POW 2026-03: Maximum non-positivity

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We introduce new definition that generalizes the condition of problem.

Definition) $(a_1, \cdots, a_n) \in \mathbb{R}^n$ with $n \geq 3$ is V_n-tuple if it satisfies

$a_1 \leq a_2 \leq \cdots \leq a_n$, $\displaystyle\sum_{1\leq i\leq n} a_i > 0$, $\displaystyle\sum_{1\leq i<j\leq n} a_i a_j > 0$, and $\displaystyle\sum_{1\leq i<j<k\leq n} a_i a_j a_k > 0$.

Lemma 1) (a_1, a_2, a_3) is V_3-tuple if and only if $a_1 > 0$.

Proof) If part is trivial. To prove the only if part, let (a_1, a_2, a_3) be a V_3-tuple.

From $a_1a_2a_3 > 0$, we get only two possible cases: $a_1 \leq a_2 < 0 < a_3$ and $0 < a_1 \leq a_2 \leq a_3$.

For the first case, let $f(x) = a_1a_2 + (a_1 + a_2)x$. Then $f'(x) = a_1 + a_2 < 0$ and

$f(-a_1 - a_2) = a_1a_2 - (a_1 + a_2)^2 = -(a_1^2 + a_1a_2 + a_2^2) = -(a_1 + a_2/2)^2 - 3a_2^2/4 < 0$, so

$f(x) < 0$ for all $x \geq -a_1 - a_2$. As $a_3 \geq -a_1 - a_2$ from $\displaystyle\sum_{1\leq i\leq 3} a_i > 0$, it is contradiction.

Therefore, $a_1 > 0$. ■

Lemma 2) If $(a_1, \cdots, a_n)$ is V_n-tuple, then $(a_1 + c, \cdots, a_n + c)$ is also V_n-tuple for all $c \geq 0$.

Proof) $a_1 + c \leq a_2 + c \leq \cdots \leq a_n + c$, $\displaystyle\sum_{1\leq i\leq n} a_i + c = nc + \sum_{1\leq i\leq n} a_i > 0$.

$$\sum_{1\leq i<j\leq n} (a_i + c)(a_j + c) = \sum_{1\leq i<j\leq n} (a_ia_j + ca_i + ca_j + c^2) = \sum_{1\leq i<j\leq n} (a_ia_j) + C \sum_{1\leq i\leq n} (a_i) + C' > 0$$

for some nonnegative constant C, C'.

Similarly, $\displaystyle\sum_{1\leq i<j<k\leq n} (a_i + c)(a_j + c)(a_k + c) = \sum_{1\leq i<j<k\leq n} (a_ia_ja_k) + C \sum_{1\leq i<j\leq n} (a_ia_j) + C' \sum_{1\leq i\leq n} (a_i)$

$+ C'' > 0$ for some nonnegative constant C, C', C''. ■

Claim) If $(a_1, \cdots, a_n)$ is V_n-tuple, then $a_{n-2} > 0$. Furthermore, there is V_n-tuple with $a_{n-3} \leq 0$.

Proof) Use induction on n. If $n = 3$, we proved it in **Lemma 1**. Assume it holds for n.

Then for $n+1$, assume $(a_1, \cdots, a_{n+1})$ is V_{n+1}-tuple.

If $a_{n-1} \leq 0$, then $(a_1 - a_{n-1}, \cdots, a_{n-2} - a_{n-1}, 0, a_n - a_{n-1}, a_{n+1} - a_{n-1})$ is also V_{n+1}-tuple by **Lemma 2**. Then $(a_1 - a_{n-1}, \cdots, a_{n-2} - a_{n-1}, a_n - a_{n-1}, a_{n+1} - a_{n-1})$ is also V_n-tuple.

But this is contradiction as $a_{n-2} - a_{n-1} \leq 0$. Therefore, $a_{n-1} > 0$ for every V_{n+1}-tuple, so induction holds. For the second statement of claim, $(0, \cdots, 0, 1, 1, 1)$ is the example. ■

By the claim, we conclude that maximum p such that there exists V_n-tuple $(a_1, \cdots, a_n)$ whose $a_p \leq 0$ is $n - 3$. For $n = 5$, $p = 2$.