the following operator $\Pi^k_K : V_K \to V_k$, which plays a key role in the product formulation:

$$b_k(\Pi^k_K v, w) = b_K(v, H^K_k w), \quad v \in V_K, w \in V_k,$$

for $k = 0, \dots, K-1$, and set $\Pi^K_K = I$; i.e., Π^k_K is the adjoint operator of H^K_k with respect to $b_k(\cdot,\cdot)$.

Lemma 2.2. *We have*

$$\Pi^{k-1}_K = P_{k-1}\Pi^k_K. \tag{2.11}$$

This relation follows from the definitions of the operators Π^k_K and P_k.

It follows from (2.11) that

$$\Pi^k_K = P_k P_{k+1} \dots P_{K-1};$$

consequently, by (2.6), we have

$$\Pi^k_K H^K_k = I \quad \text{on } V_k. \tag{2.12}$$

We can also easily see that

$$\Pi^{k-1}_K H^K_k = P_{k-1},$$

so that

$$\Pi^{k-1}_K (I - H^K_k \Pi^k_K) = 0. \tag{2.13}$$

Lemma 2.3. *$H_k^K J_k \Pi_K^k$ is symmetric with respect to the $b_K(\cdot,\cdot)$ form.*

This can be easily checked by the definition of Π_K^k since $J_k = I - R_k A_k$ is symmetric with respect to the $b_k(\cdot,\cdot)$ form.

We are now in a position to write $I - B_k^n A_k$ as a product of operators. On V_K, by (2.8), (2.11), and (2.13) we see that

$$\begin{aligned} I - H_k^K B_k^n A_k \Pi_K^k =& I - H_k^K \Pi_K^k + H_k^K (I - B_k^n A_k)\Pi_K^k \\ =& I - H_k^K \Pi_K^k + H_k^K (I - I_k B_{k-1}^n A_{k-1} P_{k-1}) J_k^{m(k)} \Pi_K^k \\ =& (I - H_k^K I_k B_{k-1}^n A_{k-1} \Pi_K^{k-1})(I - H_k^K \Pi_K^k + H_k^K J_k^{m(k)} \Pi_K^k). \end{aligned}$$

Now, let $S^k = H_k^K (I - J_k^{m(k)})\Pi_K^k$. Note that S^k maps V_K into V_K, but exploits the coarse space V_k. That is why we here use the superscript k to differ from the subscript used in other operators, which have a range in V_k. Then we have

$$I - H_k^K B_k^n A_k \Pi_K^k = (I - H_{k-1}^K B_{k-1}^n A_{k-1} \Pi_K^{k-1})(I - S^k).$$

Therefore, from a straightforward induction we obtain

$$I - B_K^n A_K = (I - S^0)\dots(I - S^K), \tag{2.14}$$

and

$$(I - B_K^n A_K)^* = (I - S^K)\dots(I - S^0), \tag{2.15}$$

by Lemma 2.3. We note that each factor in (2.14) and (2.15) is defined on all of V_K. To estimate the bound for the convergence of methods 2.1 and 2.2, it suffices to consider the operator norm of the corresponding multigrid operators. Namely, we only need to prove inequalities of the form:

$$a_K((I - B_K^s A_K)v, v) \le \delta_K a_K(v,v), \quad \forall v \in V_K, \tag{2.16}$$

or

$$a_K((I - B_K^n A_K)v, (I - B_K^n A_K)v) \le \delta_K a_K(v,v), \quad \forall v \in V_K. \tag{2.17}$$

¿From (2.10), it is sufficient to prove (2.17).

Let λ_k be the largest eigenvalue of A_k. As in the conforming case, our main assumption is the existence of linear operators $Q_K^k : V_K \to V_k, k = 0, \dots, K$, with $Q_K^K = I$, such that there exist constants C_1 and C_2 independent of k satisfying

$$\|(Q_K^k - I_k Q_K^{k-1})v\|_k^2 \le C_1 \lambda_k^{-1} b_K(v,v), \quad k = 1, \cdots, K, \tag{2.18}$$

$$b_k(Q_K^k v, Q_K^k v) \le C_2 b_K(v,v), \qquad k = 0, \cdots, K-1. \tag{2.19}$$

We also assume a standard condition for the symmetric smoothing oprator R_k; i.e., there exists a constant C_R independent of k such that

$$\frac{\|v\|_k^2}{\lambda_k} \le C_R(R_k v, v)_k, \quad v \in V_k. \tag{2.20}$$

The simplest example for R_k to satisfy (2.20) is $R_k = \lambda_k^{-1} I$ with $C_R = 1$.

We are now ready to estimate δ_K in (2.16) and (2.17). The following theorem follows the argument of the corresponding result for conforming finite elements [11]. The difference between the conforming elements treated in [11] and nonconforming elements considered here is that the multilevel spaces are nested in the former case but not in the latter case. Thus we need to bring each function defined on a coarse level to the finest level, for which we need the operators H_k^K and Q_K^k.

Theorem 2.4. *Assume that* (2.18)–(2.20) *hold, and define* B_K^s *and* B_K^n *by methods* 2.1 *and* 2.2*, respectively. Then* (2.16) *and* (2.17) *are satisfied with*

$$\delta_K = 1 - \frac{1}{CK}, \tag{2.21}$$

where $C = (1 + C_2^{1/2} + (C_R C_1)^{1/2})^2$.

Proof. Let $E^k = (I - H_k^K B_k^n A_k \Pi_K^k)^*$; the same remark for the notation on S^k applies to E^k as well. By (2.15), we have

$$E^k = (I - S^k) E^{k-1}, \quad k = 1, \dots, K. \tag{2.22}$$

Equality (2.22) also holds for $k = 0$ if we let $E^{-1} = I$. To show (2.17) with δ_K given by (2.21), it suffices to prove that

$$b_K(v, v) \le CK\{b_K(v, v) - b_K(E^K v, E^K v)\}, \quad \forall v \in V_K. \tag{2.23}$$

By (2.22), we observe that

$$\begin{aligned} b_K(E^{k-1}v, E^{k-1}v) =& b_K(E^k v, E^k v) + 2 b_K(S^k(I - S^k)E^{k-1}v, E^{k-1}v) \\ &+ b_K(S^k E^{k-1}v, S^k E^{k-1}v) \\ =& b_K(E^k v, E^k v) + b_K(S^k(I - S^k)E^{k-1}v, E^{k-1}v) \\ &+ b_K(S^k E^{k-1}v, E^{k-1}v). \end{aligned}$$

Since the operator $S^k(I - S^k)$ is symmetric and positive semi-definite with respect to the $b_K(\cdot, \cdot)$ form by Lemma 2.3 and the fact that the spectrum of J_k is in $[0, 1]$, we have

$$b_K(S^k E^{k-1}v, E^{k-1}v) \le b_K(E^{k-1}v, E^{k-1}v) - b_K(E^k v, E^k v).$$

Summing on k leads to

$$\sum_{k=0}^{K} b_K(S^k E^{k-1}v, E^{k-1}v) \leq b_K(v,v) - b_K(E^K v, E^K v),$$

which, together with (2.23), implies that it suffices to show that

$$b_K(v,v) \leq CK \sum_{k=0}^{K} b_K(S^k E^{k-1}v, E^{k-1}v). \tag{2.24}$$

Toward that end, let $\bar{Q}^k = H_k^K Q_K^k$, $k = 0, \ldots, K$, with $\bar{Q}^K = I$. Then we write

$$v = \sum_{k=1}^{K} (\bar{Q}^k - \bar{Q}^{k-1})v + \bar{Q}^0 v.$$

Hence we see that

$$\begin{aligned} b_K(v,v) &= \sum_{k=1}^{K} b_K(v, (\bar{Q}^k - \bar{Q}^{k-1})v) + b_K(v, \bar{Q}^0 v) \\ &= \sum_{k=1}^{K} b_K(E^{k-1}v, (\bar{Q}^k - \bar{Q}^{k-1})v) + b_K(v, \bar{Q}^0 v) \\ &\quad + \sum_{k=1}^{K} b_K((I - E^{k-1})v, (\bar{Q}^k - \bar{Q}^{k-1})v). \end{aligned} \tag{2.25}$$

We first bound the first term on the right-hand of (2.25). By the definitions of $\bar{Q}^k$ and Π_K^k, we have

$$\begin{aligned} \sum_{k=1}^{K} b_K(E^{k-1}v, (\bar{Q}^k - \bar{Q}^{k-1})v) &= \sum_{k=1}^{K} b_k(\Pi_K^k E^{k-1}v, (Q_K^k - I_k Q_K^{k-1})v) \\ &= \sum_{k=1}^{K} (A_k \Pi_K^k E^{k-1}v, (Q_K^k - I_k Q_K^{k-1})v)_k \\ &\leq \sum_{k=1}^{K} \|A_k \Pi_K^k E^{k-1}v\|_k \|(Q_K^k - I_k Q_K^{k-1})v\|_k, \end{aligned}$$

so that, applying the Cauchy-Schwarz inequality, (2.18), and (2.20), we see that

$$\begin{aligned}
&\sum_{k=1}^{K} b_K(E^{k-1}v, (\bar{Q}^k - \bar{Q}^{k-1})v) \\
&\quad \le (C_1)^{1/2} b_K^{1/2}(v,v) \sum_{k=1}^{K} \lambda_k^{-1/2} \|A_k \Pi_K^k E^{k-1} v\|_k \\
&\quad \le (C_1 C_R K)^{1/2} b_K^{1/2}(v,v) \left(\sum_{k=1}^{K} (R_k A_k \Pi_K^k E^{k-1}v, A_k \Pi_K^k E^{k-1} v)_k \right)^{1/2} \\
&\quad = (C_1 C_R K)^{1/2} b_K^{1/2}(v,v) \left(\sum_{k=1}^{K} b_k(R_k A_k \Pi_K^k E^{k-1}v, \Pi_K^k E^{k-1} v)_k \right)^{1/2}.
\end{aligned}$$

Since the spectrum of J_k is in $[0,1]$, we see that

$$\begin{aligned}
b_k(R_k A_k \Pi_K^k E^{k-1} v, \Pi_K^k E^{k-1} v)_k &= b_k((I - J_k)\Pi_K^k E^{k-1}v, \Pi_K^k E^{k-1} v) \\
&\le b_k((I - J_k^{m(k)})\Pi_K^k E^{k-1}v, \Pi_K^k E^{k-1} v) \\
&= b_K(H_k^K (I - J_k^{m(k)})\Pi_K^k E^{k-1}v, E^{k-1} v) \\
&= b_K(S^k E^{k-1} v, E^{k-1} v).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\sum_{k=1}^{K} b_K(E^{k-1}v, (\bar{Q}^k - \bar{Q}^{k-1})v) \\
&\quad \le (C_1 C_R K)^{1/2} b_K^{1/2}(v,v) \left(\sum_{k=1}^{K} b_K(S^k E^{k-1} v, E^{k-1} v) \right)^{1/2}.
\end{aligned} \tag{2.26}$$

We next bound the remaining terms in (2.25). We use summation by parts to have

$$\begin{aligned}
&\sum_{k=1}^{K} b_K((I - E^{k-1})v, (\bar{Q}^k - \bar{Q}^{k-1})v) + b_K(v, \bar{Q}^0 v) \\
&\quad = \sum_{k=1}^{K-1} b_K((E^k - E^{k-1})v, \bar{Q}^k v) + b_K((I - E^{K-1})v, v).
\end{aligned} \tag{2.27}$$

It follows from (2.22) that

$$I - E^{K-1} = \sum_{k=0}^{K-1} S^k E^{k-1},$$

which, together with (2.27), gives

$$\begin{aligned}\sum_{k=1}^{K} b_K((I-E^{k-1})v,(\bar{Q}^k-\bar{Q}^{k-1})v)+b_K(v,\bar{Q}^0v)\\ =\sum_{k=1}^{K-1} b_K(S^kE^{k-1}v,(I-\bar{Q}^k)v)+b_K(S^0v,v).\end{aligned}\tag{2.28}$$

Since the spectrum of S^k is in $[0,1]$, by (2.19) we see that

$$\begin{aligned}\sum_{k=1}^{K} b_K((I-E^{k-1})v,(\bar{Q}^k-\bar{Q}^{k-1})v)+b_K(v,\bar{Q}^0v)\\ \le K^{1/2}(1+C_2^{1/2})\left(\sum_{k=0}^{K-1} b_K(S^kE^{k-1}v,E^{k-1})v\right)^{1/2} b_K^{1/2}(v,v).\end{aligned}\tag{2.29}$$

Now, it follows from (2.25), (2.26), and (2.29) that

$$b_K(v,v)\le K(1+C_2^{1/2}+(C_RC_1)^{1/2})^2\left(\sum_{k=0}^{K} b_K(S^kE^{k-1}v,E^{k-1}v)\right),$$

which implies the desired result with $C=(1+C_2^{1/2}+(C_RC_1)^{1/2})^2$. □

Note that Theorem 2.4 gives a bound for the convergence factor δ_K in (2.16) and (2.17). In the nonconforming finite element applications later, K is proportional to a logarithm of the mesh size. This theorem implies that the multigrid methods converge quickly with the regularity-free assumptions (2.18) and (2.19). However, we mention that the result does not give any improvement in the convergence rate when the number of smoothing steps $m(k)$ increases as in the conforming case [11]. In contrast, the convergence results in [20] do show improved convergence with increased $m(k)$ as long as the regularity and approximation hypothesis is satisfied. Also, the nonnegativity assumption on J_k can be relaxed to the assumption that the spectrum of J_k is in $[1-\omega,1]$ for some ω with $1\le\omega<2$, as in the conforming case [11]. In this case the value of C in Theorem 2.4 is replaced by

$$C=\{1+C_2^{1/2}+(C_RC_1\omega/(2-\omega))^{1/2}\}^2/(2-\omega).$$

Finally, in the case of nonsymmetric R_k, the result in Theorem 2.4 can be shown with minor modifications on the above proof provided that the following hypotheses are satisfied [9]:

(1) There exists a constant C_R independent of k such that

$$\frac{\|v\|_k^2}{\lambda_k}\le C_R(\bar{R}_kv,v)_k,\quad v\in V_k,$$

where $\bar{R}_k$ is either $(I-J_k^*J_k)A_k^{-1}$ or $(I-J_kJ_k^*)A_k^{-1}$, and

(2) There exists a constant $\theta<2$ independent of k such that

$$b_k(R_kA_kv,R_kA_kv)\le\theta b_k(R_kA_kv,v)_k,\quad v\in V_k.$$

Gauss-Seidel smoothing satisfies these hypotheses for the applications considered in the next setions, for example. In the nonsymmetric case, other smoothing iterations are also possible [40].

3. Application to the P_1-Nonconforming Element

In this and next sections, as examples of the application of the theory in the previous section, we apply the multigrid methods 2.1 and 2.2 to the P_1 and rotated Q_1-nonconforming finite elements for second-order elliptic problems without regularity assumptions. The analysis below carries over directly to the P_1 and rotated Q_1-nonconforming divergence-free elements for the stationary Stokes problem [20]. Also, the same results can be obtained for the Zienkiewicz [5] and Adini [1] nonconforming finite elements for fourth-order problems [20]. However, the Morley element [31] for the biharmonic problem cannot be handled using the present approach since we cannot control the energy norm of the iterates of the usual coarse-to-fine intergrid transfer operators for this element. In fact, as numerical tests indicate [33], the energy norm of the iterates deteriorates exponentially with the number of grid levels for the Morley element. In §5 we shall consider the application to problems which have large jumps in coefficients.

We consider the model problem

$$-\nabla \cdot (a\nabla u) = f \quad \text{in } \Omega, \tag{3.1a}$$

$$u = 0 \quad \text{on } \Gamma, \tag{3.1b}$$

using the P_1-nonconforming finite element. In this and next sections, $\Omega \subset \mathbb{R}^2$ is a simply connected bounded polygonal domain with the boundary Γ, $f \in L^2(\Omega)$, and the coefficient $a \in L^\infty(\Omega)$ satisfies

$$0 < a_1 \le a(x,y) \le a_2, \quad (x,y) \in \Omega, \tag{3.2}$$

with fixed constants a_1 and a_2. Since $a(x)$ only belongs to $L^\infty(\Omega)$, it is not known whether the solution u belongs to $H^{1+\alpha}(\Omega)$, $\alpha > 0$. Thus the theory presented in [20] (or in earlier papers [16, 42] on nonconforming multigrid methods for problems with less than full elliptic regularity) does not yield a proper convergence result of multigrid methods in this case. In the present condition on the coefficient a, (3.1) is understood in the weak sense.

Problem (3.1) is recast in weak form as follows. The bilinear form $a(\cdot,\cdot)$ is defined by

$$a(v,w) = (a\nabla v, \nabla w), \qquad v, w \in H^1(\Omega),$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ or $(L^2(\Omega))^2$ inner product, as appropriate. Then the weak form of (3.1) for the solution $u \in H_0^1(\Omega)$ is

$$a(u,v) = (f,v), \qquad \forall\ v \in H_0^1(\Omega). \tag{3.3}$$

Let $0 < h_0 < 1$ and $\mathcal{E}_{h_0} = \mathcal{E}_0$ be given, where $\mathcal{E}_{h_0}$ is a triangulation of Ω into triangles with diameter h_0. For each integer $1 \le k \le K$, let $h_k = 2^{-k}h_0$ and

$\mathcal{E}_{h_k} = \mathcal{E}_k$ be constructed by connecting the midpoints of the edges of the triangle in $\mathcal{E}_{k-1}$, and let $\mathcal{E}_h = \mathcal{E}_K$ be the finest grid. In this and next sections, we replace subscript h_k simply by subscript k.

For each k, we define the P_1-nonconforming finite element space [25]

$$\begin{aligned} V_k = \{v \in L^2(\Omega) : v|_E \in P_1(E) \text{ for all } E \in \mathcal{E}_k; v \text{ is continuous at}\\ \text{the midpoints of interior edges and vanishes}\\ \text{at the midpoints of edges on } \Gamma\}. \end{aligned}$$

Associated with each V_k, we introduce a bilinear form on $V_k \oplus H_0^1(\Omega)$ by

$$a_k(v, w) = \sum_{E \in \mathcal{E}_k} (a\nabla v, \nabla w)_E, \qquad v, w \in V_k \oplus H_0^1(\Omega),$$

where $(\cdot, \cdot)_E$ is the $(L^2(E))^2$ inner product.

Let $(\cdot, \cdot)_k$ be a quadratic form on V_k defined by

$$(v, w)_k = h_k^2 \sum_q v(q)w(q), \qquad v, w \in V_k,$$

where the summation is taken over all the midpoints in $\mathcal{E}_k$. It is not difficult to see that the norm induced by $(\cdot, \cdot)_k$ is equivalent to the standard $L^2(\Omega)$ norm $||\cdot||$; i.e.,

$$C_3\|v\| \le \|v\|_k \le C_4\|v\|, \qquad v \in V_k, \tag{3.4}$$

with C_3 and C_4 independent of k.

We consider two sets of intergrid transfer operators $I_k : V_{k-1} \to V_k$ and $T_{k-1} : V_k \to V_{k-1}$ as follows. Following [6, 13, 19], if $v \in V_{k-1}$ and q is the midpoint of an edge of a triangle in $\mathcal{E}_k$, then $I_k v \in V_k$ is defined by

$$(I_k v)(q) = \begin{cases} 0 & \text{if } q \in \Gamma, \\ v(q) & \text{if } q \notin \partial E \text{ for any } E \in \mathcal{E}_{k-1}, \\ \frac{1}{2}\{v|_{E_1}(q) + v|_{E_2}(q)\} & \text{if } q \in \partial E_1 \cap \partial E_2 \text{ for some } E_1, E_2 \in \mathcal{E}_{k-1}. \end{cases}$$

If $v \in V_k$ and q is the midpoint of an edge e of a triangle in $\mathcal{E}_{k-1}$, following [20, 33] $T_{k-1}v \in V_{k-1}$ is given by

$$(T_{k-1}v)(q) = \frac{1}{2}(v(q_1) + v(q_2)),$$

where q_1 and q_2 are the respective midpoints of the edges e_1 and e_2 in $\mathcal{E}_k$, which form the edge e in $\mathcal{E}_{k-1}$. Note that the definition of T_{k-1} automatically preserves the zero nodal values on boundary edges. Also, it can be seen that

$$T_{k-1} I_k v = v, \qquad v \in V_{k-1},\ k = 1, \dots, K. \tag{3.5}$$

That is, $T_{k-1}I_k$ is the identity operator on V_{k-1}.

As in [20], we introduce the iterated intergrid transfer operators

$$H_k^K = I_K \cdots I_{k+1} : V_k \to V_K, \quad k = 0, \dots, K-1,$$
$$Q_K^k = T_k \cdots T_{K-1} : V_K \to V_k, \quad k = 0, \dots, K.$$

By (3.5), we see that

$$Q_K^k H_k^K v = v, \qquad v \in V_k.$$

Denote the norm induced by the a_k form by $||\cdot||_{\mathcal{E},k}$. It has been shown [20] that the norms $\|\cdot\|_{\mathcal{E},k}$ and $\|\cdot\|_{1,k}$ are equivalent:

$$C_5\|v\|_{\mathcal{E},k} \le \|v\|_{1,k} \le C_6\|v\|_{\mathcal{E},k}, \quad \forall v \in V_k,\ k = 0, \dots, K, \tag{3.6}$$

with C_5 and C_6 independent of k.

To verify (2.18) and (2.19), we need the next result, which was proven in [20]. Below C indicates a generic positive constant independent of k.

Lemma 3.1. *There exist constants C independent of k such that*

$$\|Q_K^k v\|_{\mathcal{E},k} \le C\|v\|_{\mathcal{E},K}, \qquad \forall v \in V_K, \tag{3.7}$$
$$\|T_{k-1}v\|_{\mathcal{E},k-1} \le C\|v\|_{\mathcal{E},k}, \quad \forall v \in V_k. \tag{3.8}$$

Now assumption (2.19) follows from (3.6) and (3.7). To prove (2.18), observe that

$$Q_K^k - I_k Q_K^{k-1} = (I - I_k T_{k-1})Q_K^k.$$

Consequently, by (3.4), (3.7), (3.8), and the approximation properties of I_k and T_{k-1} [20], we see that

$$\begin{aligned}
\|(Q_K^k - I_k Q_K^{k-1})v\|_k &= \|(I - I_k T_{k-1})Q_K^k v\| \\
&\le C\big(\|(I - T_{k-1})Q_K^k v\| + \|(I - I_k)T_{k-1}Q_K^k v\|\big) \\
&\le C h_k \|v\|_{\mathcal{E},K}, \quad v \in V_K.
\end{aligned}$$

This, together with (3.6), implies assumption (2.18). Now we see that Theorem 2.4 holds in the case of the P_1-nonconforming element.

4. Application to the Rotated Q_1-Nonconforming Element

In this section we consider the multigrid methods 2.1 and 2.2 for numerically solving (3.1) using the rotated Q_1-nonconforming finite element.

Let h_0 and $\mathcal{E}_{h_0} = \mathcal{E}_0$ be given, where $\mathcal{E}_{h_0}$ is a partition of Ω into rectangles with length h_0 and oriented along the coordinate axes. For each integer $1 \le k \le K$, let $h_k = 2^{-k}h_0$ and $\mathcal{E}_{h_k} = \mathcal{E}_k$ be constructed by connecting the midpoints of the edges of the rectangles in $\mathcal{E}_{k-1}$, and let $\mathcal{E}_h = \mathcal{E}_K$ be the finest grid. Also, let $\partial\mathcal{E}_k$ be the set of all interior edges in $\mathcal{E}_k$.

For each k, we introduce the rotated Q_1-nonconforming space [18, 36]

$$V_k = \Big\{ v \in L^2(\Omega) : v|_E = a_E^1 + a_E^2 x + a_E^3 y + a_E^4 (x^2 - y^2),\ a_E^i \in \mathbb{R},\ \forall E \in \mathcal{E}_k; \\ \text{if } E_1 \text{ and } E_2 \text{ share an edge } e, \text{ then } \int_e v|_{\partial E_1} \mathrm{d}s = \int_e v|_{\partial E_2} \mathrm{d}s; \\ \text{and } \int_{\partial E \cap \Gamma} v|_\Gamma \mathrm{d}s = 0 \Big\}.$$

The quadratic form $a_k(\cdot,\cdot)$ has the same definitions as before, while the quadratic form $(\cdot,\cdot)_k$ is defined as follows. Let $\{\phi_k^j\}$ be the basis functions of V_k such that the edge average of ϕ_k^j equals one at exactly one edge and zero at all other edges. Then each $v \in V_k$ has the representation

$$v = \sum_j v^j \phi_k^j.$$

Now, for $v,\ w \in V_k$ we define

$$(v, w)_k = h_k^2 \sum_j v^j w^j.$$

By the uniform L^2-stability of the basis functions, we can easily show that the norm induced by $(\cdot,\cdot)_k$ is equivalent to the standard $L^2(\Omega)$ norm [24].

As in the last section, we introduce two sets of intergrid transfer operators $I_k : V_{k-1} \to V_k$ and $T_{k-1} : V_k \to V_{k-1}$ as follows. Following [2, 19], if $v \in V_{k-1}$ and e is an edge of a rectangle in $\mathcal{E}_k$, $I_k v \in V_k$ is defined by

$$\frac{1}{|e|}\int_e I_k v \mathrm{d}s = \begin{cases} 0 & \text{if } e \subset \Gamma, \\ \frac{1}{|e|}\int_e v \mathrm{d}s & \text{if } e \not\subset \partial E \text{ for any } E \in \mathcal{E}_{k-1}, \\ \frac{1}{2|e|}\int_e (v|_{E_1} + v|_{E_2}) \mathrm{d}s & \text{if } e \subset \partial E_1 \cap \partial E_2 \\ & \quad \text{for some } E_1, E_2 \in \mathcal{E}_{k-1}. \end{cases}$$

If $v \in V_k$ and e is an edge of an element in $\partial\mathcal{E}_{k-1}$, $T_{k-1} v \in V_{k-1}$ is given by [24]

$$\frac{1}{|e|}\int_e T_{k-1} v \mathrm{d}s = \frac{1}{2}\left\{ \frac{1}{|e_1|}\int_{e_1} v \mathrm{d}s + \frac{1}{|e_2|}\int_{e_2} v \mathrm{d}s \right\},$$

where e_1 and e_2 in $\partial\mathcal{E}_k$ form the edge $e \in \partial\mathcal{E}_{k-1}$. Note that the definition of T_{k-1} automatically preserves the zero average values on boundary edges. These two intergrid transfer operators satisfy the relation (3.5) as well. Further, their iterated transfer operators H_k^K and Q_K^k can be defined as in the last section. Finally, the following result remains valid [20]:

Lemma 4.1. *There exist constants C independent of k such that*

$$\|Q_K^k v\|_{\mathcal{E},k} \le C\|v\|_{\mathcal{E},K}, \qquad \forall v \in V_K,$$
$$\|T_{k-1}v\|_{\mathcal{E},k-1} \le C\|v\|_{\mathcal{E},k}, \quad \forall v \in V_k.$$

Now, with the same argument as in the previous section, we can easily show assumptions (2.18) and (2.19). Consequently, Theorem 2.4 holds for the rotated Q_1-nonconforming element.

As mentioned before, the present results also apply to mixed finite element methods for second-order elliptic problems thanks to their equivalence to nonconforming finite element methods [2, 3, 17, 19, 22]. Also, an extension to three dimensions is straightforward.

5. Application to Problems with Large Jump Coefficients

In this section, as the final example of the application of the theory in §2 we apply the multigrid methods 2.1 and 2.2 to the P_1 and rotated Q_1-nonconforming finite elements for second-order elliptic problems which may have large jumps in coefficients. We only consider the P_1-nonconforming element; a similar treatment can be given for the rotated Q_1-nonconforming element.

We again consider the model problem (3.1) on a domain $\Omega \subset \mathbb{R}^2$ which we assume to have the decomposition

$$\bar{\Omega} = \cup_{l=1}^{L} \bar{\Omega}_l,$$

where $\{\Omega_l\}$ are mutually disjoint open polygons. Let $\omega_l > 0$ be the average value of a over Ω_l. Here we consider the application where there is a large variation in $\{\omega_l\}$ but little variation of a over the polygons Ω_l. Furthermore, we assume that there are positive constants a_1 and a_2 independent of $l = 1, \ldots, L$ such that

$$a_1\omega_l(\nabla v, \nabla v)_{\Omega_l} \le (a\nabla v, \nabla v)_{\Omega_l} \le a_2\omega_l(\nabla v, \nabla v)_{\Omega_l}, \qquad \forall v \in H^1(\Omega). \tag{5.1}$$

We now apply the theory in §2 to provide convergence estimates for appropriately defined multigrid methods for (3.1) which depend solely on the constants a_1 and a_2 in (5.1) but not on the values ω_l. This implies that $\max_l \omega_l / \min_l \omega_l$ can be very large without significantly reducing the convergence rate for these multigrid methods. Toward that end, we need that the coarse triangulation $\mathcal{E}_0$ align with the boundaries of $\{\Omega_l\}$, and introduce a discrete inner product which is weighted by the coefficients $\{\omega_l\}$:

$$(v, w)_{L^2_\omega(\Omega)} = \sum_{l=1}^{L} \omega_l (v, w)_{L^2(\Omega_l)}, \quad v, w \in L^2(\Omega). \tag{5.2}$$

Let $\{\phi_k^j\}$ be the usual nodal basis functions of V_k. We also modify the smoothing operator R_k as follows:

$$R_k v = \beta \sum_j \big((v, \phi_k^j)_{L^2_\omega(\Omega)} / a_k(\phi_k^j, \phi_k^j)\big)\phi_k^j, \tag{5.3}$$

where the constant β is chosen to ensure that J_k is nonnegative. In general, it can be easily seen from the definition of J_k that it is sufficient to choose $\beta = 1/3$. With these and the same argument as in §3, the theory in §2 can be applied to produce the following result.

Proposition 5.1. *With the above inner products and smoothing operators, the rate of convergence of multigrid methods* 2.1 *and* 2.2 *satisfies*

$$\delta_K \leq 1 - \frac{1}{CK^{1+\gamma}},$$

where the constant γ equals zero if the interface has no cross points (i.e., there is no point on the interface that belongs to more than two subdomains) and otherwise it equals one.

The special form of the smoothing operator R_k in (5.3) avoids the solution of Gram matrix problems associated with the inner product defined in (5.2) thanks to the feature of the nodal basis $\{\phi_k^j\}$. Also, the above result holds for three dimensions provided that, for instance, there are no internal cross points. Finally, the above analysis can be extended to local grid refinement problems, as in the conforming case [11].

6. Numerical Results

The final section presents some numerical results to illustrate the theory developed in the earlier sections and to show a comparison between the present approach and the usual approach where the discrete equations on all levels are defined by the same discretization. We consider the model problem (3.1) on the unit square $\Omega = (0,1)^2$ and with a piecewise constant coefficient:

$$a(x,y) = \begin{cases} \mu, & (x,y) \in [1/4,1/2] \times [1/4,1/2] \cup [1/2,3/4] \times [1/2,3/4], \\ 1, & \text{otherwise.} \end{cases}$$

The finite element approximation of the solution of the resulting problem is based on the P_1-nonconforming element as in §3.1. For all of the runs, the coarse grid is of size $h_0 = 1/4$, Jacobi smoothing is applied, and the number of smoothing steps is one.

We present the convergence rate δ_K as defined in (2.16) as a function of the mesh size on the finest grid for $\mu = 1, 2, 1000, 10000$ for the symmetric $\mathcal{V}$-cycle multigrid method. The numerical results using the present and usual approaches are shown in Tables 1 and 2, respectively. A deterioration in the convergence rate for the cases of $\mu = 1000$ and $\mu = 10000$ can be clearly seen, which is approximately like $1 - .95/K$ for the former approach and $1 - .97/K$ for the latter approach. Little change is observed in the convergence rate between these two cases. For the smaller values of μ, the results seem to be bounded independently of the number of levels. Finally, little difference is numerically seen between the present and usual approaches.

$K(1/h_K)$	$\mu = 1$	$\mu = 2$	$\mu = 1000$	$\mu = 10000$
2(8)	0.35	0.35	0.53	0.53
3(16)	0.41	0.42	0.67	0.68
4(32)	0.45	0.46	0.77	0.77
5(64)	0.47	0.48	0.81	0.82
6(128)	0.48	0.48	0.84	0.85

Table 1. The convergence rate δ_K for the present approach.

$K(1/h_K)$	$\mu = 1$	$\mu = 2$	$\mu = 1000$	$\mu = 10000$
2(8)	0.22	0.23	0.52	0.53
3(16)	0.27	0.27	0.67	0.67
4(32)	0.31	0.32	0.75	0.76
5(64)	0.34	0.36	0.81	0.81
6(128)	0.36	0.37	0.83	0.84

Table 2. The convergence rate δ_K for the usual approach.

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Department of Mathematics, Box 156, Southern Methodist University, Dallas, Texas 75275–0156, USA.
E-mail address: `zchen@dragon.math.smu.edu`

Department of Mathematics, Korea Advanced Institute of Science and Technology, Taejon, Korea 305–701.
E-mail address: `dykwak@math.kaist.ac.kr`