# KAIST POW - Fall 2025

2022**** Huseyn Ismayilov, huseyn.ismaylov2005@kaist.ac.kr

**2025-16 Poisson variable:** Show that if $X$ is a Poisson random variable with parameter $\mu$, for $t > \mu + 1$, there exists a constant $c > 0$ such that $\mathbb{P}(X - \mu \geq t) \geq ce^{-2t\log(1+(t+1)/\mu)}$.

**Solution:** Let $X \sim \mathrm{Pois}(\mu)$ and fix $t > \mu + 1$. Put

$$K := \lceil \mu + t \rceil \qquad \text{and} \qquad t_K := K - \mu \in [t, t+1).$$

Then $\mathbb{P}(X - \mu \geq t) = \mathbb{P}(X \geq \mu + t) \geq \mathbb{P}(X = K)$.

For every integer $k \geq 1$,

$$k! \leq \sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{1/(12k)}$$

via Stirling's approximation. Hence

$$\mathbb{P}(X = K) = \frac{e^{-\mu}\mu^K}{K!} \geq \frac{e^{-1/(12K)}}{\sqrt{2\pi K}} \exp\Big(-\mu - K\log\frac{K}{\mu} + K\Big).$$

Write the exponent as

$$-\mu - K\log\frac{K}{\mu} + K = t_K - (\mu + t_K)\log\Big(1 + \frac{t_K}{\mu}\Big) =: g(t_K),$$

where $g(u) := u - (\mu + u)\log(1 + u/\mu)$. Since $g'(u) = -\log(1 + u/\mu) \leq 0$, $g$ is decreasing, and thus

$$g(t_K) \geq g(t+1) = (t+1) - (\mu + t + 1)\log\Big(1 + \frac{t+1}{\mu}\Big).$$

Because $t > \mu + 1$, we have $\mu + t + 1 \leq 2t$, hence

$$g(t+1) \geq (t+1) - 2t\log\Big(1 + \frac{t+1}{\mu}\Big).$$

Therefore,

$$\mathbb{P}(X \geq \mu + t) \geq \frac{e^{-1/(12K)}}{\sqrt{2\pi K}} \exp\Big((t+1) - 2t\log\big(1 + \frac{t+1}{\mu}\big)\Big).$$

Since $t > \mu + 1$, we have $K = \lceil \mu + t \rceil \leq \mu + t + 1 < 2t$ and also $t \geq 1$. Hence

$$\frac{e^{-1/(12K)}}{\sqrt{2\pi K}} e^{t+1} \geq \frac{e^{-1/12}}{\sqrt{2\pi}} \cdot \frac{e^{t+1}}{\sqrt{2t}} \geq \frac{e^{-1/12}}{\sqrt{2\pi}} \cdot \frac{e^2}{\sqrt{2}} := c > 0.$$

Combining the displays,

$$\mathbb{P}(X - \mu \geq t) \geq c\exp\Big(-2t\log\big(1 + \frac{t+1}{\mu}\big)\Big),$$

with the absolute constant

$$c = \frac{e^{2-1/12}}{\sqrt{4\pi}}.$$