POW2026-05
SEPARATING A 2-COMPONENT LINK BY SURFACES

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Problem

A link in S^3 is a smooth embedding of a finite disjoint union of circles into S^3. A link diagram is a generic projection to S^2 together with over/under data at each double point. For an oriented 2-component link $K \cup J$, the linking number $\mathrm{lk}(K, J)$ is one-half of the signed sum of the crossings between K and J.

Prove or disprove that if $\mathrm{lk}(K, J) = 0$, there there exist disjoint, compact, properly embedded, orientable surfaces $F_1, F_2 \subseteq S^3 \times I$ such that $\partial F_1 = K \times \{1\}$ and $\partial F_2 = J \times \{1\}$. Your solution should consist almost entirely of pictures. Each picture may have at most one short explanatory sentence.

Solution

We prove the statement below.

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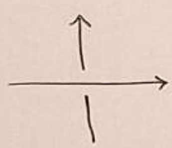
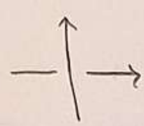
Let $K \cup J$ be a smooth link in S^3. First, we assume that each of K and J is individually the unknot. Suppose $lk(K, J) = 0$ and choose a generic projection onto the plane.

e.g.) The Whitehead link :  has linking number zero.

Convention : Given an orientation on K and J, we define the following sign conventions :



(+) crossing



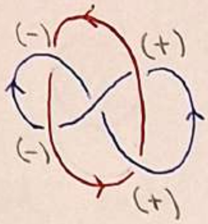
(−) crossing

We will demonstrate how to construct the surfaces F_0, F_1 for the Whitehead link, and then explain why such construction works in the general case as well.

In $S^3 \times I$, regard the last coordinate as the "time" component $t \in [0, 1]$.

Construction for Whitehead link

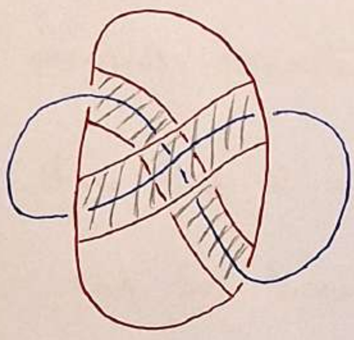
$\frac{1}{2} < t \leq 1$:



the usual Whitehead link is embedded in each time slice $S^3 \times \{t\}$.

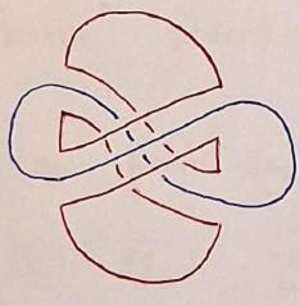
Note that since $lk = 0$, the (-) crossings and the (+) crossings can be paired up. For each pair, we may assume that the same component is the overcrossing one. (In the diagram above, the diagonal crossings form a pair.)

$t = \frac{1}{2}$:



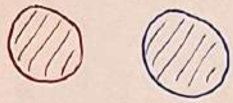
For each pair of opposite crossings, attach bands as shown. (The interior of the bands do not intersect anything else.)

$0 < t < \frac{1}{2}$:



= unlink (= O O)

Once the interior of the bands are removed, the remaining part is just the unlink.

$t = 0$:  Since the components are unlinked, each one of them bounds a disk & they can be made disjoint.

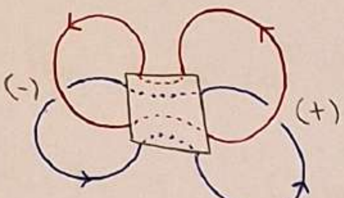
The trace of the diagrams along $t \in [0,1]$ forms two disjoint compact orientable surfaces, with boundary equal to each component of the Whitehead link at $t = 1$.

The General Construction

Looking at the previous example, we deduce that :

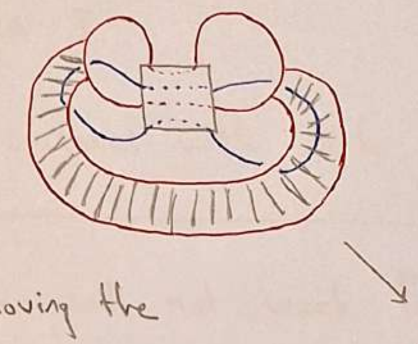
It suffices to show that a general link $L = K \cup J$ with $\mathrm{lk}(K, J) = 0$ can be turned into an unlink by applying the "attaching bands" procedure (what we did at $t = \frac{1}{2}$).

As before, we can pair each $(+)$ crossing with a $(-)$ crossing since $\mathrm{lk} = 0$. Zooming into one such pair, we have the following.



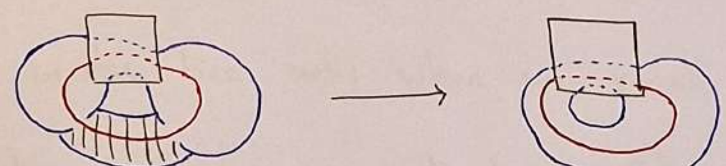
(Inside the box, the link can be very complicated. Also note that the pairing can be done so that for each pair, red crosses over blue at both crossings, or vice versa.)

Attaching band #1 :



By attaching the red band and then removing the interior, we obtain this :

Attaching band #2 :



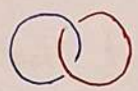
By attaching the blue band and then removing the interior, we successfully unlinked the pair of crossings !

In summary, the surfaces $F_0, F_1 \subseteq S^3 \times I$ are obtained as the trace of :

$t = 1$: Original link $L = K \cup J$

$0 < t < 1$: Repeat the "attaching bands" procedure, as explained before, until we can unlink the components K and J.

$t = 0$: Close up the surfaces with disks.

Non e.g.) The construction will not work for the Hopf link :  $(\mathrm{lk}(K,J) = \pm 1)$.

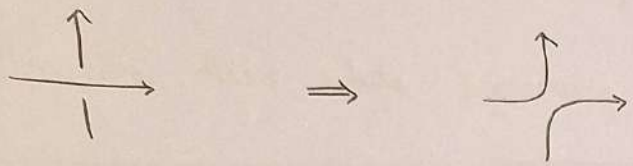
If K and J are nontrivial knots :

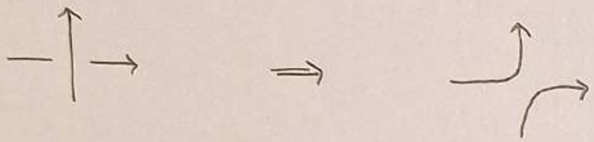
The above construction works fine until when we reach $t = 0$; we don't know whether any knot in S^3 bounds a compact orientable surface.

<u>The final claim</u> : Any knot (or link) in S^3 bounds a compact orientable surface in S^3.

Let $K \subseteq S^3$ be a smooth knot with a generic projection onto the plane. Give K an orientation.

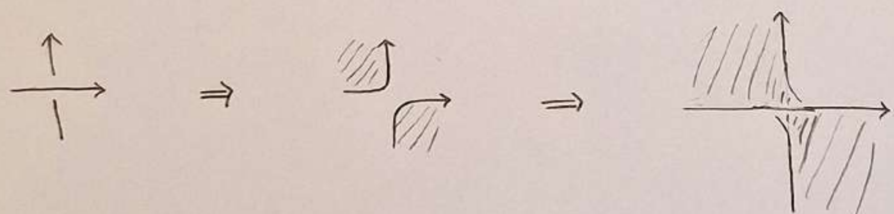
At each crossing, we smooth it out as follows.

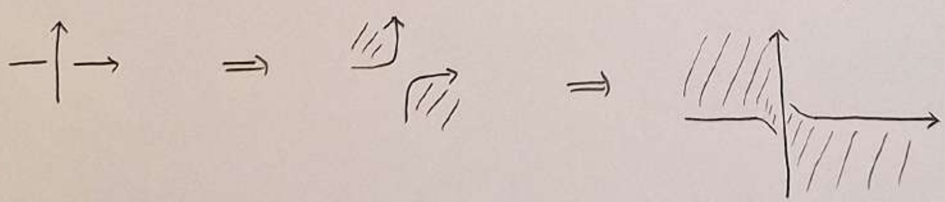




Then the knot diagram becomes a disjoint collection of circles; we may attach to them pairwise disjoint disks.

Now reconnect each crossing point with a twist:





This way, we obtain the desired compact orientable surface with boundary K. (The Seifert surface)

This surface is orientable since the circles were oriented and also the twists account for the change in orientation. In other words, each disk admits an orientation which induces the orientation of its boundary circle, and the twists glue the disks while preserving orientation.