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Problem

Prove that for every positive integer k there exists a positive integer n such that

$$\frac{(n+1)(n+2)\cdots(2n-k)}{n(n-1)\cdots(n-k+1)}$$

is an integer and $k = o(n)$ for such n.

Solution

For an integer $N \geq k+21$, let $n = N! + N$. We claim that

$$M := \frac{(n+1)(n+2)\cdots(2n-k)}{n(n-1)\cdots(n-k+1)} = \frac{(2n-k)!(n-k)!}{n!n!}$$

is an integer for such n. It suffices to prove that $\nu_p(M) \geq 0$ for every prime p dividing $n(n-1)\cdots(n-k+1)$, where ν_p denotes the p-adic valuation of a rational number. For such prime p, $p|n-m$ for some $0 \leq m \leq k-1$. Since

$$N-m|N!+(N-m) = n-m,\ 20 \leq N-k \leq N-m \leq \frac{N!}{20} \leq \frac{n-m}{20}$$

$n-m$ has a divisor between 20 and $(n-m)/20$, so the largest prime divisor of $n-m$ cannot exceed $(n-m)/20$. Thus

$$p \leq \frac{n-m}{20} \leq \frac{n}{20} \tag{1}$$

Let $s_p(m)$ be the sum of the digits in the base-p expansion of $m \in \mathbb{N}$. Legendre's formula

$$\nu_p(m!) = \frac{m-s_p(m)}{p-1}$$

and the inequality

$$0 \leq s_p(m) = (m \bmod p) + \left(\left\lfloor \frac{m}{p} \right\rfloor \bmod p\right) + s_p\left(\left\lfloor \frac{m}{p^2} \right\rfloor\right) \leq 2p + \frac{m}{p^2} \leq 2p + \frac{m}{4}$$

yield

$$\frac{3}{4}m - 2p \leq (p-1)\nu_p(m!) \leq m \tag{2}$$

Finally, we will use the inequality

$$n \geq N(N-1) \geq (k+21)(k+20) \geq 40k \tag{3}$$

$$\begin{aligned}
(p-1)\nu_p(M) &= (p-1)\nu_p((2n-k)!) + (p-1)\nu_p((n-k)!) - 2(p-1)\nu_p(n!) \\
&\geq \left(\frac{3}{2}n - \frac{3}{4}k - 2p\right) + \left(\frac{3}{4}n - \frac{3}{4}k - 2p\right) - 2n \quad (\because \text{Inequality } 2) \\
&= \frac{n}{4} - \frac{3}{2}k - 4p \\
&\geq \frac{n}{4} - \frac{3}{2} \cdot \frac{n}{40} - 4 \cdot \frac{n}{20} \quad (\because \text{Inequality } 1 \text{ and } 3) \\
&\geq 0
\end{aligned}$$

Hence $M \in \mathbb{Z}$ whenever $n = N! + N$ and $N \geq k + 21$. Since n can be arbitrarily large, *a fortiori* we can choose n such that $k = o(n)$.