KAIST POW - Fall 2025

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Denote $P = \{(x, y, z) \in \mathbb{R}^3 : 10 < x, y, z < 31\}$. Suppose a function $f(v) : \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ satisfies:

(a) $f(\lambda v) = \lambda^{25} f(v)$ for all $v \in P$ and $0 < \lambda \in \mathbb{R}$,

(b) $f(v + w) \geq f(v)$ for every $v, w \in P$,

(c) $f(v)$ is locally bounded.

Show that $f(v)$ is locally Lipschitz in P.

Solution

The problem is wrong.

Counterexample

Define, for $v = (x, y, z) \neq 0$,

$$\phi\left(\frac{v}{\|v\|}\right) = \begin{cases} 2 & \text{if } x = y = z \\ 1 & \text{otherwise} \end{cases}$$

and set

$$f(v) = \|v\|^{25} \phi\left(\frac{v}{\|v\|}\right) \quad (x, y, z > 0).$$

with $f(0, 0, 0) = 0$. We claim that f satisfies condition (a), (b) and (c).

(a) For all $v \in P \subset \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ and $0 < \lambda \in \mathbb{R}$,

$$f(\lambda v) = \|\lambda v\|^{25} \phi\left(\tfrac{\lambda v}{\|\lambda v\|}\right) = \lambda^{25} \|v\|^{25} \phi\left(\tfrac{v}{\|v\|}\right) = \lambda^{25} f(v).$$

(b) For $v, w \in P$, since $x, y, z > 10$, we know that all coordinates are positive and thus $v \cdot w > 0$. Then

$$\|v+w\|^2 > \|v\|^2 + \|w\|^2 > \|v\|^2 + (10\sqrt{3})^2$$

Combining this with $\|v\| \leq 31\sqrt{3}$, we get

$$\frac{\|v+w\|^2}{\|v\|^2} > 1 + \frac{(\sqrt{3}\,10)^2}{(\sqrt{3}\,31)^2} = 1 + \frac{100}{961}.$$

Using Bernoulli's inequality, we finally have

$$\frac{\|v+w\|^{25}}{\|v\|^{25}} > \left(1 + \frac{100}{961}\right)^{25/2} \geq 1 + \frac{25}{2} \cdot \frac{100}{961} = 1 + \frac{1250}{961} > 2$$

and thus

$$f(v+w) = \|v+w\|^{25}\,\phi\Big(\tfrac{v+w}{\|v+w\|}\Big) \geq \|v+w\|^{25} \cdot 1 > 2\|v\|^{25} \geq \|v\|^{25}\phi\Big(\tfrac{v}{\|v\|}\Big) = f(v).$$

(c) On any bounded subset of $\mathbb{R}^3$, both $\|v\|^{25}$ and ϕ are bounded, so f is locally bounded.

However, f is not continuous in P. For $p = (t, t, t) \in P \quad (10 < t < 31)$,

$$\lim_{\substack{v \to p \\ x \neq y}} f(v) = \|p\|^{25} \neq f(p) = 2\|p\|^{25}$$

Since locally Lipschitz functions are continuous by the definition, f cannot be locally Lipschitz in P.