POW2026-06 Polynomial integrals

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1 Problem

Let $f(x)$ be a function such that

$$(1-x^2)f''(x) - 2xf'(x) + \alpha(\alpha+1)f(x) = 0$$

for some $\alpha \notin \mathbb{N}$. Define $P_n(x) = \frac{d^n}{dx^n}(x^2-1)^n$ for $n = 0, 1, \dots$. Compute $\int_{-1}^{1} f(x)P_n(x)\,dx$.

2 Solutioin

Note: $f(x)$ is a solution of Legendre's ODE, so $f(x) \in C^\infty((-1,1))$ (or we can have this result by Fuchs's Theorem).

By the solution of Legendre's ODE, $P_n(x)$ is a solution of

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

We can alternatively check it by direct computation. Let $v(x) = (x^2-1)^n$. Then, $P_n(x) = v^{(n)}(x)$. Observe that $v'(x) = 2nx(x^2-1)^{n-1} \implies (x^2-1)v'(x) = 2nxv(x)$. Differentiating both sides $n+1$ times using Leibniz's rule:

$$\frac{d^{n+1}}{dx^{n+1}}((x^2-1)v'(x)) = (x^2-1)v^{(n+2)}(x) + (n+1)2xv^{(n+1)}(x) + \frac{n(n+1)}{2}\cdot 2v^{(n)}(x)$$
$$\frac{d^{n+1}}{dx^{n+1}}(2nxv(x)) = 2nxv^{(n+1)}(x) + 2n(n+1)v^{(n)}(x)$$

Equating the two results:

$$\begin{aligned}&(x^2-1)v^{(n+2)}(x) + 2(n+1)xv^{(n+1)}(x) + n(n+1)v^{(n)}(x)\\&= 2nxv^{(n+1)}(x) + 2n(n+1)v^{(n)}(x)\\&\implies (1-x^2)v^{(n+2)}(x) - 2xv^{(n+1)}(x) + n(n+1)v^{(n)}(x) = 0\end{aligned}$$

$\therefore (1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0.$

By this expression, we have $\frac{d}{dx}((1-x^2)P_n'(x)) = -n(n+1)P_n(x)$.

From the original ODE for $f(x)$, we have:

$$(1-x^2)P_n(x)f''(x) - 2xP_n(x)f'(x) = -\alpha(\alpha+1)f(x)P_n(x)$$

$$\implies \frac{d}{dx}((1-x^2)P_n(x)f'(x)) - (1-x^2)P_n'(x)f'(x) = -\alpha(\alpha+1)f(x)P_n(x)$$

Integrating both sides from -1 to 1:

$$\int_{-1}^{1}\left[\frac{d}{dx}((1-x^2)P_n(x)f'(x))\right]dx - \int_{-1}^{1}(1-x^2)P_n'(x)f'(x)\,dx = -\alpha(\alpha+1)\int_{-1}^{1} f(x)P_n(x)\,dx$$
$$\implies \left[(1-x^2)P_n(x)f'(x)\right]_{-1}^{1} - \int_{-1}^{1}(1-x^2)P_n'(x)f'(x)\,dx = -\alpha(\alpha+1)\int_{-1}^{1} f(x)P_n(x)\,dx$$

The boundary term evaluates to 0. Integrating the remaining term by parts again:

$$-\int_{-1}^{1}(1-x^2)P_n'(x)f'(x)\,dx = -\left[(1-x^2)P_n'(x)f(x)\right]_{-1}^{1} + \int_{-1}^{1}\frac{d}{dx}((1-x^2)P_n'(x))f(x)\,dx$$
$$= -n(n+1)\int_{-1}^{1}P_n(x)f(x)\,dx \;(\because \frac{d}{dx}((1-x^2)P_n'(x)) = -n(n+1)P_n(x))$$

Thus, we have:

$$-n(n+1)\int_{-1}^{1}P_n(x)f(x)\,dx = -\alpha(\alpha+1)\int_{-1}^{1}P_n(x)f(x)\,dx$$

$$\therefore (\alpha(\alpha+1)-n(n+1))\int_{-1}^{1}P_n(x)f(x)\,dx = 0$$
$$\therefore (\alpha-n)(\alpha+n+1)\int_{-1}^{1}P_n(x)f(x)\,dx = 0$$

If $\alpha \neq -n-1$, then since $\alpha \notin \mathbb{N}$, the scalar term is non-zero, so $\int_{-1}^{1}P_n(x)f(x)\,dx = 0$.

Suppose $\alpha = -n-1$. Then $(1-x^2)f''(x) - 2xf'(x) + n(n+1)f(x) = 0$. In this case, we have the general solution:

$$f(x) = aP_n(x) + bQ_n(x) \quad \text{on } -1 < x < 1, \text{ for } a, b \in \mathbb{R}$$

where $P_n(x)$ is a Legendre polynomial(given as this problem) and $Q_n(x)$ is a series solution which has infinity value at $x = -1, 1$. Since $f(x)$ is continuous on $\mathbb{R}$ (since it has a second derivative in original ODE), we must have $b = 0$, so $f(x) = aP_n(x)$.

$$\therefore \int_{-1}^{1}f(x)P_n(x)\,dx = a\int_{-1}^{1}P_n^2(x)\,dx$$

Observe that:

$$\int_{-1}^{1}P_n^2(x)\,dx = \left[v^{(n)}(x)v^{(n-1)}(x)\right]_{-1}^{1} - \int_{-1}^{1}v^{(n+1)}(x)v^{(n-1)}(x)\,dx$$

Note: $v(x) = (x^2-1)^n = (x-1)^n(x+1)^n$. By Leibniz's rule,

$$v^{(n-1)}(x) = \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{d^k}{dx^k}(x-1)^n \cdot \frac{d^{n-1-k}}{dx^{n-1-k}}(x+1)^n$$

Thus, every term contains $(x-1)$ and $(x+1)$ as factors, meaning $v^{(n-1)}(1) = v^{(n-1)}(-1) = 0$. Similarly, $v^{(k)}(1) = v^{(k)}(-1) = 0$ whenever $0 \leq k < n$.

$$\therefore \int_{-1}^{1}P_n^2(x)\,dx = -\int_{-1}^{1}v^{(n+1)}(x)v^{(n-1)}(x)\,dx$$

Repeating this integration by parts process n times:

$$\int_{-1}^{1}P_n^2(x)\,dx = -\int_{-1}^{1}v^{(n+1)}(x)v^{(n-1)}(x)\,dx$$
$$= \int_{-1}^{1}v^{(n+2)}(x)v^{(n-2)}(x)\,dx$$
$$= \cdots = (-1)^n\int_{-1}^{1}v^{(2n)}(x)v(x)\,dx$$
$$= (-1)^n(2n)!\int_{-1}^{1}(x^2-1)^n\,dx$$
$$= (-1)^n(2n)!\int_{-1}^{1}(x-1)^n(x+1)^n\,dx$$

Using repeated integration by parts on the remaining integral:

$$\begin{aligned}\int_{-1}^{1} (x-1)^n (x+1)^n \, dx &= (-1)^n \int_{-1}^{1} n! \frac{n!}{(2n)!} (x+1)^{2n} \, dx \\ &= \frac{(-1)^n n! n!}{(2n)!} \left[\frac{1}{2n+1} (x+1)^{2n+1} \right]_{-1}^{1} \\ &= \frac{(-1)^n n! n!}{(2n+1)!} 2^{2n+1}\end{aligned}$$

Therefore:

$$\int_{-1}^{1} P_n^2(x) \, dx = (-1)^n (2n)! \cdot \frac{(-1)^n n! n!}{(2n+1)!} 2^{2n+1} = \frac{n! n!}{2n+1} 2^{2n+1}$$

$$\therefore \int_{-1}^{1} f(x) P_n(x) \, dx = a \frac{n! n!}{2n+1} 2^{2n+1} \quad \text{(since } f(x) = a P_n(x))$$

To find a, we evaluate $P_n(x)$ at $x = 1$:

$$P_n(x) = \frac{d^n}{dx^n} ((x-1)^n (x+1)^n) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k} (x-1)^n \cdot \frac{d^{n-k}}{dx^{n-k}} (x+1)^n$$

At $x = 1$, only the $k = n$ term is non-zero:

$$P_n(1) = n! 2^n \implies f(1) = a n! 2^n \implies a = \frac{f(1)}{n! 2^n}$$

$$\therefore \int_{-1}^{1} f(x) P_n(x) \, dx = \frac{f(1)}{n! 2^n} \cdot \frac{n! n!}{2n+1} 2^{2n+1} = \frac{2^{n+1}}{2n+1} n! f(1)$$

To sum up,

$$\int_{-1}^{1} f(x) P_n(x) \, dx = \begin{cases} 0 & \text{if } \alpha \neq -n-1 \\ \frac{2^{n+1}}{2n+1} n! f(1) & \text{if } \alpha = -n-1 \end{cases}$$