

POW2026-04
VOTING SYSTEM

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PROBLEM

Let n be an odd positive integer, and let

$$f : \{-1, 1\}^n \rightarrow \{-1, 1\}.$$

Interpret $x_i = 1$ as voter i voting for candidate A , and $x_i = -1$ as voter i voting for candidate B . The value $f(x_1, \dots, x_n)$ is the choice. Find all functions f satisfying the following properties:

1. Anonymity: For every permutation $\sigma \in S_n$,

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

2. Neutrality:

$$f(-x_1, \dots, -x_n) = -f(x_1, \dots, x_n).$$

3. Monotonicity: If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ satisfy $x_i \leq y_i$ for all $i = 1, \dots, n$, then $f(x) \leq f(y)$.

SOLUTION

Remark (Spoiler). There is precisely one such f which outputs 1 if more voters voted for candidate A , and outputs -1 if more voters voted for candidate B . This precisely models the *majority rule*.

Proposition 1. Let $n > 0$ be any (odd or even) integer and let $S = \{(a, b) \mid a, b \in \mathbb{Z}_{\geq 0}, a + b = n\}$. Then there are canonical bijections

$$\begin{aligned} \left\{ \begin{array}{l} \text{Anonymous functions} \\ f : \{-1, 1\}^n \rightarrow \{-1, 1\} \end{array} \right\} &\simeq \left\{ \text{Functions } g : S \rightarrow \{-1, 1\} \right\}, \\ \left\{ \begin{array}{l} \text{Anonymous neutral functions} \\ f : \{-1, 1\}^n \rightarrow \{-1, 1\} \end{array} \right\} &\simeq \left\{ \begin{array}{l} \text{Functions } g : S \rightarrow \{-1, 1\} \text{ such that} \\ g(b, a) = -g(a, b) \end{array} \right\}, \\ \left\{ \begin{array}{l} \text{Anonymous neutral monotone} \\ \text{functions } f : \{-1, 1\}^n \rightarrow \{-1, 1\} \end{array} \right\} &\simeq \left\{ \begin{array}{l} \text{Functions } g : S \rightarrow \{-1, 1\} \text{ such that} \\ g(b, a) = -g(a, b) \text{ and } g \text{ is monotone} \\ \text{in the first argument} \end{array} \right\}. \end{aligned}$$

Proof. The first bijection sends an anonymous function f to $g : S \rightarrow \{-1, 1\}$ defined as

$$g(a, b) = f(\underbrace{1, \dots, 1}_{\times a}, \underbrace{-1, \dots, -1}_{\times b}).$$

Conversely, any $g : S \rightarrow \{-1, 1\}$ induces a function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ by

$$f(x_1, \dots, x_n) = g(a, b)$$

where $a = \#\{x_i = 1\}$ and $b = \#\{x_i = -1\}$. Such f is clearly anonymous since permuting the x_i 's does not change the number of 1's (resp. -1 's) in the input.

Under the first correspondence, f is neutral if and only if $g(b, a) = -g(a, b)$ for any $(a, b) \in S$. In one direction, we have

$$g(b, a) = f(\underbrace{1, \dots, 1}_{\times b}, \underbrace{-1, \dots, -1}_{\times a}) = -f(\underbrace{-1, \dots, -1}_{\times b}, \underbrace{1, \dots, 1}_{\times a}) = -f(\underbrace{1, \dots, 1}_{\times a}, \underbrace{-1, \dots, -1}_{\times b}) = -g(a, b).$$

In the other direction, if $(x_1, \dots, x_n) \in \{-1, 1\}^n$ with $a = \#\{x_i = 1\}$ and $b = \#\{x_i = -1\}$, then

$$f(-x_1, \dots, -x_n) = f(\underbrace{1, \dots, 1}_{\times b}, \underbrace{-1, \dots, -1}_{\times a}) = g(b, a) = -g(a, b) = -f(\underbrace{1, \dots, 1}_{\times a}, \underbrace{-1, \dots, -1}_{\times b}) = -f(x_1, \dots, x_n).$$

This proves the second bijection.

Moreover, under the second correspondence, f is monotone if and only if g is monotone in the first argument (i.e., if $a_1 \leq a_2$ then $g(a_1, b_1) \leq g(a_2, b_2)$). If f is monotone, then for $(a_1, b_1), (a_2, b_2) \in S$ with $a_1 \leq a_2$, we have

$$g(a_1, b_1) = f(\underbrace{1, \dots, 1}_{\times a_1}, \underbrace{-1, \dots, -1}_{\times b_1}) \leq f(\underbrace{1, \dots, 1}_{\times a_2}, \underbrace{-1, \dots, -1}_{\times b_2}) = g(a_2, b_2).$$

Conversely, suppose g is monotone in the first argument, and choose $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \{-1, 1\}^n$ with $x_i \leq y_i$ for all $i = 1, \dots, n$. Let $a_1 = \#\{x_i = 1\}$, $b_1 = \#\{x_i = -1\}$, $a_2 = \#\{y_i = 1\}$ and $b_2 = \#\{y_i = -1\}$. Since $a_1 \leq a_2$, we have

$$f(x_1, \dots, x_n) = f(\underbrace{1, \dots, 1}_{\times a_1}, \underbrace{-1, \dots, -1}_{\times b_1}) = g(a_1, b_1) \leq g(a_2, b_2) = f(\underbrace{1, \dots, 1}_{\times a_2}, \underbrace{-1, \dots, -1}_{\times b_2}) = f(y_1, \dots, y_n).$$

This proves the third bijection. \square

The previous proposition essentially says that the value of an anonymous function f only depends on “the number of votes for each candidate”.

Theorem 1. *Let $n > 0$ be an odd integer and define S as before. If $g : S \rightarrow \{-1, 1\}$ satisfies $g(b, a) = -g(a, b)$ and is monotone in the first argument, then g must be of the form*

$$g(a, b) = \begin{cases} 1 & \text{if } a > n/2 \\ -1 & \text{if } a < n/2. \end{cases}$$

Of course, the above function satisfies the desired conditions.

Proof. If $(a, b) \in S$ with $a > n/2$, then $a > b$ and it follows that $g(a, b) = -g(b, a) \geq -g(a, b)$. Hence $g(a, b) = 1$. A similar argument works when $a < n/2$ to force $g(a, b) = -1$. \square

Corollary 1. *When $n > 0$ is odd, there is only one function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ which is anonymous, neutral, and monotone:*

$$f(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } \#\{x_i = 1\} > n/2 \\ -1 & \text{if } \#\{x_i = 1\} < n/2. \end{cases}$$

Remark. When n is even, there is no such voting system. In fact, there is no anonymous neutral function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$. For if $n = 2k$ and such f exists, then the corresponding function $g : S \rightarrow \{-1, 1\}$ must satisfy $g(k, k) = -g(k, k)$ which is not possible. This reflects the intuition that no decision can be made when candidates A and B receive the same number of votes.

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