POW 2025-17 : Combinatorial Identity

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Problem.

$$\sum_{k=0}^{n-1} \binom{z}{k} \frac{x^{n-k}}{n-k} = \sum_{k=1}^{n} \binom{z-k}{n-k} \frac{(x+1)^k - 1}{k}.$$

Proof. We first compare the coefficients of x^i on both sides, for a fixed integer i with $1 \le i \le n$. On the left-hand side, the coefficient of x^i appears only when $n-k=i$, i.e. $k=n-i$. Hence, the coefficient of x^i is

$$\binom{z}{n-i}\frac{1}{i}.$$

On the right-hand side, we expand

$$(x+1)^k - 1 = \sum_{j=0}^{k} \binom{k}{j} x^j - 1 = \sum_{j=1}^{k} \binom{k}{j} x^j,$$

so that

$$\sum_{k=1}^{n} \binom{z-k}{n-k} \frac{(x+1)^k - 1}{k} = \sum_{k=1}^{n} \binom{z-k}{n-k} \frac{1}{k} \sum_{j=1}^{k} \binom{k}{j} x^j = \sum_{j=1}^{n} (\sum_{k=j}^{n} \binom{z-k}{n-k} \frac{1}{k} \binom{k}{j}) x^j.$$

Therefore, the coefficient of x^i is

$$\sum_{k=i}^{n} \binom{z-k}{n-k} \frac{1}{k} \binom{k}{i} = \frac{1}{i} \sum_{k=i}^{n} \binom{z-k}{n-k} \binom{k-1}{i-1},$$

since $\frac{1}{k}\binom{k}{i} = \frac{1}{i}\binom{k-1}{i-1}$.

Thus, for the coefficient of x^i, the desired identity is equivalent to

$$\binom{z}{n-i} = \sum_{k=i}^{n} \binom{z-k}{n-k} \binom{k-1}{i-1}. \tag{1}$$

Now we regard both sides of (1) as polynomials in z. Then, (LHS)-(RHS),difference of both sides of (1), is also a polynomial in z of degree at most $n-i$. Therefore, it suffices to prove (1) for $n-i+1$ distinct values of z. We choose sufficiently large integers $z \geq n$, for which we can give a combinatorial interpretation.

Fix an integer $z \geq n$. Then $\binom{z}{n-i} = \binom{z}{z-n+i}$ counts the number of subsets of $\{1, 2, \ldots, z\}$ of size $z-n+i$.

We count the same set of subsets in another way. For a subset $A \subset \{1, \ldots, z\}$ of size $z-n+i$, let k denote the i-th smallest element of A. Then k must satisfy $i \leq k \leq n$. For any such k, the number of subsets A of size $z-n+i$ whose i-th smallest element is k is

$$\binom{k-1}{i-1}\binom{z-k}{z-n},$$

because we must choose $i-1$ elements from $\{1, \ldots, k-1\}$ and $z-n$ elements from $\{k+1, \ldots, z\}$.

Summing over all k from i to n gives

$$\binom{z}{z-n+i} = \sum_{k=i}^{n} \binom{k-1}{i-1}\binom{z-k}{z-n}.$$

But this is exactly

$$\binom{z}{n-i} = \sum_{k=i}^{n} \binom{k-1}{i-1}\binom{z-k}{n-k}.$$

Thus (1) holds for all integers $z \geq n$.

It shows both sides of (1) agree at infinitely many integer points $z \geq n$, so they must be identical polynomials. Therefore (1) holds for all z, and consequently the original identity is valid.