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1 Problem

Let

$$Q_n = \{0,1\}^n$$

be the n-dimensional discrete cube, viewed as a graph in which two vertices are adjacent if they differ in exactly one coordinate. For a subset $A \subseteq Q_n$, let $\partial_e A$ denote the set of edges with one endpoint in A and the other endpoint in $Q_n \setminus A$. Then

$$|\partial_e A| \geq |A|\big(n - \log_2 |A|\big).$$

2 Solution

1. Reduce to bounding internal edges:

   (a) If $A = \varnothing$, then it is trivial. So we assume $A \neq \varnothing$.

   (b) Let $e(A)$ be the number of edges of Q_n whose endpoints both lie in A.

   (c) Since all vertices of Q_n have degree n, we have

   $$n|A| = 2e(A) + |\partial_e A|.$$

   So

   $$|\partial_e A| = n|A| - 2e(A).$$

   (d) Therefore, it is enough to prove

   $$2e(A) \leq |A| \log_2 |A|.$$

2. Conditional entropy equation for $e(A)$:

(a) Define a random variable

$$X = (X_1, \dots, X_n)$$

with uniform distribution on A:

$$\mathbb{P}(X = x) = \frac{1}{|A|} \qquad \forall x \in A.$$

(b) Since X is uniformly distributed, we have

$$H(X) = \log_2 |A|.$$

(c) For each $i \in \{1, \dots, n\}$, define

$$X_{-i} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n).$$

(d) For $y = (y_1, \dots, y_{n-1}) \in \{0, 1\}^{n-1}$, define

$$A_i(y) = \{t \in \{0, 1\} : (y_1, \dots, y_{i-1}, t, y_i, \dots, y_{n-1}) \in A\}.$$

Then we have

$$|A_i(y)| \in \{0, 1, 2\}.$$

(e) If $\mathbb{P}(X_{-i} = y) > 0$, then $A_i(y) \neq \varnothing$, so

$$|A_i(y)| \in \{1, 2\}.$$

(f) Fix i and condition on $X_{-i} = y$. Let's calculate the conditional entropies:

i. If $|A_i(y)| = 1$, then X_i is determined so

$$H(X_i \mid X_{-i} = y) = 0.$$

ii. If $|A_i(y)| = 2$, then X_i can be both 0 and 1. Since X is uniformly distributed on A, both events have probability $\frac{1}{2}$ conditioned on $X_{-i} = y$. Therefore

$$H(X_i \mid X_{-i} = y) = 1.$$

(g) Then for every y with $\mathbb{P}(X_{-i} = y) > 0$,

$$H(X_i \mid X_{-i} = y) = \mathbf{1}_{\{|A_i(y)|=2\}}.$$

(h) We can express the conditional entropy by averaging over all possible values of X_{-i}.

$$H(X_i \mid X_{-i}) = \sum_{y\in\{0,1\}^{n-1}} \mathbb{P}(X_{-i}=y)H(X_i \mid X_{-i}=y) = \sum_{y\in\{0,1\}^{n-1}} \mathbb{P}(X_{-i}=y)\mathbf{1}_{\{|A_i(y)|=2\}}.$$

(i) The right-hand side is the probability

$$|A_i(X_{-i})| = 2$$

occurs. Therefore

$$H(X_i \mid X_{-i}) = \mathbb{P}(|A_i(X_{-i})| = 2).$$

(j) For $x \in Q_n$, let x^{-i} be the point obtained from x by flipping the i-th coordinate.

For $X \in A$, the condition $|A_i(X_{-i})| = 2$ is equivalent to

$$X^{-i} \in A.$$

So we have

$$H(X_i \mid X_{-i}) = \mathbb{P}(X^{-i} \in A).$$

(k) Summing over i, we get

$$\sum_{i=1}^{n} H(X_i \mid X_{-i}) = \sum_{i=1}^{n} \mathbb{P}(X^{-i} \in A).$$

(l) Since X is uniformly distributed on A, we can calculate the right-hand side:

$$\sum_{i=1}^{n} \mathbb{P}(X^{-i} \in A) = \frac{1}{|A|}\sum_{x\in A} \left|\{i \in \{1,\dots,n\} : x^{-i} \in A\}\right|.$$

(m) The right-hand side counts each edge with both endpoints in A exactly twice. So we have

$$\sum_{i=1}^{n} H(X_i \mid X_{-i}) = \frac{2e(A)}{|A|}.$$

3. Bounding entropy sum:

(a) Conditioning on more information does not increase entropy.

Since X_{-i} contains $X_1, \dots, X_{i-1}$, we have

$$H(X_i \mid X_{-i}) \le H(X_i \mid X_1, \dots, X_{i-1}).$$

(b) Therefore

$$\sum_{i=1}^{n} H(X_i \mid X_{-i}) \leq \sum_{i=1}^{n} H(X_i \mid X_1, \ldots, X_{i-1}).$$

(c) By the chain rule of entropy,

$$\begin{aligned}
\sum_{i=1}^{n} H(X_i \mid X_1, \ldots, X_{i-1}) &= H(X_1) + \sum_{i=2}^{n} H(X_i \mid X_1, \ldots, X_{i-1}) \\
&= H(X_1) + \sum_{i=2}^{n} \Big(H(X_1, \ldots, X_i) - H(X_1, \ldots, X_{i-1}) \Big) \\
&= H(X_1, \ldots, X_n) \\
&= H(X).
\end{aligned}$$

(d) Since $H(X) = \log_2 |A|$, we have

$$\sum_{i=1}^{n} H(X_i \mid X_{-i}) \leq \log_2 |A|.$$

4. Conclusion:

(a) Combining the equation from part 2 and inequality from part 3, we have

$$\frac{2e(A)}{|A|} = \sum_{i=1}^{n} H(X_i \mid X_{-i}) \leq \log_2 |A|.$$

(b) Therefore

$$2e(A) \leq |A| \log_2 |A|.$$

and we proved the goal from part 1.